problems with group completion paper

June 2, 1971: On group completion.

In ~~connection with~~ writing the paper about group completion of simplicial monoids, I find that there are several points that are not well understood.

Simplest case: Let M be a simplicial monoid and $\bar{M}$ its group completion. Assume M is connected and good for group completion. Then I prove, using the Zeeman comparison theorem for the map of Eilenberg-Moore spectral sequences associated to $M \to \bar{M}$, that

$$H_*(M,k) \xrightarrow{\sim} H_*(\bar{M},k)$$

for any field k. In virtue of the spec. seq.

$$E^2_{pq} = \mathrm{Tor}_p^{k[\pi_1 M]}(H_q(\tilde{M},k),k) \Longrightarrow H_{p+q}(M,k)$$
$$\|$$
$$\mathrm{Tor}_p^{k[\pi_1 M]}(k,k) \otimes H_q(\tilde{M},k)$$

(= because $\pi_1 M$ acts trivially on $H_*(\tilde{M},k)$, as M is a monoid) and the similar one for $\bar{M}$, one has ~~that~~ again by the comparison theorem that

$$H_*(\tilde{M},k) \xrightarrow{\sim} H_*(\tilde{\bar{M}},k).$$

~~It follows~~ This being true for any field k, it follows that $M \to \bar{M}$ is a quasi-isomorphism, in particular that

$$H_*(M,L) \xrightarrow{\sim} H_*(\bar{M},L)$$

for any local coefficient system L on $\bar{M}$.

(But using the comparison theorem is pretty heavy-handed, and it would be desirable to have a good "categorical" understanding of what was happening. ~~...~~ Thus if $\bar{M}$ ~~... is~~ admits calculation by right fractions, the canonical map

$$\begin{array}{ccccc} B(M^2, M) & \xrightarrow[\text{augmentation}]{\text{horizontal}} & M^2_M & \xrightarrow{\sim} & \bar{M} \\ \| & & \| & & \text{induced by} \\ \text{nerve of } M \text{ acting} & & \text{orbit} & & (s,t) \mapsto st^{-1} \\ \text{diagonal on the} & & \text{space of} & & \\ \text{right of } M^2 & & \text{action} & & \end{array}$$

is acyclic and on the other hand the spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^{H_*(M)}(H_*(M^2), k)_q \Longrightarrow H_{p+q}(B(M^2, M), k)$$

degenerates yielding

$$H_*(M, k) \xrightarrow{\sim} H_*(B(M^2, M), k)$$

because $H_*(M^2)$ is free over $H_*(M)$ as latter is a sub-Hopf algebra (with inversion). This argument is more satisfying and might ~~...~~ generalize to a monoid in a topos, ~~...~~ provided the correct analogue of $B(M^2, M)$ could be found. For example if M is free, one would want to replace $B(M^2, M)$ by a simplicial gadget resolving $\bar{M}$.
)

Next suppose that M is good and that the localization of $H_*(M,\mathbb{Z})$ with respect to $\pi_0 M = S$ admits calculation by right fractions. Then the same is true of $H_*(M,k)$ for any [redacted] field k. (One [redacted] shows that left multiplication by an element [redacted] of S on $H_*(M,A).S^{-1}$ is invertible for any abelian group A, ~~starting~~ from $A=\mathbb{Z}$ using dévissage.) Hence one knows by the theorem of the group completion paper that

$$H_*(M,k).S^{-1} \xrightarrow{\sim} H_*(\bar{M},k)$$

$$\varinjlim_{s} H_*(M_s,k) \xrightarrow{\sim} H_*(\bar{M}_e,k)$$

for k any field. By devissage its true for [redacted] k replaced by any abelian group. <u>Problem:</u> Can k be replaced by any local coeff. system L on $\bar{M}$?

Observe first that if L is a loc. coeff. system on $\bar{M}$, then L can be pulled back via the canonical map $M \to \bar{M}$ [redacted] and there is a canonical map

$$H_*(M,L) \longrightarrow H_*(\bar{M},L).$$

[redacted] Unfortunately I see no natural action of S on either of these, nor does it seem possible to define an S action on L covering that (only defined up to homotopy) on $\bar{M}$. So it doesn't seem to make sense to generalize the first isom.

[redacted]

Next observe that for each $s \in S$, there is a canonical homomorphism

$$\begin{tikzcd}
\pi_1(M_s) \arrow[r, dashed] \arrow[d] & \pi_1(\overline{M}_e) \arrow[d, "\cong"] \\
H_1(M_s, \mathbb{Z}) \arrow[r] & H_1(\overline{M}_e, \mathbb{Z})
\end{tikzcd}$$

where the bottom arrow is induced by the canonical map $M_s \longrightarrow \overline{M}_s$ followed by right multiplication by s^{-1} on $H_*(\overline{M}, \mathbb{Z})$. This means that given a local coefficient system L on $\pi_1(\overline{M}_e)$ one obtains a local coefficient system on M_s unique up to isom., but not necessarily up to canonical isom., so again there is a problem. Thus the problem on page 3 doesn't seem to make sense.

If M acts to the right on a simplicial set X, let $B(X, M)$ be the diagonal of the nerve of the simplicial category with objects X and arrows $X \times M$:

$$\begin{tikzcd}
X \times M \times M \arrow[r, shift left=2, "\mu \times id"] \arrow[r, "id \times \mu" description] \arrow[r, shift right=2, "pr_{12}"'] & X \times M \arrow[r, shift left=1, "\mu"] \arrow[r, shift right=1, "pr_1"'] & X
\end{tikzcd}$$

Then one has a spectral sequence

$$E^1_{pq} = H_q(X \times M^p; L) \Longrightarrow H_{p+q}(B(X, M); L)$$

where L is any local coefficient system on $B(X, M)$, i.e. any local coeff. system on X endowed with a compatible action of M.

Now $X = M^2$ with diagonal multiplication action. Then there are maps

$$B(M^2, M) \xrightarrow[\text{augmentation}]{\text{horizontal}} \underset{\substack{\| \\ \text{orbit} \\ \text{space}}}{M^2_M} \xrightarrow{(m_1, m_2) \mapsto m_1 m_2^{-1}} \overline{M} \longleftarrow G\overline{W}(M)$$

If M is good the last map is a quis of Kan complexes, whence there is a map of simplicial sets

$$(*) \qquad B(M^2, M) \longrightarrow G\overline{W}(M)$$

unique up to canonical homotopy. ~~Problem: Show this last map is a quis when~~ In general if M is not good, we choose a free resolution $P \to M$, and $B(P^2, P) \longrightarrow B(M^2, M)$ is a quis, whence one obtains a well-defined map $(*)$ in the derived category of pointed semi-simplicial sets. <u>Problem</u>: Show $(*)$ is an isomorphism in derived category when the localization of $H_*(M, \mathbb{Z})$ can be calculated by right fractions.

It suffices to prove that the map

$$B(M^2, M) \longrightarrow \overline{M}$$

is a quis when M is a free simplicial monoid. First consider homology with coefficients in a field k. One uses the spec. seq. at bottom of page 4

$$E^2_{pq} = \mathrm{Tor}_p^{H_*(M)}\left(H_*(M^2), k\right)_q$$

$$\qquad\qquad \Vert$$

$$\mathrm{Tor}_p^{H_*(\bar{M})}\left(H_*(M^2)S^{-1}, k\right)_q$$

Let $R = H_*(M)$, and let $\Delta : R \longrightarrow R \otimes R$ be the diagonal. Then if C is the filtering category belonging to S and its right multiplications, $C \times C$ is also filtering and $\Delta : C \longrightarrow C \times C$ is cofinal, whence

$$(R \otimes R)S^{-1} = RS^{-1} \otimes RS^{-1}.$$

But we know that $RS^{-1} = H_*(\bar{M})$ ~~and hence~~ that

$$\bar{R} \otimes \bar{R} = (\bar{R} \otimes 1) \tilde{\otimes} (\Delta \bar{R})$$

(corresponds to $G \times G$ being the semi-direct product of ΔG acting on by $G \times e$). Hence

$$E^2_{pq} = \mathrm{Tor}_p^{H_*(\bar{M})}\left(H_*(\bar{M}^2), k\right)_* = \begin{cases} 0 & p > 0 \\ H_*(\bar{M}) & p = 0 \end{cases}$$

and the map $\mathrm{Tor}_{0,*} \longrightarrow H_*(\overline{M})$ is clearly the map induced by the map $B(M^2, M) \longrightarrow \overline{M}$. This shows therefore that the last map induces isoms. on homology with coeffs. in any field, hence with arbitrary (constant) coeffs.

If we want to extend this argument, it seems necessary to show that if $\nu: M^2 \to \overline{M}$ sends $s,t \longmapsto st^{-1}$, and if L is a local coeff. system of k-modules on $\overline{M}$, k a field, then

$$H_*(M^2, \nu^* L).S^{-1} \xrightarrow{\sim} H_*(\overline{M}^2, \overline{\nu}^* L)$$

where $\overline{\nu}: \overline{M}^2 \longrightarrow \overline{M}$ sends $s,t \mapsto st^{-1}$. In addition, to solve the problem on page 5, we need to show that $B(M^2, M)$ and $\overline{M}$ have the same fundamental groupoids. (Actually we would be done if we knew $B(M^2, M)$ were simple.)

If L is a local coeff. system over M^2 endowed with M-action

$$\begin{array}{ccc} L \times M & \xrightarrow{\nu} & L \\ \downarrow & & \downarrow \\ M^2 \times M & \xrightarrow{\mu} & M^2 \end{array}$$

then one has isos.

$$L_{xy} \xrightarrow[\simeq]{\cdot m} L_{xm,ym}$$

Letting m_t run over a loop representing an element of $\pi_1(M, m)$ one sees ~~these~~ isos. show that the loop (xm_t, ym_t)

acts trivially on $L_{xm,ym}$, hence the composite

$$\pi_1(M,m) \longrightarrow \pi_1(M,xm) \times \pi_1(M,ym) \xrightarrow[L_{xm,ym}]{\text{action on}} \mathrm{Aut}(L_{xm,ym})$$

is zero. If we $x=y=e$ this shows that the action of $\pi_1(M,m) \times \pi_1(M,m)$ on L_{mm} kills the diagonal subgroup, hence that the action is abelian. Indeed the quotient of G^2 by the normal subgroup generated by ΔG is G_{ab}. Consequently it seems we've proved that the identity component of $B(M^2, M)$ has an abelian fundamental group.

June 4, 1970: Galois cohomology

Suppose $F \subset E$ is a finite Galois extension of fields with $\pi = Gal(E/F)$. One wants to relate $K_* F$ and $H^*(\pi, K_* E)$. The first thing one tries for is a spectral sequence

$$E_2^{pq} = H^p(\pi, K_{-q} E) \Longrightarrow K_{-p-q} F.$$

The motivation: suppose $X \longrightarrow Y$ is a Galois covering with group π, and let h^* be a generalized cohomology theory. Then the canonical map

$$P_\pi \times^\pi X \longrightarrow X/\pi = Y$$

is a heq whence we can think of Y as being fibred over $B\pi$ with fibre X. Then we can filter $B\pi$ with respect to its skeletal filtration obtaining a spectral sequence

$$E_2^{pq} = H^p(\pi, h^q(X)) \Longrightarrow h^{p+q}(Y)$$

which will converge at least if $B\pi$ is a finite dimensional complex.

__Digression:__

Suppose we consider the localization situation.

$$\mathrm{Modf\,tors}(A) \longrightarrow \mathrm{Modf}(A) \longrightarrow \mathrm{Modf}(K),$$

A d.v.r. Let G be a group. Then

$$\mathrm{Modftors}(A)_G \hookrightarrow \mathrm{Modf}(A)_G \longrightarrow \mathrm{Modf}(K)_G$$

is no longer exact, however the quotient category may be identified with the full subcat. $\mathcal{I}$ of repns V of G over K such that V contains a lattice stable under G. (To give a G-homomorphism from $L \otimes K$ to $L' \otimes K$ is the same thing as giving a G-morphism $\pi^n L \to L'$ for some n.)

Now consider the functor

$$\mathcal{I} \hookrightarrow \mathrm{Modf}(K)_G$$

and its effect on the Q-categories. ~~Observe $\mathcal{I}$ is a thick subcategory~~ Then

$$Q(\mathcal{I}) \hookrightarrow Q(\mathrm{Modf}(K)_G)$$

is fully faithful and is a crible, that is, if $\exists M \to I \in \mathcal{I}$ then $M \in \mathcal{I}$. Given a representation of G on a K-vector space V, we would like to consider the subquotients which are in $\mathcal{I}$. The category of these is discrete up to homotopy.

$\mathrm{Modf}(K)_G$ is an abelian category in which every object has finite length; $\mathcal{I}$ being a full subcategory has the same property. So without changing homotopy types, I should be able to replace these by their semi-simple subcategories.

$$\mathcal{I}_{ss} \hookrightarrow \left(\mathrm{Modf}(K)_G\right)_{ss}$$

So I conclude that the induced maps on homotopy groups will be injective onto a direct summand. In particular we should have

$$\begin{tikzcd}
{} \arrow[r, "\partial"] & K_i(k,G) \arrow[r] & K_i(A,G) \arrow[r] & K_i(\mathcal{I}) \arrow[r, "\partial"] \arrow[d, shift right] & K_{i-1}(k,G) \\
& & & K_i(K,G) \arrow[u, shift right] &
\end{tikzcd}$$

In particular, in dimension zero we have

$$\begin{tikzcd}
R_k(G) \arrow[r] \arrow[dr, "0"] & R_{A\,\mathrm{mods}}(G) \arrow[r] \arrow[d] & R_{\substack{K\mathrm{mods}\\ \text{with } L}}(G) \arrow[r] \arrow[dl, dashed] \arrow[dd, shift right] & 0 \\
& R_k(G) & & \\
& & R_K(G) \arrow[uu, shift right] &
\end{tikzcd}$$

In the case of a complete curve C over an alg closed field k with function field K, we have the exact sequence

$$\longrightarrow \check{K}_{i+1}F \xrightarrow{\partial} K_i k \otimes Div \longrightarrow \tilde{K}_i C \longrightarrow \tilde{K}_i F$$

and we want to prove

$$\tilde{K}_i F = K_{i-1}k \otimes (F^*/k^*)$$

or equivalently $\cancel{\tilde{K}_i C = K_{i-1}k \otimes Pic}$

$$\cancel{\to \mathrm{Tor}_1(K_{i-1}k, Pic) \to \tilde{K}_i C \to K_i k \otimes Pic}$$

$$0 \longrightarrow K_i k \otimes Pic \longrightarrow \tilde{K}_i C \longrightarrow \mathrm{Tor}_1(K_{i-1}k, Pic) \longrightarrow 0$$

Now the approach I want to use ~~[illegible]~~ would try to take advantage of the isomorphism

$$R_C(G) = R_k(G) \otimes K_0 C$$

for all G.

What I am after is an exact sequence

$$\xrightarrow{\partial} F^*/k^* \otimes K_i k \longrightarrow D \otimes K_i k \xrightarrow{\alpha} \tilde{K}_i C \xrightarrow{\partial} F^*/k^* \otimes K_{i-1} k$$

and perhaps I can get this from the long exact sequence for a localization by inventing an abelian category $\mathcal{A}$ with $K_* \mathcal{A} = D \otimes K_* k$ and a Serre subcategory $\mathcal{B}$ with $K_* \mathcal{B} = F^*/k^* \otimes K_* k$.

There is an interesting confusion here; the map α above is the sum of the transfers:

$$\begin{array}{ccc}
\bigoplus_{P \in C} K_*(k(P)) & \xrightarrow{\Sigma_i \,\mathrm{tr.}} & K_*(C) \\
\wr\| & & \downarrow \\
D \otimes K_*(k) & \xrightarrow{\alpha} & \tilde{K}_*(C)
\end{array}$$

whereas the embedding $\mathrm{Pic} \otimes K_*(k) \subset K_*(C)$ we seek should be induced by multiplication

$$(L, \; V \ k\mathrm{mod}) \longmapsto L \otimes_k V.$$

The relation is as follows. Transfer is

$$V \longmapsto \mathcal{O}_P \otimes V$$

which in ~~this~~ virtue of the exact sequence

$$0 \to \mathcal{O}(P)^{-1} \to \mathcal{O} \to \mathcal{O}_P \to 0$$

(in general for $D \geq 0$, we have

$$0 \to \mathcal{O}(D)^{-1} \to \mathcal{O} \to \mathcal{O}_D \to 0 \;)$$

is equivalent to

$$V \longmapsto (1 - \mathcal{O}(P)^{-1}) \otimes V$$

Thus up to sign and transpose it has the same image in $\tilde{K}_*(C)$.

C curve

If $H^1(E) = 0$, does it follow that

$$\bigoplus_{D \geq 0} \Gamma(E(D))$$

is a free, or projective $\bigoplus_{D \geq 0} \Gamma(\mathcal{O}(D))$

module. never.

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that's what's peculiar.

<u>example:</u>

$$M \longmapsto H^i(M(D)).$$ module over $\bigoplus_{D} \Gamma(\mathcal{O}(D))$

but it is slightly peculiar.

H^0 reasonable in the positive direction

H^1 ———————— negative ————.

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k finite then probably $\bigoplus_{D} \Gamma(\mathcal{O}(D))$ is a noetherian ring. <u>No</u> wrong growth. wrong eigenvalues.

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am after exact sequence (transfer)?

$$F^*/k^* \otimes K_i k \longrightarrow \underline{D} \otimes K_i k \longrightarrow \tilde{K}_i C \longrightarrow F^*/k^* \otimes K_{i-1} k$$

Thus what I would want to do is to perhaps to invent an abelian category

FUNDAMENTAL PROBLEM:

To understand the h-factorization of a map f

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "i"] \arrow[dr, "f"'] & & \mathcal{E} \arrow[dl, "p"] \\ & \mathcal{C}' & \end{tikzcd}$$

as intrinsically as possible.

__Ex:__ Given $L \in D_{lc}(\mathcal{C}) \xleftarrow[i^*]{\sim} D_{lc}(\mathcal{E})$, we can recover $Rp_*((i^*)^{-1}L) = \mathbb{H}f_*(L)$ by the adjunction fmla

$$\mathrm{Hom}_{D_{lc}(\mathcal{C}')}(L', \mathbb{H}f_*(L)) = \mathrm{Hom}_{D_{lc}(\mathcal{C})}(f^*L', L)$$

Thus we can recover the cohomology groups of the homotopy-fibres:

$$H^q(\mathcal{E}_y, L) = \mathbb{H}^q f_*(L)_y$$

but only for L a local system on $\mathcal{C}$.

__Question:__ Can you recover $\underline{\pi}_\bullet \mathcal{E}_y$ intrinsically?

Better: given $x \in \mathcal{C}$, find $\pi_1(\mathcal{E}_y, x)$.

July 8, 1971: Quasi-fibrations.

Let $f: X \longrightarrow Y$ be a morphism of simplicial sets, and let F be a sheaf of abelian groups on X, that is, a contravariant functor from the fibred category Δ/X to Ab, or an abelian group object of ~~the category of simplicial sets over X.~~ Then working with these "simplicial" sheaves we have

$$R^q f_*(F)_y = H^q(X_y, F)$$

~~where~~

$$X_y = \Delta(d) \times_Y X$$

where d = degree of y and $\Delta(d) \longrightarrow Y$ is the canonical map associated to y.

If F is locally constant on X and if f is a fibration, one knows that $R^q f_*(F)$ is a locally constant sheaf on Y for each q.

Indeed given a map $y' \longrightarrow y$ in Δ/Y have

$$\begin{tikzcd} \Delta(d') \arrow[dr, "y'"] \arrow[d] & \\ & Y \\ \Delta(d) \arrow[ur, "y"'] & \end{tikzcd}$$

hence have a map $X_{y'} \longrightarrow X_y$ which is a weak equivalence, hence have isomorphism

$$H^*(X_y, F) \xrightarrow{\sim} H^*(X_{y'}, F).$$

Suppose now that F is locally constant on X and that $R^q f_*(F)$ is loc. const. on Y for all q. Then choose a factorization

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

with i a weak equivalence and f' a fibration. Let F' be the locally constant sheaf on X with $i^* F' = F$. Then we have a map of Leray spectral sequences

$$\begin{tikzcd} {'E_2^{pq}} = H^p(Y, R^q f'_*(F')) \arrow[r, Rightarrow] \arrow[d] & H^{p+q}(X', F') \arrow[d, "\cong"] \\ E_2^{pq} = H^p(Y, R^q f_*(F)) \arrow[r, Rightarrow] & H^{p+q}(X, F) \end{tikzcd}$$

If $\pi_1 Y = 0$, then the Zeeman comparison thm. shows that

$$R^q f'_*(F') \xrightarrow{\sim} R^q f_*(F)$$

for all q. The same conclusion can be drawn in general by reducing to

$$\begin{tikzcd} \tilde{X} \arrow[rr] \arrow[dr] & & \tilde{X}' \arrow[dl] \\ & \tilde{Y} & \end{tikzcd}$$

where $\tilde{Y}$ is ~~the~~ a universal covering of Y and $\tilde{X}, \tilde{X}'$ are the induced coverings

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suppose given maps of spaces ~~say~~

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

with Y fibrant over S, and $X \to Y$ a weq. Factor $X \to Y$ in the standard way but over S.

$$X \longrightarrow X \times_Y (Y/S)^I \longrightarrow Y$$

where $(Y/S)^I$ is the space of paths in the fibres of $Y \to S$. Claim second map is a fibration if X fibrant over S:

$$\begin{tikzcd}
X \times_Y (Y/S)^I \arrow[r] \arrow[d] & Y^I \arrow[d, "\text{fibn.}"] \\
X \times_S Y \arrow[r] \arrow[d, "\text{fibn if } X/S \text{ fibrant}"] & Y \times_S Y \\
Y &
\end{tikzcd}$$

Since $X \to Y$ a weq, follows that $X \times_Y (Y/S)^I \to Y$ is a trivial fibration, hence if Y cofibrant it has a section and hence we have a map $Y \to X$ (over S) $\ni$ $Y \to X \to Y$ is homotopic to id_Y (over S). Similarly if X cofibrant we get a homotopy on the other side, so $X \to Y$ is a fiber homotopy equivalence over S

Summary of work related to Friedlander's thesis - July, 1971.

(I.) A Verdier-style theorem for the Leray spectral sequence.

1.) Motivation: Given a map $f: X \longrightarrow Y$ (of topoi) and a sheaf F on X, there is the Leray spectral sequence

$$E_2^{pq} = H^p(Y, R^q f_*(F)) \Longrightarrow H^{p+q}(X, F).$$

Suppose we have a ~~hypercovering~~ Q of Y which is fine enough to compute the cohomology $H^p(Y, R^q f_*(F))$ in the sense that

$$(1) \qquad H^+(Q_r, R^q f_*(F)) = 0 \qquad \forall q, r$$

so that the canonical edge homo

$$(2) \qquad H^p(r \mapsto \Gamma(Q_r, R^q f_*(F))) \xrightarrow{\sim} H^p(Y, R^q f_*(F)).$$

is an isomorphism. On the other hand (1) implies that the Leray spectral sequence for the map $f^{-1}Q_r \longrightarrow Q_r$ and sheaf $F/_{f^{-1}Q_r}$ ~~trivially degenerates~~:

$$E_2^{pq} = H^p(Q_r, R^q f_*(F)) \Longrightarrow H^{p+q}(f^{-1}Q_r, F)$$

degenerates, yielding a canon. isom.

$$(3) \qquad \Gamma(Q_r, R^q f_*(F)) \xleftarrow{\sim} H^q(f^{-1}Q_r, F)$$

To compute the latter simplicially one needs a hypercovering $P_{r\cdot} = \{P_{rs}\}$ of $f^{-1}Q_r$ for each r. Indeed, it is easily seen that hypercoverings of the form $f^{-1}Q_r \times_X \mathcal{U}$, with ~~$\mathcal{U}$~~ $\mathcal{U}$ a hypercovering of X are not sufficiently fine to compute $H^*(f^{-1}Q_r, F)$; this is because the sheaf on X associated to the presheaf

$$U \longmapsto H^*(f^{-1}Q_r \times_X U, F)$$

is $R^* j_*(j^*F)$, where $j: f^{-1}Q_r \longrightarrow X$ is the canonical map, and this sheaf need not be zero.

Therefore we must consider bisimplicial objects $P = \{P_{rs}\}$ of X with an augmentation

$$P \longrightarrow f^{-1}Q$$

~~[illegible]~~ such that $P_{r\cdot}$ is a hypercovering of $f^{-1}Q_r$ for each r. Assume we can find a P which is fine enough in the sense that

(4) $$H^+(P_{rs}, F) = 0 \qquad \forall r, s.$$

Then the canonical edge homo. gives an isom.

(5) $$H^q(s \longmapsto \Gamma(P_{rs}, F)) \xrightarrow{\sim} H^q(f^{-1}Q_r, F)$$

Combining the isos. (2), (3), (5) we obtain

$$H^p(r \longmapsto H^q(s \longmapsto \Gamma(P_{rs}, F))) \xrightarrow{\sim} H^p(Y, R^q f_*(F)).$$

showing the E_2-term of the Leray spectral sequence is the same as that of the spectral sequence associated to the bicosimplicial abelian group $\Gamma(P_{rs}, F)$. Observe that even without the acyclicity assumptions (1) + (4) there is a map

(6) $$H^p(r \mapsto H^q(s \mapsto \Gamma(P_{rs}, F))) \longrightarrow H^p(Y, R^q f_*(F))$$

suggesting that the simplicial spectral sequence always maps to the Leray spec. sequence.

The program is now as follows: Produce such a map, ~~the~~ make the pairs (Q,P) into a category, and show in the limit the simplicial spectral sequence becomes isomorphic to the Leray spec. seq.

2.) Morphisms of cohomology and Leray spectral sequences.

Let $f: X \longrightarrow Y$ be a morphism of topoi, F an object of ~~[illegible]~~ X, G an object of ~~[illegible]~~ Y, and let $u: F \to G$ be an arrow over f, i.e. a map $f^*G \to F$ in X, or equivalently an arrow $G \to f_* F$ in Y. Write

$$(f,u) : (X,F) \longrightarrow (Y,G)$$

To (f,u) is associated a morphism of cohomology

$$(f,u)^* : H^q(Y,G) \longrightarrow H^q(X,F)$$

defined as follows: If $G \to J^\cdot$ is an injective resolution, then $f^*(G) \longrightarrow f^*(J^\cdot)$ is a resolution, hence there is a ~~map~~ dotted arrow $\tilde{u}$ unique up to homotopy:

$$\begin{tikzcd} f^*G \arrow[r] \arrow[d, "u"'] & f^*J^\cdot \arrow[d, dashed, "\tilde{u}"] \\ F \arrow[r] & I^\cdot \end{tikzcd}.$$

Then $\tilde{u}$ furnishes a morphism of complexes

(1) $$\Gamma(Y, J^\cdot) \longrightarrow \Gamma(X, I^\cdot)$$

whose effect on cohomology is $(f,u)^*$ by defn.

Prop 2.1: If $G \xrightarrow{\sim} f_* F$ and $R^q f_*(F) = 0$ for $q > 0$, then

$$(f,u)^* : H^n(Y, G) \xrightarrow{\sim} H^n(X, F) \qquad \forall n.$$

For in this case $f_* I^\cdot$ is an injective resolution of G, hence we can take $J^\cdot = f_* I^\cdot$ whence (1) is an ~~isom~~

[Alternative interpretation of $(f,u)^*$: Composition

(2) $$H^n(Y, G) \xrightarrow{u_*} H^n(Y, f_* F) \longrightarrow H^n(X, F)$$

where second map is the edge homo. in Leray spec. seq. of (f, F).]

Now suppose we have a square of topoi

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

(2-commutative ?) and a map

$$(g', u) : (X', F') \longrightarrow (X, F).$$

Then we have a map

$$(g, v) : \left(Y', \underline{\underline{R}} f'_*(F')\right) \longrightarrow \left(Y, Rf_*(F)\right)$$

as follows. If $F \longrightarrow I^{\cdot}$ and $F' \longrightarrow I'^{\cdot}$ are injective resolutions, there exists dotted arrow

$$\begin{tikzcd} g'^* F \arrow[r] \arrow[d, "u"] & g'^* I^{\cdot} \arrow[d, dotted, "\tilde{u}"] \\ F' \arrow[r] & I'^{\cdot} \end{tikzcd}$$

Applying f_* to the "adjoint" of $\tilde{u}$ furnishes a map

$$f_* I^{\cdot} \longrightarrow f_* g'_* I'^{\cdot} = g_* f'_* I'^{\cdot}$$

which adjoints to the desired map

$$v : g^* \underline{\underline{R}} f_*(F) \longrightarrow \underline{\underline{R}} f'_*(F').$$

As g^* is exact, it carries the Postnikov decomposition of $\underline{\underline{R}} f_*(F)$ into that of $g^* \underline{\underline{R}} f_*(F)$, and the latter gets mapped by v to the Postnikov decomposition of $\underline{\underline{R}} f'_*(F')$. Hence (g, v) induces a map of spectral sequences

$$
\begin{array}{lcl}
E_2^{pq}(f,F) = H^p(Y, R^q f_*(F)) & \Longrightarrow & H^{p+q}(X,F) \\
\downarrow & (3) & \downarrow (g';u)^* \\
E_2^{pq}(f',F') = H^p(Y', R^q f'_*(F')) & \Longrightarrow & H^{p+q}(X',F') .
\end{array}
$$

~~The map on abutments is $(g';u)^*$, that of the E_2-terms is $(g', \mathcal{H}^q(v))^*$ where~~

$$\mathcal{H}^q(v) : \quad g^* R^q f_*(F) \longrightarrow R^q f'_*(F')$$

~~is the effect of v on the homology sheaves of degree q.~~

<u>Prop. 2.2</u>: Assume

i) $F \xrightarrow{\sim} g'_* F'$ and $R^+ g'_*(F') = 0$

ii) $R^+ g_*(R^q f'_*(F')) = 0$ all q.

Then (3) is an isomorphism of the Leray spec. seqs of (f,F) and (f',F').

~~Alternative description of v: Set $h = fg' = gf'$. Then have~~

$$\underline{\underline{R}}f_*(g'_*F') \to \underline{\underline{R}}f_*\underline{\underline{R}}g'_*F' = \underline{\underline{R}}h_*(F') = \underline{\underline{R}}g_*\underline{\underline{R}}f'_*(F')$$

~~which induces a~~

Alternative description of v. Adjoint to

$$\begin{array}{ccc} \underline{\underline{R}}f_*(g'_* F') & \longrightarrow & \underline{\underline{R}}f_* \underline{\underline{R}}g'_*(F') = \underline{\underline{R}}g_* \underline{\underline{R}}f'_*(F') \\ \uparrow u & & \\ \underline{\underline{R}}f_*(F) & & \end{array}$$

where ~~the~~ second map is canonical arrow $g'_*F' \longrightarrow \underline{\underline{R}}g'_*F'$. Its effect on homology ~~sheaves~~ of degree q is the composite

$$R^q f_*(F) \xrightarrow{u_*} R^q f_*(g'_* F') \xrightarrow[\text{homo}]{\text{edge}} R^q (fg')_*(F')$$

(4)

$$= R^q (gf')_*(F') \xrightarrow[\text{homo}]{\text{edge}} g_* R^q f'_*(F')$$

Hypothesis i) implies ~~the~~ first two maps are isos.; hypothesis ii) implies ~~second~~ last map is an iso., hence

(5) $\quad v^q: \quad R^q f_*(F) \longrightarrow g_* R^q f'_*(F')$

is an isom.

Using preceding prop. and hyp. ii) again we get that $E_2^{pq}(f,F) \xrightarrow{\sim} E_2^{pq}(f',F')$. q.e.d.

(Lesson: ~~Always~~ ~~first~~ After defining the map of spectral sequences (3), ~~just~~ state that the map on E_2-terms is ~~that~~ $(g, v^q)^*$ with v^q defined by (4). Proof of prop. is then immediate.)

3). Categories in a topos.

Let X be a topos and let C be a category object in X. We denote by $C^{\wedge}$ the category of objects F over ~~of~~ of X over ObC endowed with a right action of $Ar\,C$:

$$F \times_{ObC} Ar C \longrightarrow F$$

Thus if X is the category of sets, F may be identified with a contravariant functor from C to sets. $C^{\vee}$ will denote the category of F over ObC with ~~a~~ a left action of $Ar\,C$, i.e. analogues of covariant functors.

~~The forgetful functor from $C^{\wedge}$ to $X_{/ObC}$ commutes with~~ ~~$C^{\wedge}$ is closed under inductive and projective limits and the functor~~

One shows that inductive limits and projective limits in $C^{\wedge}$ "calculate themselves" in the induced topos $X_{/ObC}$, i.e. that $C^{\wedge}$ is closed under such limits and that the forgetful functor from $C^{\wedge}$ to $X_{/ObC}$ commutes with these limits. Since $C^{\wedge}$ has generators

$$U \times_{ObC} Ar\,C$$

with U running over a set of generators for $X_{/ObC}$, one sees by Giraud's criterion that $C^{\wedge}$ is a topos.

Now given a morphism $f : \mathbf{C} \longrightarrow \mathbf{C}'$ of category objects in X, there is induced a functor

$$f^* : \mathbf{C}'^\wedge \longrightarrow \mathbf{C}^\wedge$$

$$f^*(F') = F' \times_{ObC'} ObC \qquad \text{(obvious right ArC action)}$$

which commutes with ~~the~~ inductive and projective limits, hence admits adjoints

$$\begin{tikzcd} C^\wedge \arrow[r, shift left=2, "f_!"] \arrow[r, shift right=2, "f_*"'] & C'^\wedge \arrow[l, "f^*"'] \end{tikzcd}$$

by general nonsense. The pair (f^*, f_*) constitute a morphism of topoi from $C^\wedge$ to $C'^\wedge$.

To obtain formulas for $f_!$, f_* we observe that for any $F \in C^\wedge$

$$F \times_{ObC} ArC \times_{ObC} ArC \underset{id \times \mu}{\overset{\mu \times id}{\rightrightarrows}} F \times_{ObC} ArC \xrightarrow{\mu} F$$

is right exact, (it is the beginning of the standard resolution

$$(i_! i^*)^2 F \rightrightarrows i_! i^* F \longrightarrow F$$

where i^* is the forgetful functor $C^\wedge \longrightarrow X_{/ObC}$, and $i_!$ is its left adjoint

$$i_! G = G \times_{ObC} ArC .)$$

Now

$$f_!(G \times_{ObC} ArC) = G \times_{ObC'} ArC'$$

($f_! \lambda_! = \lambda'_! (\mathrm{Ob} f)_!$ or by direct calculation), hence

$$f_!(F) = \mathrm{Coker}\left\{ F \times_{\mathrm{Ob}\, C} \mathrm{Ar}\, C \times_{\mathrm{Ob}\, C'} \mathrm{Ar}\, C' \rightrightarrows F \times_{\mathrm{Ob}\, C'} \mathrm{Ar}\, C' \right\}.$$

(In the case where X is sets, this formula says

$$f_!(F)_y = \varinjlim_{y \to f(x)} F(x) = \mathrm{Coker}\left\{ \coprod_{x_1 \leftarrow x_0,\, f(x_0) \leftarrow y} F(x_1) \rightrightarrows \coprod_{f(x) \leftarrow y} F(x) \right\}).$$

For f_*, we have a formula of the form

$$f_*(F) = \mathrm{Ker}\left\{ \underline{\mathrm{Hom}}_{/\mathrm{Ob}\, C}(\mathrm{Ar}\, C' \times_{\mathrm{Ob}\, C'} \mathrm{Ob}\, C, F) \rightrightarrows \underline{\mathrm{Hom}}_{/\mathrm{Ob}\, C}(\mathrm{Ar}\, C' \times_{\mathrm{Ob}\, C'} \mathrm{Ar}\, C, F) \right\}$$

corresponding to the formula

$$(f_* F)_y = \varprojlim_{f(x) \to y} F(x) = \mathrm{Ker}\left\{ \prod_{y \leftarrow f(x)} F(x) \rightrightarrows \prod_{\substack{y \leftarrow f(x) \\ x \leftarrow x'}} F(x') \right\}$$

in the case of ordinary categories.

Suppose now that y is a point of $\mathrm{Ob}\,C'$; as the "stalk at y" commutes with finite projective limits and arbitrarily inductive limits, we have that

$$f_!(F)_y = \varinjlim_{y \to f(x)} F_x$$

the limit being taken over the category $y\backslash C$ with objects

$\mathrm{Ob}(y\backslash C) = (\mathrm{Ob}\,C \times_{\mathrm{Ob}\,C'} \mathrm{Ar}\,C')_y$ ($\mathrm{Ar}\,C'$ over $\mathrm{Ob}\,C'$ via s)

and morphisms

$\mathrm{Ar}(y\backslash C) = (\mathrm{Ar}\,C \times_{\mathrm{Ob}\,C'} \mathrm{Ar}\,C')_y$.

A similar formula for $f_*(F)_y$ does not exist in general, because the "stalk at y" functor is not compatible with <u>Hom</u>'s. (There is however a possibility if the category of pairs $(x, f(x) \to y)$ is finite.)

4) Cohomology for category objects in a topos.

Let $f: C \longrightarrow C'$ be a morphism of category objects in the topos X. Denote by $C^\wedge_{ab}$ the abelian objects of $C^\wedge$. There are adjoint functors

$$\begin{tikzcd} C^\wedge_{ab} \arrow[r, shift left=2, "f_{!ab}"] & C'^\wedge_{ab} \arrow[l, "f^*"'] \arrow[l, phantom, shift right=3, ""] \end{tikzcd} \quad C^\wedge_{ab} \xrightarrow{f_*} C'^\wedge_{ab}$$

and in general $f_{!ab}$ is not the $f_!$ of the preceding section. In the following, however, we work almost exclusively with abelian object and drop the ~~ab~~ subscript "ab" where convenient.

Since $C^\wedge$ is a topos, $C^\wedge_{ab}$ has enough injectives, so the derived functors $R^q f_*$ are defined.

<u>Prop 4.1</u>: The left-derived functors $L_q f_!$ exist.

Proof: Let $Ob\,C$ be regarded as a category object in X with ~~...~~ all arrows the identity, and let

$$i : Ob\,C \longrightarrow C$$

denote the evident morphism. The corresponding morphism of topoi

$$i : (Ob\,C)^\wedge \longrightarrow C^\wedge$$

may be ~~...~~ identified with the canonical morphisms from the induced topos $C^\wedge/_{Ar\,C}$ to $C^\wedge$, hence one knows ~~...~~ by general nonsense that

$$i_! : (\mathrm{Ob}\, C)^\wedge_{ab} \longrightarrow C^\wedge_{ab}$$

is exact. (In fact we have intuitively

$$i_! F = F \otimes_{\mathbb{Z}[\mathrm{Ob}\, C]} \mathbb{Z}[\mathrm{Ar}\, C]$$

and precisely that, if x is a point of $\mathrm{Ob}\, C$, then

$$(i_! F)_x = \coprod_{x \to x'} F_{x'} \qquad \text{[illegible]}.)$$

Let $C.(F)$ ~~[illegible]~~ denote the standard resolution of F with resp. to ~~[illegible]~~ $(i_!, i^*)$:

$$(1) \qquad \cdots \rightrightarrows (i_! i^*)^2 F \rightrightarrows i_! i^* F \quad .$$

~~[illegible]~~ We claim

$$(2) \qquad L_q f_!(F) = H_q\left(f_!\, C.(F)\right).$$

~~[illegible]~~ The left-side is a homological functor of F because $i_!$ and $f_! i_! = i'_! (\mathrm{Ob} f)_!$ are exact; it is co-effaceable in positive degrees because F is a quotient of $i_! i^* F$ and $C.(i_! M)$ is contractible. q.e.d.

If F is a complex in $C^\wedge$ bounded above let $\mathbb{L} f_!(F)$ denote the total complex associated to the double complex $f_!\, C.(F)$. ~~[illegible]~~ The two spectral sequences ~~[illegible]~~ are (using exactness of $f_!\, C.(?)$ for the second)

$$(3) \qquad E^1_{pq} = L_q f_!(F_p) \implies H_{p+q}(\mathbb{L}f_!(F))$$

$$(4) \qquad E^2_{pq} = L_p f_!(H_q F) \implies H_{p+q}(\mathbb{L}f_!(F)),$$

showing that $\mathbb{L}f_!$ preserves quasi-isos., hence induces

$$\mathbb{L}f_! \; : \; D^-(C^\wedge) \longrightarrow D^-(C'^\wedge).$$

We want to show this is ~~left~~ adjoint to $\mathbb{R}f_*$ in a suitable sense.

<u>Lemma 4.2</u>: If M is abelian over $\mathrm{Ob}\, C$ and if I' is injective in $C'^\wedge$, then

$$\mathrm{Ext}^+_{C^\wedge}(i_! M, f^* I') = 0.$$

Proof: Let $I^\cdot$ be an injective resolution of f^*I', whence

$$\mathrm{Ext}^q_{C^\wedge}(i_! M, f^*I') = H^q \mathrm{Hom}_{C^\wedge}(i_! M, I^\cdot)$$
$$= H^q \mathrm{Hom}_{\mathrm{Ob}C}(M, i^* I^\cdot)$$

Now $i_!$ being exact, entails i^* preserves injectives, hence $i^* I^\cdot$ is an injective resolution of $i^* f^* I' = (\mathrm{Ob} f)^* i'^* I$. This last is injective as both $(\mathrm{Ob} f)_!$ and $i'_!$ are exact, so done.

~~Let $F' \in D^-(C'^\wedge)$ and $G' \in D^+(C'^\wedge)$, and let $f^*G \to I$ be an injective resolution and~~

$$\cancel{P = C_\cdot(f^*F') \longrightarrow F'}$$

~~the resolution constructed above. Then we have a~~

<u>Prop. 4.3</u>: If $F \in D^-(C^\wedge)$ and $G' \in D^+(C'^\wedge)$, then

$$\mathbb{R}\mathrm{Hom}(\mathbb{L}f_!(F), G') = \mathbb{R}\mathrm{Hom}(F, f^*G').$$

Proof: Can suppose G' injective whence

$$\mathbb{R}\mathrm{Hom}(\mathbb{L}f_!(F), G') = \mathrm{Hom}(f_!\, C(F), G')$$
$$= \mathrm{Hom}(C(F), f^*G')$$

But the latter <u>equals</u> $\mathbb{R}\mathrm{Hom}(F, f^*G')$ by 4.2.

In down to earth terms, the preceding proposition means that there is a spectral sequence

(5) $$E_2^{pq} = \mathrm{Ext}^p(L_q f_!(F), G') \Longrightarrow \mathrm{Ext}^{p+q}(F, f^*G')$$

for F in $C^\wedge$ and G' in $C'^\wedge$.

<u>Prop. 4.4</u>: $\mathbb{L}f_! \circ f^* \xrightarrow{\sim} \mathrm{id}$ on $D^-(C'^\wedge)$

$\iff$ $\mathbb{R}f_* \circ f^* \xleftarrow{\sim} \mathrm{id}$ on $D^+(C'^\wedge)$.

$\iff$ $\mathbb{R}\mathrm{Hom}(F', G') \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}(f^*F', f^*G')$ $\quad \forall\ F' \in D^-,\ G' \in D^+$

Proof: Observe there is commutatively in

$$\begin{tikzcd} \mathbb{R}\mathrm{Hom}(F', G') \arrow[r] \arrow[d] \arrow[dr] & \mathbb{R}\mathrm{Hom}(\mathbb{L}f_! f^*F', G') \arrow[d, equal] \\ \mathbb{R}\mathrm{Hom}(F', \mathbb{R}f_* f^*G') = & \mathbb{R}\mathrm{Hom}(f^*F', f^*G'). \end{tikzcd}$$

because if $P = C(f^*F')$ and I is an injective resolution of G', which is assume injective, this square is the square of complexes

$$\begin{tikzcd}
\mathrm{Hom}^\bullet(F', G') \arrow[r] \arrow[d] & \mathrm{Hom}^\bullet(f_! P, G') = \mathrm{Hom}^\bullet(P, f^* G') \arrow[dd] \\
\mathrm{Hom}^\bullet(F', f_* I) \arrow[d, equal] & \\
\mathrm{Hom}^\bullet(f^* F', I) \arrow[r] & \mathrm{Hom}^\bullet(P, I)
\end{tikzcd}$$

So the prop. follows from the fact that the functors $\mathbb{R}\mathrm{Hom}(F', ?)$ $F' \in D^b$ are conservative on D^+ (resp. other way) (in fact, enough to worry about F' of the form $F[n]$). q.e.d.

In down-to-earth terms, the spectral sequence (5) shows that $L_q f_!(f^* F') = F'$ if $q = 0$ and 0 if $q > 0$

$\Longrightarrow$

(6) $\quad \mathrm{Ext}^n(F', G') \simeq \mathrm{Ext}^n(f^*F', f^*G')$ for all G', n.

~~This map is also obtained from the edge homomorphism of the spectral sequence~~

~~$E_2^{pq} = \mathrm{Ext}^p(F', R^q f_*(f^*G')) \Longrightarrow \mathrm{Ext}^{p+q}(f^*F', f^*G')$~~

~~and the canonical map $G \to f_* f^* G$.~~

But considering the sheaves associated to both sides considered as presheaves on $C'^\wedge$ (put $F' = \mathbb{Z}[U]$), we obtain

$$G' \simeq f_*(f^* G') \qquad n = 0$$

$$0 \simeq R^n f_*(f^* G') \qquad n > 0.$$

Conversely given these last formulas the spectral sequence

$$\mathrm{Ext}^p(F', R^q f_*(f^*G')) \Longrightarrow \mathrm{Ext}^{p+q}(f^*F', f^*G')$$

degenerates yielding the isom. (6). Taking G' to be injective, (5) degenerates, yielding

$$\mathrm{Hom}(L_q f_!(f^*F'), G') = \mathrm{Ext}^q(f^*F', f^*G'),$$

and as there are enough injectives to detect things, on has

$$\begin{aligned} L_q f_!(f^*F) &= 0 \qquad q > 0 \\ &= F' \qquad q = 0. \end{aligned}$$

(following should follow 4.1)

<u>Prop. 4.5</u>: Let y be a point of $\mathrm{Ob}\, C'$ and $y|C$ the category of arrows $y \to f(x)$. Then

$$L_q f_!(F)_y = L_q \varinjlim_{y \to f(x)} (F_x).$$

(immediate from formulas on page 13, and fact that $i_!$, i^* compatible with stalk functor)

~~Cor. 4.6: $Rf_* \circ f^* = \mathrm{id}$ on $D^+(\tilde{C'})$ if $\mathrm{Ob}\, C'$ has enough points y and if for each such point $y|C$ has trivial homology.~~

(The following should be rewritten using the concepts on p. 22.)

~~Prop 4.6: Suppose that ...~~

Cor 4.6: Assume that the inclusion $\mathcal{C}_y \to y\backslash\mathcal{C}$ has a right adjoint (which is the case for ordinary categories when f is fibred). Then

$$L_q f_!(F)_y = L_q \varinjlim_{\mathcal{C}_y} (F|\mathcal{C}_y)$$

Consequently, if ~~...~~ this holds ~~for~~ and if $\mathcal{C}_y$ is acyclic for enough points y of $\mathrm{Ob}\,\mathcal{C}'$, then $\mathbb{L}f_! \circ f^* = id$.

Beware: It is not enough to know that the categories $y\backslash\mathcal{C}$ are acyclic, since this does not imply that the homology of a functor of the form $(y \to f(x)) \mapsto F(f(x))$ will be zero. Example: Take a map of simplicial sets $f: U \to V$ which is an aspherical fibration. Then for $f: \Delta^\circ/U \longrightarrow \Delta^\circ/V$, one knows that f_* is exact, but $f_* f^* \neq id$, yet $\mathbb{L}f_!(\mathbb{Z}) = \mathbb{Z}$, so all fibres $\Delta^\circ/f^*(v)$ are acyclic.

Problem: Find a good notion of fibrant for $f: \mathcal{C} \to \mathcal{C}'$. It should imply ~~...~~ the first hypothesis of 4.6.

5) Simplicial objects in a topos.

Let $Q = \{Q_n\}$ be a simplicial object in X. ~~But it gives rise to two category objects in X which we will denote by Δ/Q and Δ°/Q respectively. Each has $\coprod Q_n$ for its objects~~ Let Δ/Q be the category object of X with ~~objects~~

$$\mathrm{Ob}(\Delta/Q) = \coprod_n Q_n$$

$$\mathrm{Ar}(\Delta/Q) = \coprod_{[m]\to[n]} Q_n = \Big(\coprod_n Q_n\Big) \times_{\mathrm{Ob}\,\Delta} (\mathrm{Ar}\,\Delta).$$

(It is the analogue of the fibred category over Δ defined by Q.). The opposed category object will be denoted Δ°/Q.

$(\Delta/Q)^{\wedge}$ is the category of simplicial objects of X over Q, simplicial sheaves over Q as we shall call them.

$(\Delta^{\circ}/Q)^{\wedge}$ is the category of Deligne-style sheaves on Q, cosimplicial sheaves as we shall call them.

<u>Prop. 5.1:</u> Let $f: P \longrightarrow Q$ be a morphism of simplicial objects. Then

$\mathbb{L} f_! = f_!$ computed dimension-wise for simplicial sheaves

$\mathbb{R} f_*$ computed dimension-wise for cosimplicial sheaves.

<u>Proof:</u> The first morally is a consequence of the fact that the dimension-wise $f_!$ carries simplicial sheaves over P to simp-

sheaves over Q. $f_!$ is exact, because for any map $U \xrightarrow{g} V$ in X, $g_! :$ $X_{/U} \to X_{/V}$ is exact. In the case of cosimplicial sheaves the dimension-wise f_* works. Since quite generally an injective of $C^\wedge$ is injective as a sheaf over $\mathrm{Ob}\,C$ (exactness of $i_!$), it follows that injectives of $(\Delta^\circ/U)^\wedge$ are dimension-wise injectives; hence $\mathbb{R}f_*$ also computed dimension-wise.

~~Remark: The last argument is quite general and shows that given a map of "simplicial topoi" $f: U \to V$, then $\mathbb{R}f_*$ is computed dimension-wise for Deligne-style cosimplicial sheaves.~~

~~Now given a simplicial object Q in X we consider the morphisms of topoi~~

$$\cancel{(\Delta^\circ)^\wedge \longleftarrow (\Delta^\circ/Q)^\wedge \longrightarrow X}$$

~~Prop. 5.2: If Q is a simplicial object in X, then~~

Remark: The last argument is quite general and shows that for any map of "simplicial topoi" $f: U \to V$, $\mathbb{R}f_*$ is computed dimension-wise for the cosimplicial sheaves. In particular, given a simplicial object Q of X, for the canonical map $f: (\Delta^\circ/Q)^\wedge \longrightarrow (\Delta^\circ)^\wedge$, we have that $\mathbb{R}f_*$ is calculated dimension-wise. In particular the Leray spectral sequence for f takes the form

(1) $$E_2^{pq} = \check{H}^p(n \mapsto H^q(Q_n, F_n)) \Longrightarrow H^{p+q}((\Delta^\circ/Q)^\wedge ; F)$$

since

$$H^p((\Delta^\circ)^\wedge, M) = \check{H}^p(n \mapsto M_n)$$

as one sees using the Dold-Puppe ~~correspondence~~ equivalence between $(\Delta^\circ)^\wedge_{ab}$ and cochain complexes.

Suppose now that Q is a hypercovering of X, or more generally that, $\mathbb{Z}Q$ is a resolution of $\mathbb{Z}$ in X. Then using 4.6 one finds that the map

$$(\Delta^\circ/Q)^\wedge \xrightarrow{f} X$$

is universally acyclic, i.e. $\mathbb{R}f_* \circ f^* = id$. In particular

$$H^n(X, F) \xrightarrow{\sim} H^n((\Delta^\circ/Q)^\wedge ; f^*F)$$

so (1) takes the form

(2) $$E_2^{pq} = \check{H}^p(n \mapsto H^q(Q_n, F)) \Longrightarrow H^{p+q}(X, F).$$

We omit the verification that this spectral sequence coincides with the one used by Verdier (hypercohomology, using that $\mathbb{Z}Q$ is a resolution of $\mathbb{Z}$).

Addition to §4: A map $f : C \longrightarrow C'$ satisfies

$$H^*(C'^\wedge, F') \xrightarrow{\sim} H^*(C^\wedge, f^*F')$$

for all F' in $C'^\wedge$ iff $\mathbb{L}f_!(\mathbb{Z}) = \mathbb{Z}$, or equivalently iff all the categories $y|C$ are acyclic. Indeed this map is the map

$$\operatorname{Ext}^*(\mathbb{Z}, F') \longrightarrow \operatorname{Ext}^*(\mathbb{L}f_!(\mathbb{Z}), F')$$

induced by $\mathbb{L}f_!(\mathbb{Z}) \longrightarrow \mathbb{Z}$.

It seems reasonable to call a map f ∋

$$H^*(C'^\wedge, F') \xrightarrow{\sim} H^*(C^\wedge, f^*F') \quad \text{all } F'$$

<u>acyclic</u>, and a map ∋ $\mathbb{R}f_* \circ f^* = \mathrm{id}$ <u>universally acyclic</u>.

More generally given a triangle of topoi

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "gf"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

(2-commutative, comme toujours), we should call f <u>S-acyclic</u> if

$$\mathbb{R}g_* \xrightarrow{\sim} \mathbb{R}(gf)_* f^* .$$

Then 'acyclic' occurs when $S =$ sets and 'universally acyclic' occurs when $S = Y$.

6). Suppose that $f : X \to Y$ is a morphism of topoi, that Q is a simplicial object of Y acyclic over Y, and that P is a bisimplicial object of X endowed with ~~as~~ a vertical augmentation $P \to f^*Q$ which is acyclic vertically ($\forall n$, $P_{n\cdot} \to f^*Q_n$ acyclic.). We consider the square of topoi

$$(1)\qquad \begin{tikzcd} ((\Delta^2)^\circ/P)^\wedge \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ (\Delta^\circ/Q)^\wedge \arrow[r, "g"'] & Y \end{tikzcd} .$$

<u>Lemma 6.1</u>: $$g^* R^q f_*(F) \xrightarrow{\sim} R^q f'_*(g'^* F) \quad \forall q.$$

Proof. It is clear (by arguments $\overbrace{\text{similar to ones used}}$ before on p.20) that $R^q f'_*$ is computed dimension-wise horizontally, i.e. given a bicosimplicial G over P

$$R^q f'_*(G)_n = R^q f'_{n*}(G_n)$$

where $f'_n : (\Delta^\circ/P_n)^\wedge \longrightarrow Y_{/Q_n}$ is induced by f'.

Now f'_r factors into

$$(\Delta^\circ/P_r)^\wedge \xrightarrow{h_r} X_{/f^*Q_r} \xrightarrow{\tilde{f}} Y_{/Q_r}$$

where the first map is univ. acyclic as P_r is acyclic over f^*Q_r. Thus

$$\begin{aligned} R^q f'_*(g'^*F)_r &= H^q(R\tilde{f}\circ Rh_r\circ h_r^*(F/f^*Q_r)) \\ &= R^q\tilde{f}\,(F|f^*Q_r) \\ &= R^q f_*(F)\,\|\,Q_r \end{aligned}$$

proving the lemma.

<u>Prop. 6.2</u>: The Leray spectral sequence of (f, F) is isomorphic to that of (f', g'^*F).

Proof. Apply 2.2. i) holds because P being acyclic over f^*Q, and the latter acyclic over $X \Rightarrow P$ acyclic over X.

Now consider the square

(2)
$$\begin{tikzcd} ((\Delta^2)^\circ)^\wedge \arrow[d, "f''"] & ((\Delta^2)^\circ/P)^\wedge \arrow[l, "h'"'] \arrow[d, "f'"] \\ (\Delta^\circ)^\wedge & (\Delta^\circ/Q)^\wedge \arrow[l, "h"] \end{tikzcd}$$

Rh'_* and Rh_* are computed dimension-wise. The Leray spectral sequence of f'' takes the form

$$E_2^{pq} = \check{H}^p(r \mapsto \check{H}^q(s \mapsto M_{rs})) \Longrightarrow \check{H}^{p+q}(n \mapsto M_{nn})$$

and is undoubtedly isomorphic to the spectral sequence of a bicosimplicial abelian group. (We will not try to prove this.) From the square (2) we get a map of Leray spectral sequences

$$\begin{array}{lccc} & \check{H}^p(r \mapsto \check{H}^q(s \mapsto \Gamma(P_{rs}, F))) & \Longrightarrow & \check{H}^{p+q}(n \mapsto \Gamma(P_{nn}, F)) \\ (3) & \downarrow & & \downarrow \\ & H^p(Y, R^q f_*(F)) & \Longrightarrow & H^{p+q}(X, F) \end{array}$$

where we have used 6.2 to identify the Leray spectral sequences of (f', g'^*F) and (f, F).

The map on abutments in (3) is clearly the edge homomorphism of the spectral sequence (2) of §5, p.21. It should not be hard to identify the map on E_2-terms with the map (6) described in section 1, but we omit this. Assuming this we in any case can apply the argument of §1, or 2.2 to obtain

Prop. 6.3: Assume that i) $H^+(P_{rs}, F) = 0$ ii) $H^+(Q_r, R^q f_*(F)) = 0$ for all Q. Then (3) is an isom.

7) Review of Verdier's theorem.

Given a topos X, let J_X be the category of hypercoverings of X. It is the full subcategory of $\underline{\mathrm{Hom}}(\Delta^\circ, X)$ consisting of Q such that the map $Q \longrightarrow X$ (= ~~[illegible]~~ constant simplicial object) is special (i.e. its stalks are aspherical fibrations of simplicial sets. More generally one defines J_U for any simplicial object U in X to be the category of $Q \longrightarrow U$ which are special. Two maps $Q' \rightrightarrows Q$ ~~[illegible]~~ in J_U are called strictly homotopic if there is a strict homotopy $Q' \longrightarrow Q^I$ between them ~~[illegible]~~ compatible with the trivial homotopy of U. $\mathrm{Ho}\, J_U$ is the quotient category of J_U with homotopy classes of maps, two maps being homotopic if they are connected by a chain of strictly homotopic maps.

<u>Lemma 7.1</u>: $\mathrm{Ho}\, J_U$ is cofiltering.

Proof: i) Given Q, Q' special over U, they are both dominated by $Q \times_U Q'$, which is special over U.

ii) Given two arrows $u, v : Q' \rightrightarrows Q$ over U, consider the ~~[illegible]~~ diagram

$$\begin{tikzcd}
Q'' \arrow[r, dashed] \arrow[d, dashed] & Q^I \times_{U^I} U \arrow[rr] \arrow[d] & & Q^I \arrow[d, "\alpha"] \\
Q' \arrow[r, "(u,v)"] & Q \times_U Q = Q^{\dot I} \times_{U^{\dot I}} U \arrow[rr] \arrow[d] & & Q^{\dot I} \times_{U^{\dot I}} U^I \arrow[d] \\
 & U \arrow[rr] & & U^I
\end{tikzcd}$$

where all the squares are cartesian. One knows that α is special, hence Q'' is special over Q, hence Q'' is special over U and $Q'' \longrightarrow Q'$ equalizes u, v up to homotopy over U.

Given Q in J_U, consider the spectral sequence (1) of §5:

$$(1) \qquad E_2^{pq} = \check{H}^p(n \longmapsto H^q(Q_n, F_n)) \Rightarrow H^{p+q}((\Delta^\circ/Q)^\wedge; F).$$

for any cosimplicial F over Q. We take F of the form f^*L, where $f : Q \longrightarrow U$ is the structural map and L is a cosimplicial sheaf over U. We want the map induced by f

$$(2) \qquad H^*((\Delta^\circ/U)^\wedge, L) \longrightarrow H^*((\Delta^\circ/Q)^\wedge, f^*L)$$

to be an isomorphism, that is, we want f to be acyclic. By page 22 it suffices to have $\mathbb{L}f_!(\mathbb{Z}) = \mathbb{Z}$, or equivalently, to have ~~the~~ acyclicity of the category $y|C$ for any point y of $\operatorname{Ob} C' = \coprod U_n$, where we use the notation of p. 22. We can reduce to the case where $f: Q \longrightarrow U$ is the map of fibres over the image of y in X. Now $C = \Delta^\circ/Q$, $C' = \Delta^\circ/U$ and $y|C$ is the category of diagrams

$$\begin{tikzcd} \Delta(d^\circ x) \arrow[r, "x"] \arrow[d] & Q \arrow[d, "f"] \\ \Delta(d^\circ y) \arrow[r, "y"] & U \end{tikzcd}$$

hence $\quad y\backslash C = \Delta^\circ / f^{-1}(y)$ where $f^{-1}(y) = \Delta(d_y^\circ) \times_{U} Q$. As f is an aspherical fibration all of these fibre simplicial sets are contractible, hence acyclic. So we have proved.

<u>Prop 7.2</u>: Let $f : Q \longrightarrow U$ be a map of simplicial objects in a topos X with enough points x such that for each point y of U the fibre simplicial set $f^{-1}(y)$ is acyclic (in particular if f is special). Then for all F in $(\Delta^\circ/U)^\wedge$ we have

$$H^*((\Delta^\circ/U)^\wedge, F) \xrightarrow{\sim} H^*((\Delta^\circ/Q)^\wedge, f^*F)$$

i.e. $f : (\Delta^\circ/Q)^\wedge \longrightarrow (\Delta^\circ/U)^\wedge$ is acyclic. (Actually this isomorphism will also holds for the higher direct image sheaves relative to X, i.e. X-acyclic (p. 22)).

Scholium: If $f : X \to Y$ is a map of simplicial sets such that $f^{-1}(y)$ is acyclic $\forall y$, then f is acyclic for cosimplicial sheaves, but <u>not</u> universally acyclic.

Returning to the case of a map $f : Q \longrightarrow U$ in J_U and a cosimplicial sheaf F over U, we have from (1) and 7.2 a spectral sequence

$$E_2^{pq}(Q) = \check{H}^p(n \mapsto H^q(Q_n, f_n^* F_n)) \Rightarrow H^{p+q}((\Delta^\circ/U)^\wedge ; F)$$

The E_2 term is a homotopy invariant as a functor on J_U, hence becomes a functor on $\operatorname{Ho} J_U$. As the latter is filtering, we can form the limit spectral sequence

$$E_2^{pq}(\infty) = \varinjlim_{Q \in \mathrm{Ho}\, J_U} \check{H}^p\big(n \mapsto H^q(Q_n, f_n^* F_n)\big) \Longrightarrow H^{p+q}\big((\Delta^\circ/U)^\wedge; F\big).$$

We are now going to show $E_2^{pq}(\infty) = 0$ for $q > 0$.

<u>Lemma 7.4</u>: Given a simplicial object Q and a surjective map $S \longrightarrow Q_p$, there exists a special map $Q' \longrightarrow Q$ such that $Q'_p \longrightarrow Q_p$ factors through $S \longrightarrow Q_p$.

Proof: Let $j : pt \longrightarrow \Delta$ be the functor with image $[p]$. Then have adjoint functors

$$X = \underline{\mathrm{Hom}}(pt, X) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \underline{\mathrm{Hom}}(\Delta^\circ, X)$$

where $j^* Q = Q_p$ and

$$(j_* S)_n = \varprojlim_{p \to n} S = \prod_{\mathrm{Hom}(\Delta(p), \Delta(n))} S.$$

Claim that j_* transforms surjections into special maps. In effect formation of j_* commutes with fibres, so reduce to case where X is sets. But then $j_* S' \longrightarrow j_* S$ has the RLP with respect to an injection $L \subset K$ iff $S' \to S$ has the RLP wrt $L_p \subset K_p$; this is the case if $S' \to S$ is surjective.

So now given a surjective map $S \longrightarrow Q_p = j^* Q$

let Q' be the pull-back

$$\begin{tikzcd} Q' \arrow[r] \arrow[d] & Q \arrow[d] \\ j_*S \arrow[r] & j_*j^*Q \end{tikzcd}$$

so that $Q' \longrightarrow Q$ is special. Then by properties of adjoint functors we have

$$\begin{tikzcd} j^*Q' \arrow[r] \arrow[d] & j^*Q \arrow[d, "id"] \\ S \arrow[r] & j^*Q \end{tikzcd}$$

showing $Q'_p \to Q_p$ factors through $S \longrightarrow Q_p$. q.e.d.

Suppose now that $z \in E_2^{pq}(Q)$, and let $c \in H^q(Q_p, f_p^*F_p)$ be a cocycle representing c. If $q > 0$, there is a surjective map $h: S \longrightarrow Q_p$ such that $c \in \operatorname{Ker}\{h^*: H^q(Q_p, f_p^*F_p) \longrightarrow H^q(S, h^*f_p^*F_p)\}$. By 7.4 there is a map $Q' \to Q$ in J_U such that $Q'_p \to Q_p$ factors through S. It follows that c dies on being pulled back to Q', hence z becomes zero in $E_2^{pq}(Q')$. Thus $E_2^{pq}(\infty) = 0$ for $q > 0$ and the limit spectral sequence degenerates yielding

<u>Verdier's theorem 7.5</u>: For any cosimplicial sheaf F on U we have

$$\operatorname*{lim.ind.}_{Q \in Ho(J_U)} H^n\left(\nu \mapsto \Gamma(Q_n, f_n^*F_n)\right) \xrightarrow{\sim} H^n\left((\Delta^\circ/U)^\wedge; F\right).$$

($f: Q \longrightarrow U$ structural map.)

The usual form of Verdier's theorem is the special case when $\mathcal{U}$ is the constant simplicial object X and F is a sheaf on X:

$$\varinjlim_{Q \in J_X} H^n(\nu \mapsto \Gamma(Q_n, F)) \xrightarrow{\sim} H^n(X; F).$$

<u>Remark</u>: Instead of $J_{\mathcal{U}}$ one can work with the subcategory $J_{\mathcal{U}}^s$ with the same objects and special maps for morphisms. The homotopy category $\operatorname{Ho} J_{\mathcal{U}}^s$ is again cofiltering and $\operatorname{Ho} J_{\mathcal{U}}^s \longrightarrow \operatorname{Ho} J_{\mathcal{U}}$ is cofinal.

8) A generalized Verdier theorem for the Leray spectral sequence.

Given $f : X \to Y$ a morphism of topoi (having enough points), let I_f be the category of triples (P, Q, u) where Q is a hypercovering of Y, P is a bisimplicial object of X, $u : P \to f^*Q$ is a vertical augmentation (map of bisimplicial objects, where f^*Q is regarded as constant in the vertical direction), and where $u : P_{r\cdot} \to f^*Q_r$ is a hypercovering for each r. Put another way, P is a hypercovering of f^*Q in the category $\underline{\mathrm{Hom}}(\Delta^0, X)$; this is immediate from the fact that surjective = dimension-wise surjective. Evident morphisms.

For each object (P, Q, u) we have a morphism of Leray spectral sequences

$$(1)\qquad \begin{array}{ccccc} E_2^{pq}(P,Q,u) & = & \check{H}^p(r \mapsto \check{H}^q(s \mapsto \Gamma(P_{rs}, F))) & \Rightarrow & \check{H}^{p+q}(n \mapsto \Gamma(P_{nn}, F)) \\ \downarrow & & \downarrow & & \downarrow \\ E_2(f) & = & H^p(Y, R^q f_*(F)) & \Longrightarrow & H^{p+q}(X, F) \end{array}$$

and we wish to prove that the limit of the former spectral sequences as (P, Q, u) runs over I_f exists and is isomorphic to the latter.

General remarks concerning limits of spectral sequences: Let $J \to$ (spec. seq.), $j \mapsto E(j)$ be a functor to (supposed first quadrant cohomological type) spectral sequences, and set

$$\bar{E}_r^{pq} = \varinjlim_{J} E_r^{pq}(j)$$

$$\bar{H}^n = \varinjlim_{J} H^n(j)$$

$$\bar{F}_p{}^n = \varinjlim_{J} F_p H^n(j).$$

Then $E_r(\infty)$ inherits a differential d_r and there is a canonical map

(2) $$H^{p,q}(\bar{E}_r, d_r) \longrightarrow \bar{E}_{r+1}^{p,q}$$

which is an isomorphism for $r \gg p+q$ (also $d_r = 0$). As inductive limits are right exact, there is an exact sequence

$$\bar{F}_{p+1}{}^n \longrightarrow \bar{F}_p{}^n \longrightarrow \bar{E}_\infty^{p,n-p} \longrightarrow 0$$

so that if we set

$$F_p \bar{H}^n = \mathrm{Im}\{\bar{F}_p{}^n \longrightarrow \bar{H}^n\}$$

we obtain surjections

(3) $$\bar{E}_\infty^{p,n-p} \twoheadrightarrow F_p\bar{H}^n / F_{p+1}\bar{H}^n .$$

We say that the limit of the spectral sequence functor $j \mapsto \mathcal{E}(j)$ exists if (2) and (3) are isomorphisms. A sufficient condition is that J be a filtering category, for then inductive limits over J are exact (values in any Grothendieck abelian category).

The category I_f is fibred over J_Y by the functor $(P,Q,u) \longmapsto Q$, the inverse image of $u: P \to f^*Q$ relative to a map $g: Q' \to Q$ being $f^*Q' \times_{f^*Q} P \longrightarrow f^*Q'$. Consequently

$$(4) \qquad \varinjlim_{I_f} M = \varinjlim_{Q \in J_Y} \Big(\varinjlim_{I_{f,Q}} M \Big)$$

for any $M : I_f^{\circ} \longrightarrow \mathcal{A}b$, where $I_{f,Q}$ denotes the fibre category of the functor $I_f \to I_Y$ over Q. Clearly

$$I_{f,Q} = J_{f^*Q}$$

the latter denoting hypercoverings of f^*Q in the topos of simplicial objects in X. A more sympathetic way of writing (4) is perhaps

$$(4') \qquad \varinjlim_{P \to f^*Q} M = \varinjlim_{Q \in J_Y} \; \varinjlim_{P \in J_{f^*Q}} M .$$

~~Now $E_1^{pq}(P,Q,u) = H^q(\cdot \mapsto \Gamma(P_{p\cdot}, F))$ is a homotopy invariant on J_{f^*Q},~~

Now the spectral sequence of the bicosimplicial group $\Gamma(P_{\cdot\cdot}, F)$

is a homotopy-~~xxx~~ invariant from the ~~x~~ term

$$E_1^{pq} = H^q(s \mapsto \Gamma(P_{ps}; F))$$

on for ~~xxx~~ vertical homotopies, hence ~~xxx~~ the restriction of the spectral functor $\mathcal{E}(P, Q, u)$ to J_{f^*Q} ~~xxx~~ from E_1 on comes from a spectral functor on $\mathrm{Ho}\, J_{f^*Q}$. As the latter is cofiltering, the limit spectral sequence

$$(5) \qquad \varinjlim_{P \in J_{f^*Q}} \mathcal{E}(P, Q, u) = \varinjlim_{P \in \mathrm{Ho}\, J_{f^*Q}} \mathcal{E}(P, Q, u)$$

exists. ~~xxx~~ Moreover ~~xxx~~ the morphisms (1) induce a map of this limit to the Leray spectral sequence of f. Now Verdier's theorem implies

$$\varinjlim_{P \in \mathrm{Ho}\, J_{f^*Q}} E_1^{pq}(P, Q, u) = H^q(f^*Q_p, F),$$

because $P_{p\cdot}$ is a hypercovering of f^*Q_p for each p. Hence

$$(6) \qquad \begin{array}{ccc} \varinjlim_{\mathrm{Ho}\, J_{f^*Q}} E_2^{pq}(P, Q, u) & = & H^p(\nu \mapsto H^q(f^*Q_\nu, F)) \\ \| & & \\ \varinjlim_{J_{f^*Q}} E_2^{pq}(P, Q, u) & & \end{array}$$

This shows that the limit spectral sequence (5) is a homotopy-invariant of Q, hence comes from ~~the~~ a spectral functor on the cofiltering category $\mathrm{Ho}\, J_Y$.

~~[illegible]~~

Using (4) it follows that the limit spectral sequence

(7) $$\varinjlim_{I_f} \mathcal{E}(P, Q, u)$$

~~[illegible], from E_2 on, exists. [illegible]~~
~~[illegible] Now we will assume without checking that [illegible] the limit (5)~~

~~$$E_2^{pq} = H^p(\nu \mapsto H^q(f^*Q_\nu, F)) \Longrightarrow H^{p+q}(X, F)$$~~

~~to the spectral sequence~~

exists from E_2 on. We will now assume without checking that the map from the E_2-term of (5) to that of the Leray spec. seq. is the canon. map

(8) $$H^p(\nu \mapsto H^q(f^*Q_\nu, F)) \longrightarrow H^p(Y, R^q f_*(F))$$

obtained by ~~[illegible]~~ composing ~~[illegible]~~ the canonical map

$$H^q(f^*Q_\nu, F) \longrightarrow \Gamma(Q_\nu, R^q f_*(F))$$

with the Verdier map associated to Q and the sheaf $R^q f_*(F)$. According to Verdier's theorem in the form of SGAA, for any presheaf G on Y

$$\varinjlim_{\mathrm{Ho}\, J_Y} H^n(\nu \mapsto G(Q_\nu)) \xrightarrow{\sim} H^n(Y, \tilde{G}),$$

hence we conclude that the limit of (8) over J_y is an isomorphism. So we have obtained

<u>Theorem 8.1:</u> The limit over I_f of the simplicial spectral sequences

$$E_2^{pq} = \check{H}^p(r \mapsto \check{H}^q(s \mapsto \Gamma(P_{rs}, F))) \Rightarrow \check{H}^{p+q}(\nu \mapsto \Gamma(P_{\nu\nu}, F))$$

exists and is isom. to the Leray spectral sequence of f, F.

9) An alternative version of 8.1:

This theorem that the category over which the limit is taken is not cofiltering. We now present an alternative approach.

Instead of I_f we consider its full subcategory J_f consisting of (P, Q, u) such that $u: P \to f^*Q$ is <u>special</u> as a map of bisimplicial objects of X, that is, all fibres have the RLP with respect to injective maps of bisimplicial sets.

~~Define the path space of (P, Q, u) to be (P^I, Q^I, u^I) where~~

$$\cancel{P^I = P^{(pr_1^* \Delta(1))}}$$

$$\cancel{Q^I = Q^{\Delta(1)}}$$

Define

$$(P, Q, u)^I = \left(P^{pr_1^* \Delta(1)}, Q^{\Delta(1)}, u^{pr_1^* \Delta(1)}\right)$$

where $pr_1 : \Delta^2 \longrightarrow \Delta$ is the projection. (Note

$$pr_1^* \Delta(1) = \Delta(1) \boxtimes \Delta(0) = \Delta(1,0)$$

and

$$\left(pr_1^* f^* Q\right)^{pr_1^* \Delta(1)} = pr_1^* f^*\left(Q^{\Delta(1)}\right).)$$

By a homotopy between two arrows $\alpha, \beta : (P', Q', u') \to (P, Q, u)$ we mean a map

$$h : (P', Q', u') \longrightarrow (P, Q, u)^I$$

yielding α, β at the ends. Such an h amounts to a homotopy between the two maps $Q' \rightrightarrows Q$, together with a horizontal homotopy between $P'' \rightrightarrows P$ covering the former. We call two maps in J_f homotopic if they can be joined by a sequence of homotopies and denote by $\mathrm{Ho}\, J_f$ the corresponding homotopy category.

<u>Lemma 7.2:</u> $\mathrm{Ho}\, J_f$ is cofiltering.

Same as 7.1; it is only necessary to check ~~that~~ for the map

$$(P, Q, u)^I \longrightarrow (P, Q, u)^{\dot{I}}$$

that any base change of it lies in J_f. Recall Friedlander's lemma:

<u>Lemma 7.3:</u> Given a cartesian square in the category of pairs of simplicial objects

$$\begin{array}{ccc} (U''' \to V''') & \longrightarrow & (U' \to V') \\ \downarrow & & \downarrow \\ (U'' \to V'') & \longrightarrow & (U \to V), \end{array}$$

if $U' \longrightarrow U \times_V V'$ and $U'' \longrightarrow V''$ are special then $U''' \longrightarrow V'''$ is special.

Proof of §.3: In the following all vertical faces are cartesian

$$\begin{tikzcd}
 & & U' \arrow[d, "sp"] & \\
U''' \arrow[urr] \arrow[d] & & U \times_V V' \arrow[r] \arrow[d] & V' \arrow[d] \\
U'' \times_{V'} V'' \arrow[r] \arrow[urr] \arrow[d] & V''' \arrow[urr] \arrow[d] & U \arrow[r] & V \\
U'' \arrow[r, "sp"'] \arrow[urr] & V'' \arrow[urr] & &
\end{tikzcd}$$

so it's clear.

In virtue of ~~this~~ lemma, to finish §.2 we need only show that

$$P^{\Delta(1,0)} \longrightarrow P^2 \times_{(pr_1^* f^* Q)^2} (pr_1^* f^* Q)^{\Delta(1,0)}$$

is special. However, quite generally, if $P \longrightarrow P'$ is a special map of bisimplicial objects so is

$$P^K \longrightarrow P'^K \times_{P'^L} P^L$$

for any inclusion $L \subset K$ in $(\Delta^2)^\wedge$. q.ed.

It is clear that

$$E_2^{pq}(P,Q,u) = H^p(r \mapsto H^q(s \mapsto \Gamma(P_{rs},F)))$$

is a (horizontal) homotopy invariant, hence this spectral functor comes from Ho J_f. It remains to see that

$$\varinjlim_{Ho(J_f)} E_2^{pq}(P,Q,u) = H^p(Y, R^q f_*(F)).$$

(It is necessary to be careful because pr_1^* does not carry special simplicial maps to special bisimplicial maps, since $(pr_1)_! = \pi_0^\vee$ does not preserve injections.) We factor the basic map into

(1) $$H^p(r \mapsto H^q(s \mapsto \Gamma(P_{rs},F))) \longrightarrow H^p(r \mapsto H^q(f^*Q_r, F))$$

followed by the Verdier map to $H^p(Y, R^q f_*(F))$. Suppose $\alpha \in E_2^{pq}(P,Q,u)$ goes to zero. Then by Verdier, we can refine Q until the image of α is zero under (1). Now fix Q and let P range over special things over $pr_1^* f^*Q$; call this category K and let Ho K be the corresponding homotopy category for <u>vertical</u> homotopy.

~~[crossed out]~~ Then we have to check

(i) $\mathrm{Ho}\,K$ cofiltering

(ii) the P in K get arbitrarily fine in each bidegree.

For (ii) we ~~[crossed out]~~ follow the argument of 7.4; it suffices to show for the functor $j: pt \longrightarrow \Delta^2$ with image $\Delta(p,q)$, that j_* transforms surjections to special maps, which is clear as j^* preserves injections. As for (i) we have to show that the vertical path space doesn't lead us out of K; it suffices to show that if P is special over $pr_1^* f^* Q$, then

$$P^{pr_2^*\Delta(1)} \longrightarrow P^{pr_2^*\Delta(1)^\circ} \underset{(pr_1^* f^* Q)^{pr_2^*\Delta(1)^\circ}}{\times} (pr_1^* f^* Q)^{pr_2^*\Delta(1)}$$
$$= P \times_{pr_1^* f^* Q} P$$

is special. But this is a special case of the last statement on p. 39.

It follows that

$$\varinjlim_{\mathrm{Ho}\,K} H^p(r \mapsto H^q(s \mapsto \Gamma(P_{rs}, F)))$$
$$= H^p(r \mapsto \varinjlim_{\mathrm{Ho}\,K} H^q(s \mapsto \Gamma(P_{rs}, F))) \qquad (i)$$
$$= H^p(r \longmapsto H^q(f^* Q_r, F)) \qquad (i) + (ii)$$

hence we conclude that it is possible to refine P within K so as to kill α. Similarly given an element β of $H^p(Y, R^q f_*(F))$, one first chooses Q fine enough so that β comes from $H^p(v \mapsto H^q(f^*Q_v, F))$ and then one finds P fine enough so that β comes from $E_2^{pq}(P, Q, u)$. Hence we obtain

<u>Theorem:</u> $Ho(J_f)$ is cofiltering and the limit of the simplicial spectral sequence $\mathcal{E}(P, Q, u)$ over $Ho(J_f)$ is isomorphic to the Leray spectral sequence of (f, F).

<u>Remark:</u> One of the virtues of J_f over I_f is that $\Delta^* P \longrightarrow Q$ is special, Δ being the diagonal. See following pages for the proof.

10) Special maps of bisimplicial sets

<u>Definition and proposition</u>: A map of bisimplicial sets $P \longrightarrow Q$ is called special if it satisfies the equivalent conditions:

i) RLP wrt any injective map of bisimplicial sets

ii) RLP wrt the injections

$$\underbrace{\Delta(p)\boxtimes\Delta(q)^{\circ} \cup \Delta(p)^{\circ}\boxtimes\Delta(q)}_{\substack{\| \\ [\Delta(p)\boxtimes\Delta(q)]^{\circ}}} \subset \Delta(p)\boxtimes\Delta(q)$$

Proof of the equivalence: i) $\Rightarrow$ ii) trivial.

ii) $\Rightarrow$ i) Given an ~~[illegible]~~ inclusion of bisimplicial sets $L \subset K$, let $x \in K_{pq} - L_{pq}$ with $p+q$ least. Then we have a square

$$\begin{tikzcd}
\Delta(p,q)^{\circ} \arrow[r, phantom, "\subset"] \arrow[d] & \Delta(p,q) \arrow[d] \\
L \arrow[r, phantom, "\subset"] & K
\end{tikzcd}$$

and what we have to prove is that the induced map

$$L \cup_{\Delta(p,q)^{\circ}} \Delta(p,q) \longrightarrow K$$

is injective. Denote the image by $L\langle x\rangle$. Any ~~element~~ z of $L\langle x\rangle_{rs} - L_{rs}$ ~~necessarily~~ is of the form $z = \eta^*(x)$ where $\eta : [r,s] \longrightarrow [p,q]$ is surjective. Moreover η is uniquely determined by z, since if $\eta^*(x) = \eta'^*(x)$ we choose $\varepsilon : [p,q] \longrightarrow [r,s]$ $\ni$ $\eta\varepsilon = id$; then $x = \varepsilon^*\eta'^*(x)$ $= (\eta'\varepsilon)^*(x)$, hence x being non-degenerate, this implies

$\eta'\varepsilon = \mathrm{id}$; this being true for all ε implies $\eta = \eta'$. The uniqueness of η shows that $\Delta(p,q)_{rs} - \Delta(p,q)^\circ_{rs} \xrightarrow{\sim} L\langle x\rangle_{rs} - L_{rs}$, so done.

Let $\Delta : \Delta \longrightarrow \Delta^2$ be the diagonal functor and

$$\Delta^\wedge \;\underset{\Delta_*}{\overset{\Delta_!}{\underset{\longleftarrow\;\Delta^*}{\rightrightarrows}}}\; (\Delta^2)^\wedge$$

the associated functors between simplicial and bisimplicial sets.

<u>Prop</u>: If $P \longrightarrow Q$ is a special map in $(\Delta^2)^\wedge$ then $\Delta^*P \longrightarrow \Delta^*Q$ is a special map (aspherical fibration) in $\Delta^\wedge$.

<u>Proof</u>: Have to show $\Delta^*P \to \Delta^*Q$ has RLP wrt $\Delta(r)^\circ \longrightarrow \Delta(r)$ for any $r \geq 0$; ~~equivalently has~~ equivalently have to show $P \to Q$ has RLP wrt $\Delta_!(\Delta(r)^\circ) \longrightarrow \Delta_!(\Delta(r))$, hence it suffices to show this last map is injective. Have

$$\Delta_!(\Delta(r)) = \Delta(r) \boxtimes \Delta(r) \overset{\text{defn.}}{=} \Delta(r,r)$$

and we have a coequalizer situation

$$\coprod_{0\leq i<j\leq r} \Delta(r-2) \underset{(\partial_{j-1})}{\overset{(\partial_i)}{\rightrightarrows}} \coprod_{0\leq i\leq r} \Delta(r-1) \xrightarrow{(\partial_i)} \Delta(r)^\circ$$

Proposition: $\Delta_!$ carries monomorphisms into monomorphisms.

Proof. Since $\Delta_!$ commutes with ind. limits and since any mono. in $\mathbb{\Delta}^\wedge$ is built up for the inclusions $\Delta(r)^\circ \longrightarrow \Delta(r)$, $r \geq 0$, using ind. limits, it suffices to show $\Delta_!$ carries this inclusion into a mono. We have the formula

$$\Delta_! \Delta(n) = \Delta(n) \boxtimes \Delta(n)$$

as both sides represent the functor $P \longmapsto P_{nn}$ from $(\mathbb{\Delta}^2)^\wedge$ to sets.

First a general observation. Let $\{S_i \to S,\ i \in I\}$ be a family of morphisms in a topos and form the diagram

$$\coprod_{i,j} S_i \times_S S_j \rightrightarrows \coprod_i S_i \longrightarrow S$$

Then the cokernel of the pairs is canonically isomorphic to the image of the arrow at the right.

We apply this to the family of morphisms in $\Delta^\wedge$ given by the face maps $\partial_i : \Delta(r-1) \longrightarrow \Delta(r)$ for $0 \le i \le r$. Here

$$S_i \times_S S_j = \begin{cases} \Delta(r-2), & pr_1 = \partial_{j-1}, pr_2 = \partial_i \quad \text{if} \quad i<j \\ \Delta(r-2), & \text{---} \ \partial_j, \ \text{---} \ \partial_{i-1} \quad \text{---} \quad i>j \\ \Delta(r-1), & \text{---} \ id, \ \text{---} \ id \quad \text{---} \quad i=j \end{cases}$$

and the image of the map $\coprod S_i \longrightarrow S$ is $\Delta(r)^\circ$. Since $\Delta_!$ commutes with inductive limits we have that $\Delta_!(\Delta(r)^\circ)$ is the cokernel of the pair

$$\coprod_{i,j} \Delta_!(S_i \times_S S_j) \rightrightarrows \coprod_i \Delta_! S_i$$

But

$$\Delta_!(S_i \times_S S_j) \xrightarrow{\sim} \Delta_! S_i \times_{\Delta_! S} \Delta_! S_j$$

because ~~because in the case of the simplicial sets $\Delta(n)$ the functor $\Delta_!$ coincides with~~ For simplicial sets of the form $K = \Delta(n)$ we have $\Delta_! K \xrightarrow{\sim} K \boxtimes K$ and the latter functor commutes with fibred products. Consequently applying our general observation to the family $\{\Delta_! S_i \longrightarrow \Delta_! S\}$, we see that $\Delta_!(\Delta(r)^\circ)$ is canonically isomorphic to the image of this family. Therefore $\Delta_!(\Delta(r)^\circ)$ maps monomorphically to $\Delta_! \Delta(r)$, proving the proposition.

Corollary: If $P \to Q$ is a special map of bisimplicial sets, then $\Delta^* P \longrightarrow \Delta^* Q$ is a special map of simplicial sets.

Proof: $\Delta^* P \to \Delta^* Q$ has RLP wrt $L \hookrightarrow K$ $\Longleftrightarrow$ $P \to Q$ has RLP wrt $\Delta_! L \hookrightarrow \Delta_! K$. But proposition $\Longrightarrow$ $\Delta_! L \to \Delta_! K$ inj. if $L \to K$ is.

~~We will want the fact that~~

~~Let K be a contractible complex. Then $\Delta_* K$ is a special~~

If $X \longrightarrow Y$ is a special map in $\Delta^\wedge$, then $\Delta_*(X) \longrightarrow \Delta_*(Y)$ is special in $(\Delta^2)^\wedge$, because Δ^* preserves monomorphisms. Consequently if $K \to e$ is special $\Delta_*(K) \longrightarrow \Delta_*(e) = e$ is special in $(\Delta^2)^\wedge$. Thus $\Delta^* \Delta_*(K)$ is a special simplicial set dominating K.

Remark: The functor $\Delta_* : \Delta^\wedge \longrightarrow (\Delta^2)^\wedge$ is analogous to the singular complex functor from spaces to simplicial spaces, because

$$\boxed{(\Delta_* X)_{pq} = \varinjlim_{\Delta(n) \to \Delta(p)\times\Delta(q)} X_n = \mathrm{Hom}_{\Delta^\wedge}(\Delta(p)\times\Delta(q), X) = (X^{\Delta(p)})_q}$$

$$
\begin{aligned}
(\Delta_* X)_{pq} &= \mathrm{Hom}_{\Delta^\wedge}\left(\Delta^*(\Delta(p,q)), X\right) \\
&= \mathrm{Hom}_{\Delta^\wedge}\left(\Delta(p) \times \Delta(q), X\right) \\
&= \left(X^{\Delta(p)}\right)_q
\end{aligned}
$$

Consequently Δ^* is an analogue of the geometric realization of a simplicial space. The preceding corollary is thus ~~the~~ an analogue of the fact that the geometric realization of a fibration is a fibration.