May 1, 1971. K-theory.

Yesterday I defined the category $\mathcal{E}$ whose objects are the fin. gen. projective R-modules P with

$$\mathrm{Hom}_{\mathcal{E}}(P,P') = \left\{(\theta, Q, u) \;\middle|\; \begin{array}{l} u : P \hookrightarrow P' \\ Q \subset P' \ni Q \oplus uP = P' \\ \theta \in \mathrm{Aut}(P') \end{array}\right\} \Big/$$

modulo equivalence relation

$$(\theta\lambda, Q, u) = (\theta, \lambda Q, u) \qquad \lambda \in \mathrm{Aut}^0(u)$$

B denote the category of fin. gen. proj. R-modules with direct injections for morphisms. The functor $f : \mathcal{E} \longrightarrow \mathcal{B}$ sending $P \longmapsto P$, $(\theta, Q, u) \longmapsto u$ is cofibrant.

I hope to be able to show that the homotopy groups of $\mathcal{E}$ with basepoint the 0 R-module coincide with my $K_i(R)$ $i \geq 1$. For this it will be necessary to compute the fundamental group of $\mathcal{E}$ and show it is abelian, and to show the homology of $\mathcal{E}$ coincides with that of $GL(R)$. Because $f : \mathcal{E} \longrightarrow \mathcal{B}$ is cofibrant, I know (sauf erreur) that

$$\mathbb{L}f_!(F)(P) = H_*(\mathrm{Aut}(P), F_P)$$

hence the Leray spectral sequence for f in homology will be

$$E^2_{pq} = H_p(\mathcal{B}, P \longmapsto H_q(\mathrm{Aut}(P), F_P)) \Longrightarrow H_{p+q}(\mathcal{E}, F).$$

Here we are taking derived functors in $\mathcal{E}^{\vee}_{ab} = \mathrm{Hom}(\mathcal{E}, \mathcal{Ab})$. A functor $F : \mathcal{E} \longrightarrow \mathcal{Ab}$ may be identified with a family of functors

$$F_P : \mathcal{E}_P \longrightarrow \mathcal{Ab} \qquad P \in \mathrm{Ob}\,\mathcal{B}$$

together with for each $u : P \to P'$ in $\mathcal{B}$ a map

$$\bar{u} : F_P \longrightarrow (\tilde{u})^* F_{P'} ,$$

where $\tilde{u} : \mathcal{E}_P \to \mathcal{E}_{P'}$ is the cobase change by u, such that the following compatibility conditions hold for $P \xrightarrow{u} P' \xrightarrow{v} P''$;

$$\begin{array}{ccccc} F_P(x) & \xrightarrow{\bar{u}} & F_{P'}(\tilde{u}x) & \xrightarrow{\bar{v}} & F_{P''}(\tilde{v}\tilde{u}x) \\ & \searrow^{\overline{vu}} & & & \wr\| \\ & & & & F_{P''}(\widetilde{vu}\,x) \end{array}$$

should be commutative. Note then that given $u : P \to P'$ we have a map

$$H_*(\mathcal{E}_P, F_P) \xrightarrow[\text{induced by } \bar{u}]{} H_*(\mathcal{E}_P, \tilde{u}^* F_{P'}) \xrightarrow[\text{canon.}]{} H_*(\mathcal{E}_{P'}, F_{P'})$$

so that indeed $P \longrightarrow H_*(\mathcal{E}_P, F_P)$ is in $\mathcal{B}^{\vee}_{ab}$.

The discussion on this page above holds for any cofibred category $\mathcal{E} \to \mathcal{B}$. In this K-situation one knows that $P \longmapsto \mathrm{Aut}(P)$ is roughly a covariant functor from $\mathcal{B}$ to groups. An element of $\mathcal{E}^{\vee}_{ab}$ is roughly a G_P-module F_P for each P

together with for $u : P \longrightarrow P'$ a map $\tilde{u} : F_P \to F_{P'}$ compatible with $\tilde{u} : G_P \longrightarrow G_{P'}$. In other words F is roughly a functor from $\mathcal{B}$ to $\mathcal{Ab}$ with G-module structure.

Fundamental groupoid of $\mathcal{E}$. First of all $\mathrm{Hom}_{\mathcal{E}}(0, P') = \{(\theta, P', 0\text{-map}) \mid \theta \in \mathrm{Aut}\, P'\} \neq \phi$ so $\mathcal{E}$ is connected. Let $F : \mathcal{E} \longrightarrow \mathcal{S}ets$ be "locally constant", i.e. for each arrow α in $\mathcal{E}$, $F(\alpha)$ is an isomorphism. Then for each P, $F(P)$ is an $\mathrm{Aut}(P)$-set. Moreover the map

$$(id_P, P, 0) : 0 \longrightarrow P$$

induces an isom. $F(0) \xrightarrow{\sim} F(P)$ permitting us to define a homomorphism

$$r_P : \mathrm{Aut}(P) \longrightarrow \mathrm{Aut}(F(0))$$

by letting $r_P(\theta)$, $\theta \in \mathrm{Aut}(P)$, be the composite

$$F(0) \xrightarrow{(id_P, P, 0)_*} F(P) \xrightarrow{(\theta, 0, id_P)} F(P) \xrightarrow{(id_P, P, 0)_*^{-1}} F(0).$$

(More briefly $r_P(\theta) = (id_P, P, 0)_*^{-1} . (\theta, P, 0)_* .$)
Given (Q, u), we have a commutative diagram

$$\begin{array}{ccccccc} & \overset{(id_P, P, 0)}{\nearrow} & P & \xrightarrow{(\theta,\, Q,\, id_P)} & P & \overset{(id_P, P, 0)}{\longleftarrow} & \\ 0 & & \downarrow{\scriptstyle (id_{P'}, Q, u)} & & \downarrow{\scriptstyle (id_{P'}, Q, u)} & & 0 \\ & \underset{(id_{P'}, P', 0)}{\searrow} & P' & \xrightarrow{((Q,u)_*\theta,\, Q,\, id_{P'})} & P' & \underset{(id_{P'}, P', 0)}{\longleftarrow} & \end{array}$$

hence
$$r_P(\theta) = r_{P'}\big((Q,u)_*\theta\big).$$

Thus F(0) has a natural action of
$$\varinjlim_{P \in \mathcal{C}} \mathrm{Aut}(P)$$
taken in the category of groups. Claim this limit is abelian: Take [redacted] $P = P'$; hence u is an isomorphism and
$$(0,u)_*(\theta) = u\theta u^{-1}.$$
Thus $r_P(\theta) = r_{P'}(u\theta u^{-1})$ for any $u \in \mathrm{Aut}\,P$. Thus
$$\pi_1(\mathcal{E}, 0) = \varinjlim_{\mathcal{B}} \mathrm{Aut}(P)_{ab} = K_1(R)$$
as it should be.

May 3, 1971.

The problem is to make the preceding day's work more intrinsic by replacing the ~~category~~ group $Aut(P)$ by the category of its torsors. Thus given a direct injection $u : P \to P'$ it gives rise to a homomorphism

$$u_* : Aut(P) \longrightarrow Aut(P')$$

unique up to a canonical inner auto of $Aut(P')$. One therefore gets a functor

$$Aut(P)\text{-torsors} \longrightarrow Aut(P')\text{-torsors}$$

unique up to canonical isomorphism. The category of such functors is equivalent to the category of $Aut(P')$-torsors with a left action of $Aut(P)$. Given a homomorphism $w : G \to G'$, the corresponding G'-torsor is G' with right G'-action and left G-action through w. Thus having chosen a complement Q for uP in P', or what comes to the same thing a projection $\varepsilon : P' \to P$ for u, we may identify the desired torsor with $Aut(P')$ with $Aut(P)$ action

$$\theta^{\in Aut(P)} \text{ acting on } \theta' \text{ in } Aut(P') = \theta . (Q,u)_* \theta'.$$

Suppose given another complement αQ where $\alpha \in Aut^0(u)$, i.e. $\alpha u = u$ and $\alpha \equiv id$ on P'/uP. Then

$$(\alpha Q, u)_* \theta = \alpha . (Q,u)_* \theta . \alpha^{-1}$$

hence the torsor isomorphism is

$$\mathrm{Aut}(P') \xrightarrow{L_\alpha} \mathrm{Aut}(P')$$

(L_α = left multiplication.) More precisely to the map u is associated a torsor for $\mathrm{Aut}(P')$ with $\mathrm{Aut}(P)$ action; call this torsor E_u. Given Q we get an element $e_Q \in E_u$ and $(Q,u)_*$ is defined by

$$\theta . e_Q = e_Q . (Q,u)_* \theta$$

Given α we have

$$e_{\alpha Q} = e_Q . \alpha^{-1}$$

$$\begin{array}{ccc} & R_{e_Q} & \mathrm{Aut}\,P' \\ E_u \swarrow & & \downarrow L_\alpha \\ \nwarrow & R_{e_{\alpha Q}} & \mathrm{Aut}\,P' \end{array}$$

for then

$$\theta . e_{\alpha Q} = \theta . e_Q . \alpha^{-1} = e_Q^{\alpha^{-1}\alpha} (Q,u)_* \theta . \alpha^{-1}$$

$$= e_{\alpha Q} . (\alpha Q, u)_* \theta .$$

Thus the torsor E_u comes with a canonical isomorphism

$$\{Q \mid Q \oplus_u P = P'\} \overset{(\mathrm{Aut}^0(u))}{\times} \mathrm{Aut}\,P' \xrightarrow{\sim} E_u$$

$$(Q\ ,\ \theta') \longmapsto e_Q . \theta'$$

$$(Q, \theta) = (\alpha Q, \alpha \theta).$$

~~Not clear from this formula how Aut(P) acts.~~

~~[illegible] There is also a canonical isom.~~

Action of Aut(P) is given by

$$\theta \cdot (e_Q \cdot \theta') = e_Q \cdot ((Q,u)_* \theta \cdot \theta').$$

Funny thing: The torsor is canonically isomorphic to $\mathrm{Isom}(P', P \oplus P'/uP)$.

$$\{Q \mid Q \oplus uP = P'\} \times^{\mathrm{Aut}^0(u)} \mathrm{Aut}\, P' \longrightarrow \mathrm{Isom}(P', P \oplus P'/uP)$$
$$(Q, \theta') \longmapsto \varphi_Q \cdot \theta'$$

where φ_Q is the canonical automorphism determined by Q and u. (i.e. $e_Q \mapsto \varphi_Q$). Make Aut(P) act on $\mathrm{Isom}(P', P \oplus P'/uP)$ by composing with $\theta \oplus id_{P'/uP}$. Then

$$\theta \text{ acting on } \varphi_Q = (\theta + id_{P'/uP})\varphi_Q$$
$$= \varphi_Q \cdot (Q,u)_* \theta,$$

hence the above isomorphism is compatible with Aut(P) action. But what's funny is that given

$$P \xrightarrow{u} P' \xrightarrow{v} P''$$

there is no canonical map

$$\mathrm{Isom}(P'', P' + \mathrm{Cok}(v)) \times \mathrm{Isom}(P', P + \mathrm{Coker}\, u)$$
$$\longrightarrow \mathrm{Isom}(P'', P + \mathrm{Cok}\, vu)$$

that I can see. Indeed given α, β in first + second

$$\begin{array}{ccc} P'' & \xrightarrow{\alpha} & P' \oplus \operatorname{Coker} v \\ & & \downarrow \beta \oplus id \\ & & (P \oplus \operatorname{Coker} u) \oplus \operatorname{Coker} v \end{array}$$

in order to identify this composite with an element of the third set we need to split

$$0 \longrightarrow \operatorname{Coker} u \longrightarrow \operatorname{Coker} vu \longrightarrow \operatorname{Coker} v \longrightarrow 0$$

the exact sequence.

This tends to be confusing because by the formulas I have written, the torsor E_u is the transpose of $\operatorname{Hom}_{\mathcal{E}}(P, P')_u$, hence there should be a canonical isomorphism

$$E_u \times^{\operatorname{Aut}(P')} E_v \xrightarrow{\sim} E_{vu}$$

since $\mathcal{E}$ is a category.

Proof that $\mathcal{E}$ has the homotopy type of $B\Sigma_\infty^+$ in case of finite sets

<u>May 5, 1971</u>:

Here is the category in the Mather situation with homotopy type BG^+.

Let $\mathcal{B}$ be the category with a single object I and with $\mathrm{Hom}_{\mathcal{B}}(I,I) =$ set of maps $u: [0,1] \longrightarrow [0,1]$ such that 1) u is a diffeo. of $[0,1]$ with $[0,a]$ for some a, $0 < a \leq 1$, ii) $u(x) = x$ for x near 0. Composition is obvious. If M denotes this monoid, then G, = diffeos of $[0,1]$ with support in the interior, is contained as a submonoid. (Alternative: u should be a germ of diffeo. from $[0,1]$ to $[0,a]$ for some $0 < a \leq 1$.)

$\mathcal{B}$ is a cofiltering category: The axiom $\cdot \leftarrow \cdot \to \cdot$ is trivial, so all one must do is prove the equalizer condition. Given $u, v \in M$ one knows that they coincide in a neighborhood of 0, hence choosing w in I with image contained in this nbd., we have $uw = vw$; this establishes the axiom

$$\cdot \xrightarrow{w} \cdot \overset{u}{\underset{v}{\rightrightarrows}} \cdot$$

Given $u \in M$, let $u_*: G \longrightarrow G$ be the homomorphism which transports a diffeo of I to the image of u via u and the identity outside. Thus

$$u_*(g).u = u.g$$

$$u_*(g) = id \quad \text{outside of the image of } u$$

Then

$$v_* u_* = (vu)_*$$

for $u, v \in M$, whence $I \longmapsto G$, $u \longmapsto u_*$ is a functor from $\mathcal{B}$ to **groups**. Hence we can form a cofibred category $\mathcal{E}$ over $\mathcal{B}$ having for morphisms pairs (g, u) with $g \in G$ and $u \in M$, with composition

$$(g', u')(g, u) = (g' \cdot u'_*(g), u'u).$$

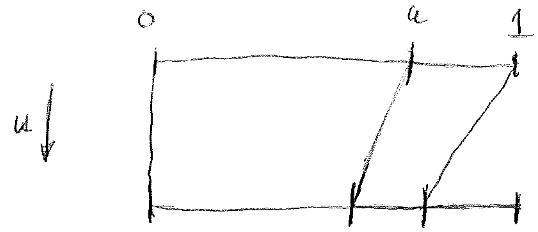
Next compute the fundamental group of $\mathcal{E}$, i.e. the group π associated to the above semi-direct product monoid $G \tilde{\times} M$. Since $\mathcal{B}$ is filtering any pair (e, u) goes to the identity of π. Hence $(g, e) \longmapsto$ its image $(\overline{g, e})$ in π is a surjection map from G to π and the kernel contains the elements $u_*(g) \cdot g^{-1}$ because

$$(e, u)(g, e) = (u_* g, e) \quad \text{and} \quad (g, e)$$

have the same image. Taking $u \in G \subset M$, whence $(u_* g) = ugu^{-1}$, we see that π is abelian, and in fact that

$$\pi = (G_{ab})_M$$

Next we show that $u_* : G \to G$ induces the identity map on $H_*(G)$ (any (constant) coefficients). Let $F \subset G$ be a finitely generated subgroup. Then the supports of all the members of F are contained in an interval $[0, a]$ with $a < 1$. There exists an element $g \in G$ such that $u_*(f) = gfg^{-1}$ for all $f \in F$;



make $g = u$ on $[0,a]$
and $g = id$ near 1.

Thus we have a commutative square

$$\begin{array}{ccc} F & \xrightarrow{in} & G \\ \downarrow{\scriptstyle in} & & \downarrow{\scriptstyle u_*} \\ G & \xrightarrow[x \mapsto gxg^{-1}]{} & G \end{array}$$

yielding

$$H_*(F) \xrightarrow{in_*} H_*(G) \underset{id}{\overset{(u_*)_*}{\rightrightarrows}} H_*(G)$$

since $H_*(G) = \varinjlim_F H_*(F)$ where F runs over the directed set of f.g. subgroups, the assertion follows. More generally one sees that is A is a $\mathbb{Z}[G_{ab}]$-module, whence $id : A \to A$ is equivariant for $u_* : G \to G$, the the induced map

$$(u_*, id)_* : H_*(G,A) \longrightarrow H_*(G,A)$$

is the identity.

Since $\mathcal{E} \longrightarrow \mathcal{B}$ is cofibred, we have a spectral sequence

$$E^2_{pq} = H_p(M, H_q(G)) \Longrightarrow H_{p+q}(\mathcal{E})$$

(coefficients in any A). Now we have just showed that $H_*(G)$ is a local coefficient system on $\mathcal{B}$. In general given a local coefficient system L on a category C

we have

$$H_*(\mathcal{C}, \mathcal{L}) \simeq H_*(\mathcal{C}^\circ, \mathcal{L}^\vee).$$

In the case at hand $\mathcal{B}$ is cofiltering, hence

$$H_*(\mathcal{B}^\circ, F) = 0$$

for any $F : \mathcal{B}^\circ \longrightarrow \mathcal{A}b$. Thus the spectral sequence degenerates, showing that the inclusion of the fibre induces an isomorphism on homology

$$H_*(G, A) \xrightarrow{\sim} H_*(\mathcal{E}, A)$$

for any $\pi_1(\mathcal{E}) \xleftarrow{\sim} G_{ab}$ - module A. Thus I have shown that the category $\mathcal{E}$ has the homotopy type of BG^+.

Analogue of the preceding for Σ_n: Let M be the monoid of injective maps $u : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\exists$ k with $u(x) = x + k$ for x larges. Then we have a map $t : M \longrightarrow \mathbb{N}$ assigning to u the integer k. Then given $u, v \in M$ with $t(u) = t(v)$ let $w \in M$ be a large translation; then $uw = vw$. I think this means that t induces a functor $\mathrm{Pro}(\mathbb{N}) \longrightarrow \mathrm{Pro}(M)$ left adjoint to the natural one. In other words given $\phi \in \mathrm{Ob}\,\mathbb{N}$ the functor $\mathrm{Hom}_{\mathbb{N}}(\phi, t(?))$ is pro-representable. Check this:

1) Let $T_a \in M$

Analogue of preceding for Σ_n. Let B be the category associated to the monoid M consisting of injective maps $u: \mathbb{N} \longrightarrow \mathbb{N}$ which are translations near ∞, i.e. $\exists\, k \ni u(x) = x+k$ for x large. Let G be the infinite symmetric group, i.e. autos. $u: \mathbb{N} \to \mathbb{N}$ which are the identity near infinity; G is the invertible elements of M here. Given u we have $u_* : G \longrightarrow G$ (this would be defined for any injection u) and $N \mapsto G$, $u \mapsto u_*$ is a covariant functor, so again we can form the cofibred category (coscinded in fact) $\mathcal{E}$, associated to the monoid $G \tilde{\times} M$. Now given $u, v \in M$ which coincide at ∞, there is a $w \in M$ ($w =$ a large translation) such that $uw = vw$. Thus in the fundamental group π of $\mathcal{E}$ we have the images $\overline{(e,u)}$ and $\overline{(e,v)}$ are equal, and hence $(u_*(g), e) = (v_*(g), e)$ because

$$(e,u)(g,e) = (u_*(g), u) = (u_*(g), e)(e,u)$$
$$(e,v)(g,e) = (v_*(g), v) = (v_*(g), e)(e,v).$$

In particular taking $u \in G$, one sees that $u_*(g) = ugu^{-1}$ must have same image as g. Thus it's clear that the fundamental group is $\pi = G_{ab} \times \mathbb{Z}$, which isn't exactly what I want.

So instead fix an infinite set S and let B be the category associated to the monoid M of injections $u: S \hookrightarrow S$. Let G be the autos. of S with finite support, and

$$u_* : G \longrightarrow G$$

the homomorphism such that

$$\begin{cases} (u_* g)(ux) = u(gx) \\ (u_* g)(x) = x \qquad \text{if } x \notin uS. \end{cases}$$

Then again we can form the cofibred category $\mathcal{E}$. To show that $H_*(G) \xrightarrow{\sim} H_*(\mathcal{E})$ constant coefficients we need that

$$u_* : H_*(G) \longrightarrow H_*(G)$$

is the identity map and that $\mathcal{B}$ is contractible, i.e. the map $\mathcal{B} \longrightarrow pt$ is a homotopy equivalence. The first we have already checked and for the second we use Mather's trick.

We first show $\pi_1(\mathcal{B}) = 0$. $\pi_1(\mathcal{B})$ is the group generated by the monoid M; denote the canonical monoid homo. $M \longrightarrow \pi_1(\mathcal{B})$ by $m \longmapsto \overline{m}$. First note that if $u, v \in M$ and $u = v$ on a subset of S of the same cardinality as S, then $\overline{u} = \overline{v}$. Indeed $\exists$ $w : S \hookrightarrow S$ with image in this subset, hence $uw = vw$. Let $u : S \longrightarrow S$ be such that $S - uS$ has the same cardinality as S. Let $\theta \in M$. Then $u_*(\theta)$ is the identity on $S - uS$, hence $\overline{u_*(\theta)} = 1$. Since

$$u_*(\theta)\, u = u . \theta$$

we have $\overline{u} = \overline{u}.\overline{\theta} \implies \overline{\theta} = 1$.

Next we show that $H_+(\mathcal{B}) = 0$, (integral homology). Given $u \in M$, let $u_* : M \longrightarrow M$ be the associated homomorphism. Claim

$(u_*)_*: H_*(\mathcal{B}) \longrightarrow H_*(\mathcal{B})$ is the identity. Indeed have

$$\mathcal{B} \underset{u_*}{\overset{id}{\rightrightarrows}} \mathcal{B} \quad (\downarrow \alpha)$$

where α is the natural transformation from the identity functor to u_* given by

$$\alpha(S) = u : S \longrightarrow S.$$

Indeed given $m \in M$, the square

$$\begin{array}{ccc} S & \xrightarrow{u} & S \\ m \downarrow & & \downarrow u_*(m) \\ S & \xrightarrow[u]{} & S \end{array}$$

is commutative. But one knows that if $f, g : \mathcal{C}_1 \rightrightarrows \mathcal{C}_2$ are functors and $\alpha : f \to g$ is a morphism of functors, then $f_* = g_* : H_*(\mathcal{C}_1) \longrightarrow H_*(\mathcal{C}_2)$; in effect α gives rise to a homotopy between the maps $\operatorname{Sing} \mathcal{C}_1 \rightrightarrows \operatorname{Sing} \mathcal{C}_2$ induced by f and g. Choose any injection $\theta : S \cup S \longrightarrow S$. Then θ gives rise to a homomorphism

$$\theta_* : M \times M \longrightarrow M$$

(or a functor $\mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$) hence to a map

$$H_*(\mathcal{B} \times \mathcal{B}) \longrightarrow H_*(\mathcal{B}).$$

As $u_* = id$ on homology one sees this map makes $H_*(\mathcal{B})$ a Hopf algebra when the coefficients are a field. Also the map is independent of the choice of θ. Now apply

Mather's trick: given a functor $\xi : \mathcal{C} \longrightarrow \mathcal{B}$ let [illegible] $P(\xi)$ be the "sum" of an infinite number of copies of ξ, i.e. defined by an injection $\mathbb{N} \times S \xrightarrow{\sim} S$. Then $\xi \oplus P(\xi) \simeq P(\xi)$, hence if c is an exponential class we have

$$c(\xi) \cdot c(P(\xi)) = c(P(\xi))$$

whence $c(\xi) = 1$.

We thus see that the monoid $G \tilde{\times} M$, i.e. the category $\mathcal{E}$ when realized has the homotopy type of BG^+.

The same sort of thing works for the infinite general linear group for G with the same $\mathcal{B}$. Want different $\mathcal{B}$:

Let R be a ring and let S be a free R-module with infinitely many generators. [illegible] [illegible] (We want $S \oplus S$ to be isomorphic to S, and $S \oplus R^n \simeq S$ for all n.) Let M be the monoid of direct injections $u : S \hookrightarrow S$, and let $\mathcal{B}$ be the corresponding category. [illegible]

Let G be the group of autos. of S ??

I think I am now in a position to prove that if $\mathcal{E}$ is the cofibred category over the category $\mathcal{B}$ of finite sets and injections constructed from the functor $P \mapsto \mathrm{Aut}(P)$, $u \mapsto u_*$, then $\mathcal{E}$ has the homotopy type of $B\Sigma_\infty^+$. Let S be an infinite set and $\mathcal{M}$ the category with S as its single object and $M = \{\text{injective maps } u : S \to S\}$ for morphisms. Put another way $\mathcal{M}$ is the full-subcategory of $\mathrm{Ind}(\mathcal{B})$ consisting of S. Now consider the functor

$$\alpha : \mathcal{B}^\vee_{ab} \longrightarrow \mathcal{M}^\vee_{ab}$$

$$F \longmapsto \varinjlim_{P \subset S} F(P)$$

The canonical homomorphism

$$\varinjlim_{P \in \mathcal{B}} F(P) \longleftarrow \varinjlim_{\mathcal{M}} \varinjlim_{P \subset S} F(P)$$

is an isomorphism; indeed any diagram can be completed

$$\begin{array}{ccc} P & \hookrightarrow & P' \\ \downarrow & & \downarrow \\ S & \overset{m}{\dashrightarrow} & S \end{array}.$$

This homomorphism extends to a morphism of homological functors:

$$H_q\left(\mathcal{M}, \varinjlim_{P\subset S} F(P)\right) \longrightarrow H_q(\mathcal{B}, F)$$

which is an isomorphism in dimension zero. To prove its an isomorphism in all dimensions we must show the left-hand homological functor is effaceable, i.e. vanishes for functors F of the form

$$F(P) = \mathbb{Z}[\mathrm{Hom}_{\mathcal{B}}(Q,P)]$$

where Q is a finite set. Then

$$\varinjlim_{P\subset S} F(P) = \mathbb{Z}[\mathrm{Inj}(Q,S)].$$

Let $\mathcal{F} \xrightarrow{f} \mathcal{M}$ be the cofibred category associated to the functor $\mathrm{Inj}(Q,S)$. Then spectral sequence for f collapses yielding the isomorphism

$$H_q(\mathcal{F},\mathbb{Z}) = H_q(\mathcal{M}, \mathbb{Z}[\mathrm{Inj}(Q,S)]).$$

However $\mathcal{F}$ is the category whose objects are injections $Q \longrightarrow S$ and whose morphisms come from $M = \mathrm{Inj}(S,S)$. Any two objects of $\mathcal{F}$ are isomorphic, hence $\mathcal{F}$ is equivalent to the full subcategory consisting of a single object $\alpha_0 : Q \hookrightarrow S$ and its endomorphisms. Choosing an isomorphism of S with $S - \alpha_0(Q)$, one sees that $\mathcal{F}$ is equivalent to $\mathcal{M}$. But we've already proved $\mathcal{M}$ is homotopically trivial, hence we've established:

$$H_*(\mathcal{B}, F) \simeq H_*(\mathcal{M}, \varinjlim_{P \subset S} F(P)).$$

Now we apply this to compute the homology of $\mathcal{E}$. First of all as it is cofibred over $\mathcal{B}$ there is a spectral sequence

$$E^2_{pq} = H_p(\mathcal{B}, P \mapsto H_q(\mathrm{Aut}(P))) \Longrightarrow H_{p+q}(\mathcal{E})$$

$$\parallel$$

$$H_p(\mathcal{M}, H_q(\mathrm{Aut}_c(S)))$$

where $\mathrm{Aut}_c(S) = \varinjlim_{P \subset S} \mathrm{Aut}(P)$. But we know that $H_*(\mathrm{Aut}_c(S))$ is a constant functor on $\mathcal{M}$, and that $\mathcal{M}$ is homotopically trivial. Thus $E^2_{pq} = 0$ for $p > 0$, and we find that

$$H_*(\mathcal{E}) \simeq H_*(\mathrm{Aut}_c(S))$$

or better the edge homomorphism of the spec. seq. is an isom.

$$\varinjlim_{\mathcal{B}} H_*(\mathrm{Aut}(P)) \xrightarrow{\sim} H_*(\mathcal{E}).$$

Let R be a ring and let $\mathcal{C}$ be the category of f.g. projective R-modules and split injections, i.e. the splitting is explicitly given. Let I be an infinite set and $S = R^{(I)}$ the free R-module generated by I. Then S gives rise to an object of Pro $\mathcal{C}$

$$\{R^{(I')}\}_{I' \text{ finite} \subset I}.$$

I should check:

<u>Lemma</u>: Let $u, v : \mathcal{C}_1 \rightrightarrows \mathcal{C}_2$ be functors and $\alpha : u \Rightarrow v$ a natural transformation, whence given $F \in \mathcal{C}_2^{\vee}$, α induces a map $\tilde{\alpha} : u^*F \longrightarrow v^*F$. Then

$$\begin{array}{ccc} H_*(\mathcal{C}_1, u^*F) & \xrightarrow{\tilde{\alpha}} & H_*(\mathcal{C}_1, v^*F) \\ & \searrow \quad \swarrow & \\ & H_*(\mathcal{C}_2, F) & \end{array}$$

commutes. In particular with constant coefficients ($F = \pi_2^* A$ so that $u^*F = \pi_1^*A = v^*F$ and $\tilde{\alpha}$ is the identity), then $u_* = v_* : H_*(\mathcal{C}_1, A) \longrightarrow H_*(\mathcal{C}_2, A)$.

<u>Proof</u>: Two morphisms of homological functors from $H_*(\mathcal{C}_1, u^*F)$ to $H_*(\mathcal{C}_2, F)$ are equal because they are equal in dimension zero.

<u>May 7, 1971:</u>

Let $\mathcal{D}$ be the category of finite linearly-ordered sets and injective monotone maps; then $\mathcal{D}$ is equivalent to the category:

$$0 \longrightarrow 1 \rightrightarrows 2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} 3 \quad \cdots$$

and $n \longrightarrow GL_n(R)$ is a covariant functor to groups whose "limit" we want to know is $BGL(R)^+$.

$\mathrm{Ind}(\mathcal{D})$ is the category of linearly ordered sets and injective monotone maps. Let S be the set of rational numbers with its natural ordering, let M be the monoid of endos. of S in the category $\mathrm{Ind}(\mathcal{D})$, and $\mathcal{M}$ the category associated to the monoid M.

Then we have a functor

$$\mathcal{D}^\vee \xrightarrow{\ l_S\ } \mathcal{M}^\vee \simeq M\text{-sets}$$

$$F \longmapsto \varinjlim_{P \subset S} F(P) \qquad \Big| \quad l_S(F) = \text{"}F(S)\text{"}$$

which is exact,

and a morphism of functors from $\mathcal{D}^\vee_{ab}$ to Ab

$$\varinjlim_{\mathcal{M}} l_S(F) \longrightarrow \varinjlim_{\mathcal{D}} F .$$

This morphism is an isomorphism. Indeed, $\forall\ P$ in $\mathcal{D}$ $\exists$ map $P \to S$, and any diagram of solid arrows

$$(*) \qquad \begin{array}{ccc} P & \longrightarrow & P' \\ \downarrow & & \vdots \\ S & \cdots\cdots\to & S \end{array}$$

can be completed as indicated. Thus one can define a map

$$\lambda_P : F(P) \longrightarrow \varinjlim_{\mathcal{M}} \ell_S(F)$$

by choosing a map $\xi : P \longrightarrow S$ and

$$F(P) \xrightarrow{\xi_*} \ell_S(F) \longrightarrow \varinjlim_{\mathcal{M}} \ell_S(F).$$

The fact that this is independent of the choice of ξ as well as the commutativity of

$$\begin{array}{ccc} F(P) & \xrightarrow{\lambda_P} & \\ \downarrow u_* & & \varinjlim_{\mathcal{M}} \ell_S(F) \\ F(P') & \xrightarrow{\lambda_{P'}} & \end{array}$$

follows from $(*)$ above.

~~Having established~~ Since ℓ_S is exact we get a morphism of homological functors on $\mathcal{D}^{\vee}_{ab}$:

$$H_q(\mathcal{M}, \ell_S(F)) \longrightarrow H_q(\mathcal{D}, F)$$

which we have seen is an isomorphism for $q = 0$. To get an isomorphism in all degrees, we need effaceability of left one, so take $F = \mathbb{Z}[\mathrm{Hom}_{\mathcal{D}}(Q, ?)]$ for a fixed Q in $\mathcal{D}$. Then $\ell_S(F)$ is the M-set $\mathbb{Z}[\mathrm{Hom}_{\mathrm{Ind}(\mathcal{D})}(Q, S)]$ and its homology over $\mathcal{M}$ is the homology of the cofibred category of injective monotone maps ~~from Q to~~ $\xi : Q \longrightarrow S$. ~~So~~ Now if Q is the ordered set $1 < \cdots < n$, then such a ξ is the same thing as n-rational numbers

$$x_1 < \cdots < x_n$$

and the morphisms are simply injective monotone maps of the rationals preserving these n-element subsets. It is clear that any two objects of this category are isomorphic, hence the category is equivalent to the full category with the single object $0<1<\cdots<n-1$. A morphism of this to itself is the same thing as a family of injective monotone maps for each of the subintervals $(-\infty, 0), (0,1), \ldots, (n-1, \infty)$, hence as each of these is isomorphic as an ordered set to S, the cofibred category is equivalent to the category defined by the monoid M^{n+1}.

It remains to show that $\mathcal{M}$ is homotopically trivial. But the same argument used before works. Denoting again by $\bar{u} \in \pi_1 \mathcal{M}$, the element belonging to $u \in M$, one sees that $\bar{u} = \bar{v}$ if $u = v$ on a non-empty interval of S, for there exists $w : S \to S$ with image in this interval. If $u : S \to S$ is such that $u(S)$ is an interval of S, then one can define $u_* : M \to M$ by

$$u_*(\theta)\, u = u\theta$$

$$u_*(\theta) = id \text{ outside of } u(S).$$

Choose such a u which is not an isomorphism, we have $\overline{u_*(\theta)} = e$ as $u_*(\theta)$ and id coincide on $S - u(S)$, hence $\bar{u} = \bar{u}\bar{\theta}$ and $\bar{\theta} = e$. So $\pi_1 M = 0$.

Next with u as above

$$(u_*)_* : H_*(\mathcal{M}) \longrightarrow H_*(\mathcal{M}) \qquad (\mathbb{Z} \text{ coeffs.})$$

is the identity as u furnishes a natural transf. from u_* to $id_{\mathcal{M}}$. Choosing an embedding $S \cup S \to S$

such that each piece of $S \cup S$ gets mapped onto an interval, one gets a Hopf algebra structure on $H_*(M)$ (field coeffs.) Choosing an embedding $S \cup S \cup S \cup \cdots\cdots \longrightarrow S$ one sees as before that the homology is trivial. Thus M is homotopically trivial.

Combining the preceding, we find that $H_*(\mathcal{D}, F) = 0$ if $l_S(F)$ [redacted] has trivial M-action, which will be the case if all the maps $P \longrightarrow P'$ in $\mathcal{D}$ have the same effect on F. So as before we can use [redacted] the cofibred category over $\mathcal{D}$ constructed from the functor $n \longmapsto GL_n(R)$ as a model for $BGL(R)^+$.

I wanted to put preceding days work on a torsor basis. effect Incomplete

May 10, 1971: More K-theory

Let $\mathcal{P}_R$ be the category of f.g. proj. R-modules. For any space X let $\mathcal{V}_X$ be the category of R-vector bundles over X, i.e. locally constant sheaves of R-modules with fibres in $\mathcal{P}(=\mathcal{P}_R)$. Given E, E' in $\mathcal{V}_X$ we consider the category whose objects are pairs (Q,u), where $Q \in \text{Ob}\, \mathcal{P}_R$ and $u: E \oplus \pi^* Q \xrightarrow{\sim} E'$ is an isom in $\mathcal{V}_X$, $\pi: X \to pt$ being the obvious map, and in which a morphism $(Q,u) \to (Q',u')$ is an isomorphism $\alpha: Q \to Q'$ such that

$$\begin{array}{ccc} E \oplus \pi^* Q & \xrightarrow{u} & \\ \downarrow{\scriptstyle id + \pi^*\alpha} & & E' \\ E \oplus \pi^* Q' & \xrightarrow{u'} & \end{array}$$

commutes. Note that, provided $X \neq \phi$, α is uniquely determined by (Q,u) and (Q',u'), hence it is reasonable to consider the set

$$I(E,E') = \begin{cases} pt & X = \phi \\ \text{iso. classes of } (Q,u) & X \neq \phi \end{cases}$$

except that we want to allow Q to vary its rank if X isn't connected. Thus we set

$$\text{Hom}_{\mathcal{E}_X}(E,E') = \varinjlim_{X = \coprod U_i} \prod_i I(E|U_i, E'|U_i)$$

so that $\text{Hom}_{\mathcal{E}_X}(E,E') = I(E,E')$ if X is connected (non-empty).

It is clear how to compose

$$E \xrightarrow{(Q,u)} E' \xrightarrow{(Q',u')} E''$$

namely, the composition is the isomorphism class represented by the object $Q'' = Q \oplus Q'$ and the isom.

$$E \oplus \pi^* Q'' \overset{\text{can.}}{\cong} (E \oplus \pi^* Q) \oplus \pi^* Q' \xrightarrow{u + id} E' \oplus \pi^* Q' \xrightarrow{u'} E.$$

This composition on I will induce a composition on $\mathrm{Hom}_{\mathcal{E}_X}(E, E')$, hence we have defined a category $\mathcal{E}_X$. Summary: If X connected ($\neq \phi$)

$$\mathrm{Ob}\, \mathcal{E}_X = R\text{-vector bundles}/X$$

$$\mathrm{Hom}_{\mathcal{E}_X}(E, E') = \text{iso classes of } (Q, u), \text{ where } Q \in \mathrm{Ob}\, \mathcal{P}_R \text{ and } u : E \oplus \pi^* Q \xrightarrow{\sim} E'.$$

(Alternative description: Suppose E is a bundle which is spanned by its global sections. If X is connected this means that E is trivial. In fact X connected ($\neq \phi$) and E arbitrary $\Rightarrow$ $\pi^* \Gamma(X, E) \hookrightarrow E$, hence if onto, $\pi^* \Gamma(X, E) \xrightarrow{\sim} E$. Thus in general one can say that $\mathrm{Hom}_{\mathcal{E}_X}(E, E')$ is the set of pairs (F, u) where $u : E \to E'$ is a vector bundle injection and where F is a subbundle of E' such that $uE \oplus F = E'$, and such that F is generated by its global sections. Unfortunately this description doesn't work for ordinary vector bundles.)

Suppose now we form the fibred category $\mathcal{E}$ over the category of spaces with $\mathcal{E}_X$ as the fibre over X. Thus the objects are pairs (X, E) with $E \in \mathrm{Ob}\, \mathcal{E}_X$ and a morphism $(X, E) \longrightarrow (Y, E')$ is a pair (f, u) where $f : X \to Y$ is a map and $u : E \longrightarrow f^* E'$ is a map in $\mathcal{E}_X$. Then we have a functor $\mathcal{E} \longrightarrow (\text{spaces})$ $(X, E) \longmapsto X$ whose limit topos is hopefully the right thing.

To prove this let S denote the [illegible] ind-object $\{\mathbf{R}^n, n \geq 0\}$ where $\mathbf{R}^n$ is viewed as a vector bundle over a point and $R^n \to R^{n+1}$ is the standard embedding with last coordinate 0, hence the complementary summand is [illegible] $R \cdot e_{n+1}$. Let M be the monoid of endomorphisms of S. What we want to show is that the following simplicial object of $\mathcal{E}^\sim$

$$\tilde{M} \times \tilde{M} \times \tilde{S} \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \tilde{M} \times \tilde{S} \;\substack{\longrightarrow\\\longrightarrow}\; \tilde{S}$$

is suitable for computing cohomology on the topos $\mathcal{E}^\sim$. Thus the problem is to show that the simplicial abelian sheaf

$$\mathbb{Z}[\tilde{M} \times \tilde{M} \times \tilde{S}] \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathbb{Z}[\tilde{M} \times \tilde{S}] \;\substack{\longrightarrow\\\longrightarrow}\; \mathbb{Z}[\tilde{S}]$$

is a resolution of $\mathbb{Z}$. ~~We have to prove exactness over~~

Meaning: S is an ind-object in $\mathcal{E}_{pt}$, hence an ind-object in $\mathcal{E}$, and $\tilde{S}$ is the inductive limit sheaf; $\tilde{M}$ is the constant sheaf associated to the [illegible] monoid M. Given $E \to X$ we want to understand the sheaf $\tilde{S}_{(X,E)}$ over X; it is the inductive limit of the sheaves ~~$R^n_{(X,E)}$. The~~

~~sheaf $R^n_{(X,E)}$ assigns to an open set U the maps~~

~~$E|U \to R^n_U$?~~

~~There is a problem here in that the representable functors on $\mathcal{E}$ are not sheaves because morphisms don't glue~~

~~$S_{(X,E)}$ where $R^n \to S$~~

$\widetilde{R}^n_{(X,E)}$: According to SGAA the sheaf associated to a system $F_{(X,E)}$ of presheaves is the system $\widetilde{F_{(X,E)}}$. Thus we want to compute the sheaf on X associated to the presheaf

$$U \longmapsto \mathrm{Hom}_{\mathcal{E}_U}\left(E|U, \pi_u^* P\right) \qquad \pi_U : U \to pt$$

where $P = R^n$ for some n. If U is a small connected nbd. of x, we have a canonical isomorphism

$$E|U \xrightarrow{\sim} \pi^* E_x$$

hence it is easy to see that

$$\mathrm{Hom}_{\mathcal{E}_U}\left(E|U, \pi_U^* P\right) = \mathrm{Hom}_{\mathcal{E}_{pt}}\left(E_x, P\right).$$

But $\mathcal{E}_{pt} = \mathcal{C}$, and we have established the acyclicity of the category with objects $\mathrm{Hom}(E_x, S)$ and morphisms coming from M. Thus the simplicial object on the preceding page is acyclic and can be used for computing cohomology. So let's compute now the cohomology of the constant sheaf $K = \mathbb{Z}_p$. We get a spectral sequence

$$E_2^{pq} = \check{H}^p\left(\nu \longmapsto H^q(\widetilde{M}^\nu \times \widetilde{S}; K)\right).$$

Now $F \mapsto H^*(\widetilde{M}^\nu \times \widetilde{S}, F)$ is the derived functors of

$$H^0(\widetilde{M}^\nu \times \widetilde{S}, F) = \prod_{M^\nu} \varprojlim_n F(R^n)$$

hence
$$E_2^{pq} = H^p(M, H^q(\tilde{S}; K))$$
where
$$H^q(\tilde{S}, F) = \begin{cases} \varprojlim F(R^n) & q=0 \\ R^1\varprojlim F(R^n) & q=1 \\ 0 & q>1 \end{cases}$$

and since $n \longrightarrow F(R^n)$ is constant for the constant sheaf
$$E_2^{pq} = \begin{cases} H^p(M, K) & q=0 \\ 0 & q>0 \end{cases}$$

showing that the topos $\tilde{\mathcal{E}}$ is ~~[illegible]~~ acyclic. ?

Surprising as this seems at first it seems correct: Let $\mathcal{E}_0 \subset \mathcal{E}$ be the full subcategory consisting of pairs (X, E) with X 1-connected. Then $\tilde{\mathcal{E}}_0 \cong \tilde{\mathcal{E}}$ yet for X 1-connected
$$\mathcal{E}^0_X \xleftarrow{\ \pi^*\ } \mathcal{E}_{pt}$$
is an equivalence of categories. Thus $\mathcal{E}$ and $\mathcal{E}_{pt}$ are equivalent as far as cohomology goes, and we already know that $\mathcal{E}_{pt}$ is acyclic.

May 12, 1971: More K-theory.

Let $\mathcal{B}$ be a category. Recall that a pseudo-functor from $\mathcal{B}$ to Cat is a gadget which assigns to each B in $\mathcal{B}$ a category $\mathcal{E}_B$ and to each arrow $u: B \longrightarrow B'$ in $\mathcal{B}$ a functor

$$u_* : \mathcal{E}_B \longrightarrow \mathcal{E}_{B'}$$

and to each pair of composable arrows

$$B \xrightarrow{u} B' \xrightarrow{v} B''$$

an isomorphism of functors

$$c_{vu} : v_* u_* \xrightarrow{\sim} (vu)_*$$

such that certain compatibility conditions are satisfied.

Suppose now that $\mathcal{B}$ is a 2-category. Then by a pseudo-functor from $\mathcal{B}$ to Cat I means a gadget assigning to each object B a category $\mathcal{E}_B$ and to each pair of objects a functor

$$\underline{\mathrm{Hom}}_{\mathcal{B}}(B, B') \xrightarrow{u \mapsto u_*} \underline{\mathrm{Hom}}_{\mathrm{Cat}}(\mathcal{E}_B, \mathcal{E}_{B'})$$

and to each

$$B \xrightarrow{u} B' \xrightarrow{v} B''$$

an isomorphism of functors

$$v_* u_* \longrightarrow (vu)_*$$

satisfying usual cocycle condition. Observe this is the same as the above except that a **2**-arrow $\alpha : u \Longrightarrow v$ is made to act as a morphism of functors $u_* \longrightarrow v_*$.

The next stage is to go to a pseudo-2-category, i.e. composition is associative up to canonical isomorphism. For simplicity we suppose $\mathcal{B}$ has only one object; it is thus up to a shift of notation a category $\mathcal{A}$ with a coherent associative operation $\oplus$. Then a pseudo-functor is a category $\mathcal{E}$ together with a functor

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \underline{Hom}_{Cat}(\mathcal{E},\mathcal{E}) \\ V & \longmapsto & V_* \end{array}$$

and for each pair V, V' of objects of $\mathcal{A}$ (these correspond to one-arrows of $\mathcal{B}$) an isomorphism

$$c_{V',V} : V'_* \circ V_* \xrightarrow{\ \sim\ } (V' \oplus V)_*$$

satisfying the compatibility condition

$$\begin{array}{ccc} V''_* \circ V'_* \circ V_* & \longrightarrow & (V'' \oplus V')_* \circ V_* \\ & & \downarrow \\ & & ((V'' \oplus V') \oplus V)_* \\ \downarrow & & \wr\wr \\ V''_* \circ (V' \oplus V)_* & \longrightarrow & (V'' \oplus (V' \oplus V))_* \end{array}$$

where the isomorphism at the lower right is the natural transf. of functors induced by the associativity data in $\mathcal{A}$.

Example: Let $\mathcal{A}$ be the category of f.g. proj. R modules with R a commutative ring. Let $\mathcal{E}$ be the category of invertible R-modules. Morphisms in both categories are isomorphisms. Given V in $\mathcal{A}$, let $\lambda(V)$ be its highest exterior power, so $\lambda(V)$ is an object of $\mathcal{E}$. Set

$$V_*(L) = \lambda(V) \otimes L$$

(an explicit $\otimes$ is supposed chosen in $\mathcal{E}$). Then from canonical isomorphism

$$\lambda(V' \oplus V) \simeq \lambda(V') \otimes \lambda(V)$$

one obtains the isomorphism $c_{V',V}$.

May 13, 1971: K-theory.

Let M be a topological monoid and consider the topological category whose objects space is $M \times M$ with M-action $m.(m_1, m_2) = (mm_1, mm_2)$. What is the fundamental groupoid of the classifying space of this topological category?

$$\begin{array}{ccc} \xrightarrow{\quad}\!\!\!\!\!\xrightarrow{\quad}\!\!\!\!\!\xrightarrow{\quad} \; M \times M \times M & \rightrightarrows & M \times M \\ (m, (m_1, m_2)) & \longmapsto & \begin{pmatrix} m_1, m_2 \\ mm_1, mm_2 \end{pmatrix} \end{array}$$

The π_0 is easy to determine; it is the cokernel of

$$\pi_0 M \times \pi_0 M \times \pi_0 M \rightrightarrows \pi_0 M \times \pi_0 M$$

and hence if $\pi_0 M$ is commutative, it will be the group associated to $\pi_0 M$.

A locally constant sheaf on $B(M \times M, M)$ will consist of a locally constant sheaf F on $M \times M$ endowed with descent for the action of M. Thus we have an action of M on F covering that on $M \times M$: such that

$$\begin{array}{ccc} F_{(x,y)} & \xrightarrow{\sim} & F_{(mx,my)} \\ z & \longmapsto & mz \end{array}$$

for each $x, y \in M$. In the example relevant to K-theory, M will be of the form $\coprod_P B\,\mathrm{Aut}(P)$, where P ranges over representatives for the isomorphism classes. Hence points of M will be the objects P and $F_{P,Q}$ will be an $\mathrm{Aut}\, P \times \mathrm{Aut}\, Q$ sets. Thus given any other object R

we will have isomorphisms

$$F_{P,Q} \xrightarrow{\sim} F_{P\oplus R,\, Q\oplus R}.$$

Thus it seems what I must consider are functors $F: \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{S}ets$ provided with isos.

$$\theta_{P,Q,R} : F(P,Q) \xrightarrow{\sim} F(P\oplus R,\, Q\oplus R)$$

satisfying the usual compatibility conditions. In particular, considering the restriction to pairs with $P=Q$, we have a functor $P \mapsto \bar{F}(P) = \bar{F}(P,P)$ together with maps

$$\bar{\theta}_{P,R} : \bar{F}(P) \xrightarrow{\sim} \bar{F}(P\oplus R)$$

compatible with automorphisms of P. We have already seen (sauf erreur) that then $\mathrm{Aut}(P)$ acts on $\bar{F}(P)$ through $K_1(\mathcal{A})$.

May 14, 1971: K-theory as a cobordism theory

Here's a new way of defining K_0. To fix the ~~[illegible]~~ ideas, let $\mathcal{A}$ be a small abelian category. Consider gadgets which assign to each object E of A an object $h(E)$ of a category $\mathcal{C}$ and for each arrow $f: E' \to E$ in $\mathcal{A}$ two morphisms

$$h(E') \underset{f_*}{\overset{f^*}{\rightleftarrows}} h(E)$$

such that the following conditions are satisfied:

i) functorality: $E \to h(E)$, $f \mapsto f_*$ is a covariant functor from $\mathcal{A}$ to $\mathcal{C}$ (resp. $E \to h(E)$, $f \mapsto f^*$ is a contravariant functor.

ii) f^*, f_* are isomorphisms

iii) If

$$\begin{array}{ccc} E_0' & \xrightarrow{g'} & E' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ E_0 & \xrightarrow[g]{} & E \end{array}$$

is bicartesian, then

$$g^* f_* = f'_* g'^* .$$

Let $i_E : 0 \to E$ and $\pi_E : E \to 0$ be the canonical maps. Let

$$e(E) = i_E^* . i_{E*} \in \mathrm{Aut}(h(0)) .$$

More generally given $f : E' \longrightarrow E$ set

$$e(f) = i^*_{E'} f^* f_* \pi^*_{E'} \in \mathrm{Aut}(h(0)).$$

so that $e(E) = e(i_E)$. Then

$$e(f) = i^*_E f_* \pi^*_{E'}$$

or

$$\boxed{f_* = \pi^*_E \, e(f) \, \iota^*_{E'}} .$$

Consequently given $E'' \xrightarrow{g} E' \xrightarrow{f} E$ we have

$$e(f)\,e(g) = \iota^*_E f_* \cancel{\pi^*_{E'} \iota^*_{E'}} g_* \pi^*_{E''}$$

$$= \iota^*_E (fg)_* \pi^*_{E''} = e(fg)$$

hence taking $E'' = 0$, $g = \iota_{E'}$ we see that

$$e(f) = e(E)\,e(E')^{-1} \qquad \text{if } f : E' \to E.$$

One sees that for the universal gadget

$$h : \mathcal{A} \longrightarrow \mathcal{C}$$

we have

$$\mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{A}, \qquad h = \mathrm{id} \text{ on objects}$$

$$\mathrm{Hom}_{\mathcal{C}}(h(E'), h(E)) \xrightarrow{\sim} K_0(\mathcal{A})$$

$$f \longmapsto e(f).$$

More slowly, from $e(fg) = e(f)e(g)$ one sees that

$$E \longmapsto e(E)$$

yields a homomorphism

$$K_0 \mathcal{A} \longrightarrow \operatorname{Aut} h(0)$$
$$\alpha \longmapsto e(\alpha)$$

using the universal property of $K_0 \mathcal{A}$. It's pretty clear that for any gadget there is a map

$$K_0 \mathcal{A} \longrightarrow \operatorname{Hom}_{\mathcal{C}}(h(E'), h(E))$$
$$\alpha \longmapsto \pi_E^{*} \cdot e(\alpha) \cdot \iota_{E'}^{*}$$

and in the universal case this is an isomorphism.

I want to carefully work out at least conjecturally a satisfying version of algebraic K-theory.

1). (From topology) Let S be a space and let $G \to S$ be a fibre bundle whose fibres G_s are topological groups. Let $BG \to S$ be its classifying space; it is the fibre bundle whose fibre over s is the classifying space BG_s constructed in one of the standard ways (Milnor, Segal). If Γ is a discrete group provided with a homomorphism

$$S \times \Gamma \to G$$

over S, then there is an induced map of classifying spaces relative to S

$$S \times B\Gamma \to BG,$$

hence a canonical map

$$B\Gamma \to \mathrm{Sect}(BG \to S) \quad (\text{another notation} = \Gamma(S,BG))$$

of $B\Gamma$ into the spce of sections of BG over S. One can ask whether there are reasonable criteria in order that the above map be a homotopy equivalence in the universal case $\Gamma = \Gamma(S,G)$. Thus we have

<u>Problem</u>: When is the canonical map

$$\boxed{B\Gamma(S,G) \longrightarrow \Gamma(S,BG)}$$

a homotopy equivalence (resp. a mod ℓ cohomology iso., etc.)?

2) Examples. Suppose $G = S \times K$ where K is a topological group. Then $\Gamma(S,G) = \mathrm{Map}(S,K)$ and $\Gamma(S,BG) \neq \mathrm{Map}(S,BG)$. Note that $\mathrm{Map}(S,K)$ comes with a natural topology such that the evaluation map

$$S \times \mathrm{Map}(S,K) \to K$$

is continuous. Thus the canonical map factors

$$S \times \mathrm{Map}(S,K)_d \to S \times \mathrm{Map}(S,K)_t \to S \times K . \qquad \begin{pmatrix} d = \text{discrete} \\ t = \text{topologized} \end{pmatrix}$$

Now it seems clear that $\mathrm{Map}(S,BK)$ is the classifying space of $\mathrm{Map}(S,K)_t$, consequently the Problem amounts to whether or not the map

$$B\mathrm{Map}(S,K)_d \to B\mathrm{Map}(S,K)_t$$

is a homotopy equivalence (resp. mod ℓ homology iso., etc.).

3) Consider this example semi-simplicially. Thus I am given a simplicial set S and G = S x K where K is a simplicial group over S. Then BG = S x BK where B means say the Eilenberg-MacLane $\bar{W}$. Now Map(S,K) is the zero-th part of the simplicial group Map(S,K), and it should be clear that there is a homotopy equivalence

$$B\underline{Map}(S,K) \quad \underline{Map}(S,BK).$$

Actually this is not quite true because the latter space need not be connected, in fact its components are $H^1(S,K)$. So the statement must be amended to read that the former is homotopy equivalent to the connected component of the basepoint in the latter. The way this is proved is to start with the principal fibration

$$K \qquad PK \qquad BK$$

~~and then~~ which then furnished a fibration

$$\underline{Map}(S,PK) \text{ ---- } \underline{Map}(S,BK)^{(0)} \qquad \text{((0) means conn. comp.)}$$

which is principal with group Map(S,K) and has contractible total space. (The topological argument is the same.)

4) The above two examples show perhaps exactly what must be ~~proved~~ done to get the cohomology of BMap(S,K) related to the cohomology of Map(S,BK), namely one must produce an acyclic principal Map(S,K) bundle, ~~with its~~ whose base has the ~~homotopy~~ homology of Map(S,BK).

5) Why does the simplicial example work? Thus somehow we work in the topos T of simplicial sets and have an object S and the group S x K over S, and we find that

$$B\underline{Map}(S,K) \text{ ---- } \underline{Map}(S,BK)^{(0)}$$

is a homotopy equivalence. (Meaning: Such things as the sheaf of maps makes sense for a general topos, so the left side is defined. Meaning of the right-sdie?) (Meaning: Map(S,K) is a group in the topos, hence its Bg is defined as g-objects. ~~It is a category object in the topos~~. BK is a topos over T. We are CONFUSED) attempt at exact seq. K theory

May 16, 1971

Let $\mathcal{A}$ be an abelian category, let $\mathcal{E}$ be the category of short exact sequences in $\mathcal{A}$, and let

$$\mathcal{E} \underset{quot}{\overset{sub}{\underset{tot}{\rightrightarrows}}} \mathcal{A}$$

be the three functors defined by

$$\begin{aligned} sub(E) &= A' \\ tot(E) &= A \\ quot(E) &= A'' \end{aligned}$$

if E is the exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0.$$

Denote by $Isob(\mathcal{C})$ the set of iso. classes of objects in the category $\mathcal{C}$, and by

$$cl : Ob\ \mathcal{C} \longrightarrow Isob(\mathcal{C})$$

the canonical map.

Set $I = Isob(\mathcal{A})$, $J = Isob(\mathcal{E})$, and denote by

$$J \underset{q}{\overset{s}{\underset{t}{\rightrightarrows}}} I$$

the maps induced by sub, tot, and quot, respectively. Let $J \times_{(q,q)} J$ be the fibred product of the map $q : J \to I$ with itself. Set

$$\pi = \operatorname{Coker}\left\{ J \times_{(g,g)} J \overset{s\times s}{\underset{t\times t}{\rightrightarrows}} I \times I \right\}$$

and denote by

$$p : I \times I \longrightarrow \pi$$

the canonical map. Let

$$pcl : \mathrm{Ob}\,\mathcal{A} \times \mathrm{Ob}\,\mathcal{A} \longrightarrow \pi$$

denote the composition of

$$cl \times cl : \mathrm{Ob}\,\mathcal{A} \times \mathrm{Ob}\,\mathcal{A} \longrightarrow I \times I$$

followed by p. It is clear that pcl makes π the quotient of $\mathrm{Ob}\,\mathcal{A} \times \mathrm{Ob}\,\mathcal{A}$ by the relation: (A,B) is related to (A',B') if $\exists$ exact sequences

$$0 \longrightarrow A \longrightarrow A' \longrightarrow C \longrightarrow 0$$

$$(*) \qquad 0 \longrightarrow B \longrightarrow B' \longrightarrow C \longrightarrow 0 .$$

Given $D \in \mathrm{Ob}(\mathcal{A})$, consider the map

$$\mathrm{Ob}\,\mathcal{A} \times \mathrm{Ob}\,\mathcal{A} \longrightarrow \mathrm{Ob}\,\mathcal{A} \times \mathrm{Ob}\,\mathcal{A}$$

$$(A, B) \longmapsto (D \oplus A, B).$$

Then if (A,B) is related to (A',B') via the exact sequences $(*)$, $(D\oplus A, B)$ is related to $(D \oplus A', B)$ by the exact sequences

$$0 \longrightarrow D\oplus A \longrightarrow D \oplus A' \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow B \longrightarrow B' \longrightarrow C \longrightarrow 0.$$

Hence there is induced a map

$$e(D) : \pi \longrightarrow \pi$$

satisfying

$$e(D)\, pcl(A,B) = pcl(D \oplus A, B).$$

Similarly one defines an ~~operator~~ endomorphism

$$e'(D) : \pi \longrightarrow \pi$$

satisfying

$$e'(D)\, pcl(A,B) = pcl(A, D \oplus B).$$

Since

$$e'(D)\, e(D)\, pcl(A,B) = e'(D)\, pcl(D \oplus A, B)$$
$$= pcl(D \oplus A, D \oplus B)$$

and this equals $pcl(A,B)$ in virtue of the exact sequences

$$0 \longrightarrow A \longrightarrow D \oplus A \longrightarrow D \longrightarrow 0$$
$$0 \longrightarrow B \longrightarrow D \oplus B \longrightarrow D \longrightarrow 0$$

it follows that $e'(D)\, e(D) = id_\pi$. Similarly $e(D)\, e'(D) = id_\pi$, and hence $e(D)$ is an isomorphism with inverse $e'(D)$.

It is clear that $e(D)$ depends only on the isomorphism class of D. Suppose

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

is an exact sequence. Then for A, B in $Ob\, \mathcal{C}$ we have

$$e(D')\,pcl(A,B) = pcl(D'\oplus A, B)$$
$$= pcl(D\oplus A, D''\oplus B)$$

in virtue of the exact sequences

$$0 \longrightarrow D'\oplus A \longrightarrow D\oplus A \longrightarrow D'' \longrightarrow 0$$
$$0 \longrightarrow B \longrightarrow D''\oplus B \longrightarrow D'' \longrightarrow 0.$$

Thus

$$e(D')\,pcl(A,B) = e(D)\,pcl(A, D''\oplus B)$$
$$= e(D)e'(D'')\,pcl(A,B)$$

so using $e'(D'')$ is the inverse of $e(D'')$, we obtain

$$\boxed{e(D) = e(D')e(D'')}.$$

This implies clearly that the map $D \longmapsto e(D)$ from $Ob(\mathcal{A})$ to the group $Aut(\pi)$ factors through a homomorphism

$$K_0\mathcal{A} \longrightarrow Aut(\pi).$$

In other words there is a unique action of $K_0\mathcal{A}$ such that if $\gamma : Ob\mathcal{A} \longrightarrow K_0\mathcal{A}$ is the canonical map

$$\gamma(D).\,pcl(A,B) = pcl(D\oplus A, B).$$

Consider the map

$$* \qquad \pi \longrightarrow K_0 \mathcal{A}$$
$$(A, B) \longmapsto \gamma(A) - \gamma(B).$$

This is a map of $K_0\mathcal{A}$ - sets where $K_0\mathcal{A}$ acts on itself by translation. Since $K_0\mathcal{A}$ acts transitively on π because

$$pcl\,(A, B) = e(A)\, e(B)^{-1} pcl(0, 0)$$

it follows the map $*$ is an isomorphism.

May 17, 1971: Mather's theorem

To develop the sheaf-theoretic viewpoint:

Assume for the sake of discussion what is meant by a Γ-structure. Ultimately it will be necessary to understand fibre bundles, such as micro-bundles and Γ-structures, where the fibre is a germ of something.

Let Γ be a topological groupoid and consider the stack of its torsors over topological spaces. Notation: TOP = gross site of top. spaces and TORS(Γ) the stack of Γ-torsors. Then TORS(Γ) is also a site in a natural way and we can consider its sheaves. These are systems ~~[illegible]~~ which associate to each (X, P), where P is a Γ-torsor over X, a sheaf $F_{(X,P)}$ over X, and to each map $f:(X',P') \longrightarrow (X,P)$ a map

$$(*) \qquad f^*(F_{(X,P)}) \longrightarrow F_{(X',P')}$$

such that (i) the habitual transitivity conditions hold for composition of morphisms of torsors, and (ii) (*) is an isomorphism if $f: X' \longrightarrow X$ is an open immersion. Call such a system "special" if (*) is an isomorphism for all maps f.

Let $Ob\,\Gamma$ and $Ar\,\Gamma$ be the spaces of objects and arrows of Γ, resp. Then

$$\begin{matrix} Ar\,\Gamma \\ \downarrow\!\downarrow \\ Ob\,\Gamma \end{matrix} \qquad \longleftarrow$$

is a Γ-torsor over $Ob\,\Gamma$. [illegible] Thus given a system $\{F_{(x,p)}, \text{etc.}\}$ one gets a sheaf

$$F_{(Ob\,\Gamma,\ ar\,\Gamma)}$$

over $Ob\,\Gamma$. There are two maps of Γ-torsors

$$\begin{array}{ccc}
ar\,\Gamma \times_{(s,t)} ar\,\Gamma & \overset{pr_2}{\underset{comp}{\rightrightarrows}} & ar\,\Gamma \\
\downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle t} \\
ar\,\Gamma & \overset{d_0 = s}{\underset{d_1 = t}{\rightrightarrows}} & Ob\,\Gamma
\end{array}$$

(scheme: upper is d_0 and arrows run $\leftarrow\!\!-\!\!\leftarrow$.) [illegible] Suppose F is special. Then we have

$$\tilde d_0 : s^* F_{(ar\,\Gamma \xrightarrow{t} Ob\,\Gamma)} \xrightarrow{\sim} F_{(ar\,\Gamma \times_{(s,t)} ar\,\Gamma \xrightarrow{pr_1} ar\,\Gamma)}$$

$$\tilde d_1 : t^* F_{(\qquad)} \xrightarrow{\sim} F_{(\qquad)}$$

Now think of $F_{(ar\,\Gamma \xrightarrow{t} Ob\,\Gamma)}$ as an étale space $\tilde F \xrightarrow{f} Ob\,\Gamma$. Then we have an isomorphism

$$ar\,\Gamma \times_{(s,f)} \tilde F \simeq ar\,\Gamma \times_{(t,f)} \tilde F$$

~~[illegible]~~ which is given by a map

$$(*) \qquad ar\,\Gamma \times_{(s,f)} \tilde F \longrightarrow \tilde F .$$

We admit that further argument will show that the operation (*) makes $\tilde{F}$ into a "covariant functor". Thereby we have a functor from the category of "special" sheaves on the site TORS(Γ) to sheaves over $Ob\,\Gamma$ with left $Ar\,\Gamma$-action. It is fairly clear that this is an equivalence of categories.

The quasi-inverse functor associates to a sheaf F over $Ob\,\Gamma$ with left $Ar\,\Gamma$-action (call these Γ-sheaves) the system

$$(P \to X) \longmapsto P \times^{Ar\,\Gamma} F .$$

(This contracted product is defined by descent using that locally P is isomorphic to $X \times_{(\xi, t)} Ar\,\Gamma$ for some $\xi : X \to Ob\,\Gamma$, whence

$$P \times^{Ar\,\Gamma} F = X \times_{(\xi, f)} F ,$$

$f : F \to Ob(\Gamma)$ being the structural map.)

It seems that Giraud's criterion implies that the category of Γ-sheaves is a topos. Since this topos admits a faithful forgetful functor to the topos of sheaves on Γ, it should, by the nonsense of glueing topoi, be given by a left exact endomorphism φ of the sheaves on Γ, φ being provided with arrows

$$\varphi \to id \ , \quad \varphi \to \varphi^2 .$$

If t is etale, so that the canonical map

$$t^*s_* \longrightarrow pr_{2*}\,pr_1^*$$

is an isomorphism for the square

$$\begin{array}{ccc} Ar\Gamma \times_{(s,t)} Ar\Gamma & \xrightarrow{pr_1} & Ar\Gamma \\ \Big\downarrow pr_2 & & \Big\downarrow s \\ Ar\Gamma & \xrightarrow{\quad t \quad} & Ob\Gamma, \end{array}$$

then I think the functor φ in question is s_*t^*. Indeed we know that an action amounts to a map $s^*F \longrightarrow t^*F$, or equivalently a map $F \longrightarrow s_*t^*F$. On the other hand, we have natural transfs.

$$s_*t^* \longrightarrow s_*\iota_*\iota^*t^* = id$$

where $i : Ob\,\Gamma \longrightarrow Ar\,\Gamma$ is the identity section, and

$$\begin{aligned} s_*t^* \longrightarrow s_*\mu_*\mu^*t^* &= s_*\,pr_{2*}\,pr_1^*t^* \\ &\cong s_*t^*s_*t^* \end{aligned}$$

(μ = composition)

by the hypothesis made above.

Suppose now that $s, t : Ar\,\Gamma \longrightarrow Ob\,\Gamma$ are etale (pseudo-group situation). Then I want to show that special sheaves can be used to compute cohomology in the following sense. Let

$$\widetilde{TORS(\Gamma)} \xleftarrow{\;f^*\;} (\Gamma\text{-sheaves})$$

be the functor

$$(f^*F)_{P\to X} = P\times^{\mathrm{Ar}\,\Gamma} F.$$

(it is the inverse image map for a morphism of topoi.) Then f^* is acyclic:

$$H^*(\widetilde{\mathrm{TORS}(\Gamma)}; f^*F) \xleftarrow{\sim} H^*(\Gamma; F) : f^*$$

the latter group denoting cohomology for the topos of Γ-sheaves. To prove this we must demonstrate the effaceability of the former functor, as the isomorphism in dimension zero is pretty clear.

The first remark is that the torsor $(\mathrm{Ar}\,\Gamma \xrightarrow{t} \mathrm{Ob}\,\Gamma)$ covers the final object of $\widetilde{\mathrm{TORS}(\Gamma)}$. Now to compute Čech simplicial object. The product

$$(\mathrm{Ar}\,\Gamma \xrightarrow{t} \mathrm{Ob}\,\Gamma)\times(\mathrm{Ar}\,\Gamma \xrightarrow{t} \mathrm{Ob}\,\Gamma)$$

has total space

$$(\mathrm{Ar}\,\Gamma)\times_{(s,s)}(\mathrm{Ar}\,\Gamma) \xrightarrow{\sim} \mathrm{Ar}\,\Gamma\times_{(s,t)}\mathrm{Ar}\,\Gamma$$

$$(u\ ,\ v) \longmapsto (uv^{-1}, v)$$

and base $\mathrm{Ar}\,\Gamma$ via $(u,v)\longmapsto uv^{-1}$. Thus the product is the torsor

$$pr_1 : \mathrm{Ar}\,\Gamma\times_{(s,t)}\mathrm{Ar}\,\Gamma \longrightarrow \mathrm{Ar}\,\Gamma.$$

It is clear then that the Čech object is

$$
\begin{array}{ccccc}
Ar\Gamma \times_{(s,t)} Ar\Gamma \times_{(s,t)} Ar\Gamma & \overset{pr_{23}}{\underset{id\times\mu}{\overset{\mu\times id}{\rightrightarrows}}} & Ar\Gamma \times_{(s,t)} Ar\Gamma & \overset{pr_2}{\underset{\mu}{\rightrightarrows}} & Ar\Gamma \\
\downarrow pr_{12} & & \downarrow pr_1 & & \downarrow t \\
Ar\Gamma \times_{(s,t)} Ar\Gamma & \overset{pr_2}{\underset{pr_1}{\overset{\mu}{\rightrightarrows}}} & Ar\Gamma & \overset{s}{\underset{t}{\rightrightarrows}} & Ob\,\Gamma .
\end{array}
$$

(Actually it is probably not necessary to know these formulas; what one needs is the fact that all maps are etale.) Denote by $(P_\cdot, X_\cdot)$ this simplicial object of TORS(Γ). Then there is a Cech spectral sequence

$$E_2^{pq} = \check{H}^p\big(\nu \longmapsto H^q(\widetilde{TORS(\Gamma)}/(P_\nu, X_\nu); f^*F)\big)$$

$$\Longrightarrow H^{p+q}(\widetilde{TORS(\Gamma)}; f^*F).$$

But the torsor P_ν is trivial, i.e. of the form

$$P_\nu = X_\nu \times_{Ob\,\Gamma} Ar\Gamma$$

for a map $X_\nu \to Ob\,\Gamma$. Hence

$$\widetilde{TORS(\Gamma)}/(P_\nu, X_\nu) \cong \widetilde{TORS(\Gamma)}/(P_\nu, X_\nu)$$

$$\cong TOP(X_\nu),$$

the point being that that the category $TORS(\Gamma)/(P_\nu, X_\nu)$ is equivalent to spaces

Now $TORS(\Gamma)/(X, P)$ is equivalent as a site to the gross site of spaces over X, hence

$$H^*(\widetilde{\mathrm{TORS}(\Gamma)}/(X,P),\ f^*F) = H^*(\mathrm{TOP}(X);\ P\times^{\Gamma}F)$$

where in the latter the sheaf $P\times^{\Gamma}F$ on X is considered as a special sheaf on the gross site. But one knows (say by the Verdier hypercovering thm.) that

$$H^*(\mathrm{TOP}(X);\ P\times^{\Gamma}F) = H^*(X,\ P\times^{\Gamma}F).$$

~~Now the torsor (P_ν, X_ν) is trivial, i.e. of the form~~

~~$$P_\nu = X_\nu \times_{\mathrm{Ob}\,\Gamma} \mathrm{Ar}\,\Gamma$$~~

~~for some map $X_\nu \longrightarrow \mathrm{Ob}\,\Gamma$, hence~~

~~$$P_\nu\times^{\Gamma}F = X_\nu \times_{\mathrm{Ob}\,\Gamma} F.$$~~

~~Now consider~~ Now the torsor (P_ν, X_ν) is trivial, i.e. of the form

$$P_\nu = X_\nu \times_{\mathrm{Ob}\,\Gamma} \mathrm{Ar}\,\Gamma$$

for some map $X_\nu \longrightarrow \mathrm{Ob}\,\Gamma$, hence

$$P_\nu\times^{\Gamma}F = X_\nu \times_{\mathrm{Ob}\,\Gamma} F.$$

~~Since $f: X_\nu \longrightarrow \mathrm{Ob}\,\Gamma$ is etale, pull back by it carries injectives into injectives, the point being that there is an extension by zero functor $f_!$ which is exact (In general if $f: X \to Y$ is etale then~~

In general if $f: X \to Y$ is etale one knows that $f^*: Sh_Y \to Sh_X$ carries injectives to injectives (SGAA). Therefore we have only to prove that if F is an injective Γ-sheaf, then F is injective as a sheaf on $Ob\,\Gamma$.

$$(\text{Sheaves on } Ob(\Gamma)) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} (\Gamma\text{-sheaves})$$

β forgets action

$$\alpha(F) = t_!\left(Ar\,\Gamma \underset{s \to Ob\,\Gamma}{\times} F\right)$$

But we have an equivalence of categories

$$(\text{Sheaves on } Ob\,\Gamma) \simeq (\Gamma\text{-sheaves over } Ar\,\Gamma_s)$$

$$F \xrightarrow{u} Ob\,\Gamma \longmapsto Ar\,\Gamma \times_{(s,u)} F$$

$$Ob\,\Gamma \underset{\xrightarrow{i} Ar\,\Gamma}{\times} G \longleftarrow G \to Ar\,\Gamma$$

and on the other hand we have already noted that based change $\mathcal{T} \to \mathcal{T}/X$ to an induced topos preserves injectives. As the composite

$$F \longmapsto Ar\,\Gamma \times_{(s,\ )} F \longmapsto Ob\,\Gamma \times_{Ar\,\Gamma} (Ar\,\Gamma \times_{(s,\ )} F)$$

just forgets the Γ-action, we are finished.

Therefore when $s,t: Ar\,\Gamma \to Ob\,\Gamma$ are etale, Γ sheaves over $Ob\,\Gamma$ are enough to compute cohomology.

May 21, 1971: Mather's theorem

Let T be a topos and let C be a category object in C. Thus C consists of two objects of T, $\mathrm{Ob}\,C$ and $\mathrm{Ar}\,C$, together with maps

$$\mathrm{Ob}\,C \xrightarrow{i} \mathrm{Ar}\,C \overset{s}{\underset{t}{\rightrightarrows}} \mathrm{Ob}\,C$$

$$m: \mathrm{Ar}\,C \times_{(s,t)} \mathrm{Ar}\,C \longrightarrow \mathrm{Ar}\,C$$

satisfying conditions guaranteeing that for each U in T $(\mathrm{Ob}\,C)(U)$ is the objects of a category $C(U)$ with arrows $(\mathrm{Ar}\,C)(U)$ such that i, s, t, m give respectively the identity, source, target, and composition for the category. Let

$$\mathrm{Ar}_\nu C = \mathrm{Ar}\,C \times_{(s,t)} \cdots \times_{(s,t)} \mathrm{Ar}\,C \qquad (\nu\text{-times})$$

whence $\nu \longmapsto \mathrm{Ar}_\nu C$ is a simplicial object in T, denoted $\mathrm{Nerv}(T)$ and called the nerve of T.

By an object of C with (left) C-action (C-object for short) we mean an object $u: F \longrightarrow \mathrm{Ob}\,C$ over $T/\mathrm{Ob}\,C$ together with a map

$$n: \mathrm{Ar}\,C \times_{(s,u)} F \longrightarrow F$$

satisfying the conditions guaranteeing that for each U in T,

$$(\mathrm{Ob}\,C)(U) \ni \xi \longmapsto \{\eta \in F(U) \mid u(\eta) = \xi\}$$

is a covariant functor from $C(U)$ to Sets. One defines a right C-action similarly. Objects with right C-action are the same as C^{o}-objects. The category of C-objects will be denoted T_C.

Composing the functor $\nu \mapsto Ar_\nu C$ from Δ° to T with the functor from Δ° to itself associating to a linearly ordered set L the set $L \amalg \{a\}$ where a is greater than all elements of L, we obtain a simplicial object $\nu \mapsto Ar_{\nu+1} C$ in T_C

$$\begin{array}{c}\xrightarrow{pr_{23}} \\ \xrightarrow{\mu\times id} \\ \xrightarrow{id \times \mu}\end{array} Ar_2 C \begin{array}{c}\xrightarrow{pr_2} \\ \xrightarrow{\mu}\end{array} Ar\, C$$

which is a resolution of the final object $(Ob\, C,\ id: Ob\, C \to Ob\, C,\ Ar\, C \times_{(s, id)} Ob\, C \xrightarrow{pr_1} Ar\, C \xrightarrow{t} Ob\, C)$ of T_C. More precisely

$$\begin{array}{c}\longrightarrow \\ \longrightarrow \\ \longrightarrow\end{array} \mathbb{Z}[Ar_2 C] \begin{array}{c}\longrightarrow \\ \longrightarrow\end{array} \mathbb{Z}[Ar\, C] \longrightarrow \mathbb{Z} \longrightarrow 0$$

is an exact sequence of abelian objects of T_C. (The exactness is proved by exhibiting a contracting homotopy.)

Since this is a resolution one has a spectral sequence of cohomology

$$E_1^{pq} = \mathrm{Ext}^q_{T_{C,ab}} (\mathbb{Z}[Ar_{p+1} C], F) \Rightarrow H^{p+q}(T_C; F)$$

$$\| $$

$$H^q(T_C / Ar_{p+1} C ; F).$$

~~[illegible]~~

<u>Proposition:</u> Let F be an abelian object of T_C ~~[illegible]~~ whose image in $T/\mathrm{Ob}\,C$ under the functor forgetting the C-action is cohomologically trivial, i.e. $H^q((T/\mathrm{Ob}C)/X, F) = 0$ for all X in $T/\mathrm{Ob}\,C$. (this will be the case if F is an injective object of $(T/\mathrm{Ob}\,C)_{ab}$.) Then there are canonical isomorphisms

$$H^p(T_C; F) = \check{H}^p(\nu \mapsto H^0(T_C/\mathrm{ar}_{\nu+1}C; F)).$$

<u>Proof:</u>

~~Take A in T_{ab}, and let $F = \mathrm{Ob}\,C \times A$ be endowed with the trivial C-action. Then~~

Since

$$\mathrm{ar}_{\nu+1}C = \mathrm{ar}\,C \times_{(s,\, v_\nu)} \mathrm{ar}_\nu C,$$

where $v_\nu : \mathrm{ar}_\nu C \longrightarrow \mathrm{Ob}\,C$ is the initial vertex map, and since there is an equivalence

$$T_C/\mathrm{ar}_{\nu+1}C \simeq T/\mathrm{ar}_\nu C$$

one knows that there are canonical isoms.

$$H^q(T_C/\mathrm{ar}_{\nu+1}C ; F) \simeq H^q(T/\mathrm{ar}_\nu C ; v_\nu^* F).$$

Now if F is cohomologically trivial as an abelian object of $T/\mathrm{Ob}\,C$, then these groups vanish for $q > 0$, hence the spectral sequence at the bottom of page 2 degenerates yielding the proposition. ~~[illegible]~~ The assertion is parentheses follows because pull-back

$T \longrightarrow T/X$ preserves injectives (proof in SGAA).

Now take A to be an abelian group in T and let $F = \mathrm{Ob}\, C \times A$ be endowed with the "trivial" C-action. Then

$$\mathrm{Hom}_{T_{C,ab}}\left(\mathbb{Z}[\mathrm{ar}_{\nu+1} C]; F\right) = \mathrm{Hom}_{T_{ab}}\left(\mathbb{Z}[\mathrm{ar}_\nu C]; A\right)$$

and so if A is injective, then the proposition implies that the cohomology $H^p(T_C; F)$ is canonically isomorphic to the cohomology of the cosimplicial abelian group obtained by applying the functor $\mathrm{Hom}_{T_{ab}}(?, A)$ to the simplicial abelian group

$$\mathbb{Z}[\mathrm{Nerv}(C)] : \qquad \mathbb{Z}[\mathrm{ar}_2 C] \rightrightarrows \mathbb{Z}[\mathrm{ar}_1 C] \rightrightarrows \mathbb{Z}[\mathrm{Ob}\, C].$$

Cor 1: A injective in T_{ab}, then

$$H^p(T_C; A) = \check{H}^p\left(\nu \longmapsto H^0(\mathrm{ar}_\nu C; A)\right).$$

Cor 2: Assume $\mathrm{Nerv}\, C$ is acyclic (this will be the case if T has enough points and each C_ξ is acyclic.) Then for any injective A in T_{ab},

$$H^p(T_C, A) = \begin{cases} A & p = 0 \\ 0 & p > 0. \end{cases}$$

In general one should define category objects C in a category $\mathcal{C}$, and C-objects in $\mathcal{C}$, and one should describe the associated semi-simplicial objects. If $\mathcal{C}$ is a topos T, then C-objects form a topos T_C. One has the standard resolution

$$\cdots\cdots \; ArC \times_{ObC} ArC \underset{pr_1}{\overset{pr_2}{\rightrightarrows}} ArC \xrightarrow{s} ObC$$

which leads to a spectral sequence

$$E_1^{pq} = H^q(ar_{p+1}C, T_C; F) \Rightarrow H^{p+q}(T_C; F).$$

Identification of E_1-term:

$$E_1^{pq} \simeq H^q(ar_p C, T; v_p^* F)$$

where $v_p : ar_p C \longrightarrow ObC$ is $t \circ pr_1$ the first vertex map. (Perhaps a better way to put this is to observe that $p \mapsto v_p^* F$ is a Deligne-style sheaf on $Nerv(C)$ such that given $\varphi: p \to q$ in Δ, $\varphi_*: \tilde{\varphi}^* F_p \longrightarrow F_q$ is an isomorphism provided $\varphi(0) = 0$.)

If $F = \pi^* A$ where $\pi: T_C \longrightarrow T$ is the canonical morphism of topoi, then

$$E_1^{pq} = H^q(ar_p C, T; A)$$

When A is injective, $E_1^{p+} = 0$, hence

$$H^p(T_C; \pi^* A) = \check{H}^p(\nu \longmapsto H^0(ar_\nu C, A)).$$

The Mather situation.

Given $n \in \mathbb{N}$ we define a topological groupoid $\mathcal{C}_n$ as follows. $\mathrm{Ob}\,\mathcal{C}_n = \mathbb{R}_{<0}$; a morphism from x to x' in $\mathcal{C}_n$ is defined to be a germ of diffeo.

$$h : [x, n] \longrightarrow [x', n]$$

such that $h(z) = z$ for z near $0, 1, \dots, n$. The set $\mathrm{Ar}\,\mathcal{C}_n$ of all morphisms in $\mathcal{C}_n$ is endowed with the topology making the source map $s : \mathrm{Ar}\,\mathcal{C}_n \longrightarrow \mathbb{R}_{<0}$ etale. More precisely, if θ is a diffeo of an open interval (a,b) containing $[0,n]$ to another such interval such that $\theta(z) = z$ for z near $0, 1, \dots, n$, then θ gives rise to a section $\tilde{\theta}$ of s over (a,b).

~~If two such sections $\tilde{\theta}_1, \tilde{\theta}_2$ coincide at some point, they coincide in a nbd. One knows this means that [illegible]~~

It is clear that every h in $\mathrm{Ar}\,\mathcal{C}_n$ is in the image of $\tilde{\theta}$ for some θ, and that if two sections $\tilde{\theta}_1, \tilde{\theta}_2$ which coincide at some point they coincide in a nbd. One knows then that s is an etale map when $\mathrm{Ar}\,\mathcal{C}_n$ is endowed with the topology having for basis sets of the form $\tilde{\theta}(U)$, where U is an open subset of $\mathbb{R}_{<0}$ and $\tilde{\theta}$ is a section of s over U constructed in the above way.

If X is a space, then $\mathcal{C}_n(X)$ is the category whose objects are continuous functions $f : X \to \mathbb{R}_{<0}$, and in which a morphism from f to f' is a continuous family of germs of diffeos. $h(x) : [f(x), n] \to [f'(x), n]$

such that $h(x)$ is the identity diffeo. near $0, 1, \ldots, n$. Here continuous means that for each x, there is a diffeo. Θ from a nbd. of $[f(x), n]$ to a nbd. of $[f'(x), n]$ such that ~~h(y) is the germ represented by Θ~~ for all y in some neighborhood of x, $h(y)$ equals the germ represented by Θ.

Let $\varphi : \{0, 1, \ldots, m\} \longrightarrow \{0, 1, \ldots, n\}$ be a strictly monotone map. ~~We propose to define a functor $\varphi^* : \mathcal{C}_n \longrightarrow \mathcal{C}_m$ which is unique up to canonical isomorphisms. φ^* is the identity map from $\mathrm{Ob}\,\mathcal{C}_n = \mathbb{R}_{\geq 0}$ to $\mathrm{Ob}\,\mathcal{C}_m = \mathbb{R}_{\geq 0}$.~~ Given $x \in \mathrm{Ob}\,\mathcal{C}_n$ and $x' \in \mathrm{Ob}\,\mathcal{C}_m$, let $\mathrm{Hom}_{/\varphi}(x, x')$ be the set of germs of diffeomorphisms

$$h : [x, \varphi(m)] \longrightarrow [x', m]$$

such that for each $i = 0, 1, \ldots, m$

$$h(z) = z - \varphi(i) + i$$

for z near $\varphi(i)$. Let Ar/φ be the disjoint union of the $\mathrm{Hom}_{/\varphi}(x, x')$ as x runs over $\mathrm{Ob}\,\mathcal{C}_n$ (resp. x' runs over $\mathrm{Ob}\,\mathcal{C}_m$). Endow Ar/φ with the natural topology so that ~~it becomes an etale space over~~ the maps

$$\mathrm{Ob}\,\mathcal{C}_m \xleftarrow{\ t\ } \mathrm{Ar}/\varphi \xrightarrow{\ s\ } \mathrm{Ob}\,\mathcal{C}_n$$

are etale.

Now observe that $\mathrm{Ar}\,\mathcal{C}_m$ lefts acts (resp. $\mathrm{Ar}\,\mathcal{C}_n$ right acts) naturally on Ar/φ.

Moreover Ar/φ is a left C_m-torsor.

Given two strictly monotone maps

$$\{0,1,..,m\} \xrightarrow{\varphi} \{0,1,..,m'\} \xrightarrow{\psi} \{0,1,..,m''\}$$

there is an evident composition

$$Ar/\varphi \times_{Ob\,C_{m'}} Ar/\psi \longrightarrow Ar/\psi\varphi$$

which induces an isomorphism

$$Ar/\varphi \times^{C_{m'}} Ar/\psi \xrightarrow{\sim} Ar/\psi\varphi.$$

of left C_m-torsors with right $C_{m''}$-action.

Let C denote the following category: $Ob\,C = \mathbb{R}_{<0} \times \mathbb{N}$; a morphism from (x,n) to (x',n') is a pair (h,φ), where $\varphi: \{0,..,n'\} \to \{0,..,n\}$ is an ~~strictly monoton~~ injective monotone map and where h is a diffeomorphism germ

$$h: [x, \varphi(n')] \longrightarrow [x', n']$$

$h(z) = z - \varphi(i) + i$
z near $\varphi(i)$
$i = 0, .., n'$.

Composition

$$(x,n) \xrightarrow{(h,\varphi)} (x',n') \xrightarrow{(h_1,\psi)} (x'',n'')$$

is defined to be the pair

$$[x, \varphi\psi(n'')] \xrightarrow{h|[x,\varphi\psi(n'')]} [x', \psi(n'')] \xrightarrow{h_1} [x'', n''].$$

Make $\mathcal{C}$ into a topological category, by giving $\mathrm{Ob}\,\mathcal{C} = \mathbb{R}_{<0} \times \mathbb{N}$ it natural top., and by making $\mathrm{Ar}\,\mathcal{C}$ into a space such that $s : \mathrm{Ar}\,\mathcal{C} \longrightarrow \mathrm{Ob}\,\mathcal{C}$ is étale.

Two functors: Define

(1)
$$\mathcal{C} \longrightarrow \Gamma$$
$$(x, n) \longmapsto x$$
$(h, \varphi) : (x, n) \to (x', n') \longmapsto$ restriction of h as a germ of diffeo. $x \longrightarrow x'$.

Let $\mathcal{B}$ be the category with

$$\mathrm{Ob}\,\mathcal{B} = \mathbb{N}$$

$$\mathrm{Hom}_{\mathcal{B}}(n, n') = \{ (h, \varphi) \mid$$

$\varphi : \{0, \ldots, n'\} \hookrightarrow \{0, \ldots, n\}$ injective monotone

$h : [0, \varphi(n')] \longrightarrow [0, n]$ diffeo. germ

$h(z) = z - \varphi(i) + i$

z near $\varphi(i)$

$i = 0, \ldots, n'$ $\}$

$\mathcal{B}$ will be regarded as a top. category with discrete topology on $\mathrm{Ob}\,\mathcal{B}$ and $\mathrm{Ar}\,\mathcal{B}$. Define

(2)
$$\mathcal{C} \longrightarrow \mathcal{B}$$
$$(x, n) \longmapsto n$$
$(h, \varphi) : (x, n) \to (x', n') \longmapsto$ (rest. of h to $[0, \varphi(n')]$, φ)

We have already associated to Γ_0 the ~~category~~ topos of Γ-sheaves. Next we want to define the appropriate kinds of sheaves for the categories $\mathcal{B}$ and $\mathcal{C}$.

First consider $\mathcal{B}$. It is a cofibred category over the category D^o where D is the category with $\mathrm{Ob}\, D = \mathbb{N}$ and whose morphism $\varphi : n \to n'$ are injective monotone maps $\{0, .., n\} \hookrightarrow \{0, .., n'\}$. The fibre $\mathcal{B}_n$ is the category associated to the group

$$G_n = G_{01} \times \cdots \times G_{n-1,n}$$

where G_{ab} = group of diffeos. of (a,b) compact support. Thus $\mathcal{B}$ is a sort of generalized simplicial category, except that there are no degeneracy operators.

(screwy idea: to what extent does a cofibred category in groupoids admit a lower central series. Thus if $i \mapsto G_i$ is a functor from a category I to groups we can form its lower central series which is a functor from I to Lie algebras. But if $i \mapsto G_i$ is a pseudo-functor what sort of sense can be made out of $\mathrm{Lie}(G_i)$. Actually since inner autos. of G_i act trivially on $\mathrm{Lie}(G_i)$ one obtains a definite functor from I to Lie algebras.) Also $i \mapsto G_i / \Gamma_r G_i$ is a pseudo-functor, but not $i \mapsto \Gamma_r G_i$ in general..)

In general suppose we have a cofibred topos $E \longrightarrow B$, by which we mean that E is bifibred over B, each fibre category E_b is a topos, and for each $u: b \to b'$ in B, the functors (u^*, u_*) constitute a morphism of topoi from E_b to $E_{b'}$. Then it is natural to consider the "Deligne-style" topos consisting of functors $F: E^\circ \longrightarrow$ Sets whose restriction to each fibre is a sheaf, or what comes to the same thing, a section $b \mapsto F_b \in E_b$ together with for each $u: b \to b'$ a map

$$u^* F_{b'} \longrightarrow F_b$$

satisfying some evident transitivity condition for composition of maps.

Now we apply this to the situation where $B = D^\circ$ and where $E_n = (B_n)^\wedge$. Thus a sheaf consists of a right G_n-set for each $n \geq 0$, together with for each arrow in E over $\varphi: m \to n$ (in D°) (a pair (h, φ) consisting of an injective monotone map $\varphi: \{0, .., n\} \longrightarrow \{0, .., m\}$ and a diffeo

$$h: [\varphi(0), \varphi(n)] \longrightarrow [0, n] \quad \ni \text{ for } 0 \leq i \leq n$$

$$h(z) = z - \varphi(i) + i \quad \text{near } i \,)$$

we should be given a map

$$(h, \varphi)^*: F_n \longrightarrow F_m$$

compatible with the action of G_m, where G_m acts on

F_n via the homomorphism

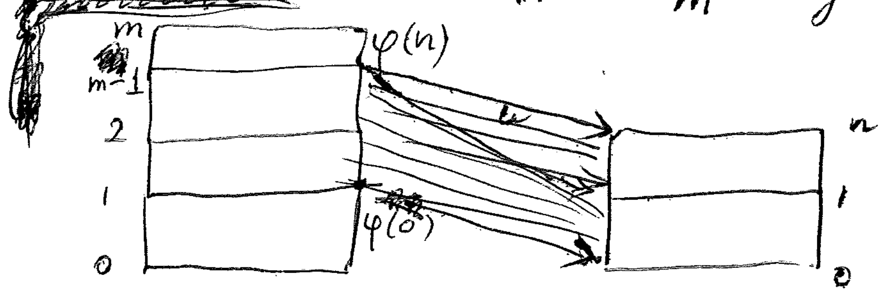
$$(h,\varphi)_* : G_m \longrightarrow G_n$$

which takes a diffeo. of $[0, m]$, restricts it to $[\varphi(0), \varphi(n)]$ and then transports it to $[0, n]$ via the diffeo. h.

Here is an interpretation of such sheaves. Consider the category whose objects are (left) G_n-torsors over X for all n. By a morphism $P_m \longrightarrow P_n$ we mean a pair $((h,\varphi), u)$, where $(h,\varphi) : m \longrightarrow n$ is a map in $\mathcal{B}$ and where $u : P_m \longrightarrow P_n$ is equivariant for the homomorphism

$$(h,\varphi)_* : G_m \longrightarrow G_n$$

(Geometrically given an injective monotone map $\varphi : n \longrightarrow m$ and a m-layered G-bundle E_m belonging to the torsor P_m (resp. an n-layered G-bundle E_n belonging to P_n) ~~and a map~~ $u : E_n \longrightarrow E_m$ lying over φ:



because P_m is the set of isos. of $[0,1] \cup \dots \cup [m-1, m]$ with the fibres of E_m, one obtains a map from P_m to P_n from u only after choosing a map

$$(h,\varphi) : \quad m] \longrightarrow \quad n]$$

lying over φ.)

Call this category $\underset{\sim}{B}(X)$; one obtains a fibred category $\underset{\sim}{B}$ over spaces. Now given $\{F_n\}$ in $B^\wedge$ ~~and a $P_m \to X$ [illegible]~~ one can associate to $P_m \to X$ the sheaf

$$F_m \times^{G_m} P_m \longrightarrow X$$

and to the map $u : P_m \longrightarrow P_n$ equivariant for $(h,\varphi)_* : G_m \longrightarrow G_n$ one can associate a morphism of sheaves

$$\begin{array}{ccc} F_n \times^{G_n} P_n & \simeq & F_n \times^{G_n} (G_n \times^{G_m} P_m) \\ \Big\downarrow{\scriptstyle ((h,\varphi),u)} & & \Big\downarrow \; = \\ F_m \times^{G_m} P_m & \longleftarrow & F_n \times^{G_m} P_m \\ & \uparrow & \\ & \text{from } (h,\varphi)^* : F_n \longrightarrow F_m & \end{array}$$

Consequently $\{F_n\}$ can be interpreted as a cartesian functor $: B(X) \longrightarrow Sh(X)$. It's pretty clear that this is an equivalence of categories between $B^\wedge$ and such cartesian functors.

Since $B \longrightarrow D^\circ$ is cofibred, one has Leray spectral sequence

$$H^*(B, F) \Longleftarrow E_2^{pq} = H^p(D^\circ, n \mapsto H^q(G_n, F_n)).$$