algebraic cycles, formulate good conjectures. main problem is really to understand the deviation of etale coh. from K-theory. Thus I conjecture that there should be a descent spectral sequence for the etale top of the form

$$E_2^{pq} = H^p(X_{et}, K_{-q}(\mathcal{O}_X)) \Rightarrow K'_{-p}(X),$$

but one knows this isn't the case.

<u>Example 1.</u> $X = \mathrm{Spec}(k)$, $k = \mathbb{F}_q$. Then X_{et} is equivalent to the category of sets with continuous π action, $\pi = \mathrm{Gal}(\bar{k}/k)$. Thus in the spectral sequence the E_2 term is

$$\begin{array}{|lll}
\hline
\mathbb{Z} & H^1(\pi,\mathbb{Z}) & H^2(\pi,\mathbb{Z}) \\
K_1(k) & 0 & \\
\boxed{K_2(k)} & 0 & \\
K_3(k) & 0 & \\
0 & 0 &
\end{array}$$

so we have the mysterious

$$H^2(\hat{\mathbb{Z}}, \mathbb{Z}) \simeq H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$$

term.

So the problem becomes how to formulate things. Idea might be as follows: On one hand we have the homotopy inverse limit

$$\underset{\pi}{\underleftarrow{\text{holim}}}\left(B_{\bar{k}}\right)$$

and on the other the theory B_k. There is a canonical map

$$B_k \longrightarrow \underset{\pi}{\underleftarrow{\text{holim}}}\left(B_{\bar{k}}\right).$$

To be a bit more general I can suppose given a hypercovering $\mathcal{U}$ of X for the flat top (fpqc). Then I get a cosimplicial spectrum

$$p \longmapsto B_{\mathcal{U}_p}$$

and I take its $\underleftarrow{\text{holim}}$, and there is a map

$$B_X \longrightarrow \underleftarrow{\text{holim}}\left(p \longmapsto B_{\mathcal{U}_p}\right).$$

(Here B_X denotes the spectrum one constructs from vector bundles on X. Possibly X should be supposed regular so that there is no reason to suppose that ~~these~~ B_X should be disconnected.)

So we have this canonical map

$$B_k \longrightarrow \underset{\overleftarrow{\pi}}{\operatorname{holim}}\; B_{\bar{k}}$$

and what we want to show is that it~~s~~ induces isomorphisms on homotopy groups π_q for q [illegible] $\geq$ some constant depending on how big k is.

~~Thus the proof~~

This is already sort of interesting from the homotopy viewpoint. It is slightly unusual to find a map $X \longrightarrow Y$ such that the fibre has vanishing homotopy groups above a certain point. Thus finitely many obstructions. So you might prove this simply by showing that if ~~a~~ one had a k-connected space K, then the maps of [illegible] K to X and to Y are the same up to homotopy. How to use k-connected? I know of ~~no method~~ nothing really like this in topology.

~~Suppose now that X is~~

4. ~~Start with a curve~~ X.

worry first about schemes finite type over $\mathbb{F}_p$.

I understand K-theory of a finite field.

Let X be a curve over $k = \overline{\mathbb{F}}_p$. Then we have

$$\to K_{i+1}F \xrightarrow{\partial} K_i k \otimes D \longrightarrow K_i X \longrightarrow K_i F \xrightarrow{\partial}$$

the localization exact sequence.

$$\begin{tikzcd}
K_i X = & K_i k \arrow[d, hook] \oplus K_i k \oplus \text{interesting part} \\
& K_i F
\end{tikzcd}$$

$$\begin{tikzcd}
K_i k \otimes F^\bullet \arrow[r] & K_i k \otimes D_0 \arrow[r] & K_i k \otimes \overset{A}{\cancel{\mathrm{Pic}(X)}} \arrow[r] & 0
\end{tikzcd}$$

basic conjecture should be set up carefully

$$\begin{tikzcd}
 \arrow[d] & K_i k \otimes F^\bullet/k^\bullet \arrow[r] \arrow[d] & K_i k \otimes D \arrow[r] \arrow[d, equal] & K_i k \otimes \mathrm{Pic}(X) \arrow[r] \arrow[d] & 0 \\
\tilde{K}_{i+1}X \arrow[r] & \tilde{K}_{i+1}F \arrow[r] & K_i k \otimes D \arrow[r] & K_i X \arrow[r] & K_i F
\end{tikzcd}$$

The point maybe is that $K_i X$ has a nat. filt.

$$\begin{tikzcd}
0 \arrow[r] & H^1(X, \mathcal{K}_{i+1}) \arrow[r] & K_i X \arrow[r] & H^0(X, \mathcal{K}_i) \arrow[r] \arrow[d, equal] & 0 \\
& \text{cokernel.} & & \mathrm{Ker}(K_i F \to K_{i-1} k \otimes D) &
\end{tikzcd}$$

5.

and the conjecture is simply that

$\vdash$ ~~$H^1(X, \mathcal{K}_{i+1}) = K$~~

the canon. maps

$$K_i k \otimes_{\mathbb{Z}} \operatorname{Pic}(X) \longrightarrow H^1(X, \mathcal{K}_i)$$

$$\operatorname{Tor}_1^{\mathbb{Z}}(K_i k, \operatorname{Pic}(X)) \longrightarrow H^0(X, \mathcal{K}_i)$$

are isos. roughly.

$$\boxed{0 \longrightarrow K_i k \otimes_{\mathbb{Z}} \tilde{K}_0(X) \longrightarrow \tilde{K}_i(X) \longrightarrow \operatorname{Tor}_1(K_{i-1}k, \tilde{K}_0 X) \to 0}$$

$$\begin{tikzcd}
 & & \arrow[d] & K_{i-1}k\otimes \arrow[d] & \\
\tilde{K}_{i+1}F \arrow[r, "\partial"] & K_i k \otimes D \arrow[r] & \tilde{K}_i(X) \arrow[r] & \tilde{K}_i(F) \arrow[r, "\partial"] & K_{i-1}k \otimes D \\
K_i k \otimes F^*/k^* \arrow[u] \arrow[r] & K_i k \otimes D \arrow[u, equal] \arrow[r] & K_i k \otimes \operatorname{Pic} \arrow[u] \arrow[r] & 0 \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
K_{i+1}k \otimes D \arrow[r] & \tilde{K}_{i+1}X \arrow[r] & \tilde{K}_{i+1}F \arrow[r] & K_i k \otimes D \arrow[d, equal] \\
0 \arrow[r] & \operatorname{Tor}_1(K_i k, \operatorname{Pic}) \arrow[r] & K_i k \otimes F^*/k^* \arrow[u] &
\end{tikzcd}$$

so what we know is that

$$\boxed{K_i k \otimes_{\mathbb{Z}} \tilde{K}_0(X) \longrightarrow H^1(X, \tilde{\mathcal{K}}_{i+1}) \to 0}$$

and

$$\operatorname{Tor}_1(K_{i-1}k, \tilde{K}_0(X)) \longrightarrow H^0(X, \mathcal{K}_i$$

6. k alg. closure of a finite field, one conjectures that

$$K_i k \otimes \underset{\text{free}}{F^\bullet/k^\bullet} = \tilde{K}_{i+1}(F)$$

~~[illegible]~~ i odd > 0.

~~[illegible]~~

true in general.

$$K_{i+1}(F) = K_{i+1} k \oplus K_i k \otimes F^\bullet$$

~~$0 \to K_{i+1} k$~~

$$0 \longrightarrow K_i k \longrightarrow K_i F \longrightarrow K_{i-1} k \otimes F^\bullet/k^\bullet \longrightarrow 0$$

Take direct limit as F ~~varies over~~ goes to its alg closure. Then

~~[illegible]~~ $$\tilde{K}_{i+1} F = K_i k \otimes \underset{\substack{\uparrow \\ \mathbb{Q}\text{ vector space}}}{F^\bullet/k^\bullet} \qquad i \geq 0$$

~~$K_0 F = \mathbb{Z}$~~

so $$\boxed{\begin{array}{ll} K_1 F = F^\bullet & \text{has lots of } \mathbb{Q}\text{-stuff} \\ K_i F = K_i k & \text{for } i \geq 2. \end{array}}$$

k alg. of a finite field

~~[illegible]~~ X curve over k, F function field

$\bar F$ its alg. closure. Then I want to look at the Galois spectral sequence. So what happens is this.

$$\tilde{K}_i \bar{F} = K_{i-1}k \otimes F^\bullet \qquad i \geq 2$$

$$K_1 F = F^\bullet$$

$$K_0 F = \mathbb{Z}.$$

$\pi = Gal(\bar F / F)$. π should be of coh. dim 1. (Tsen theory)

$$H^1(\pi, K_{i-1}k \otimes \bar F^\bullet) \qquad i \equiv 0 \ (2) \quad i > 0$$

$$K_{i-1}k \cong \mathbb{Q}/\mathbb{Z}'$$

$$K_{2i-1}k = \bigoplus_{\ell \neq p} V_\ell^{\otimes i} / T_\ell^{\otimes i}$$

$$0 \longrightarrow H^1(\pi, K_{i-1}k \otimes \bar F^\bullet) \longrightarrow \underset{\underline{Br(F). = 0}}{\underset{\|}{H^2(\pi, \bar F^\bullet)}} \longrightarrow \overset{0}{\cancel{H^2(\pi, \mathbb{Q} \otimes \bar F^\bullet)}}$$

$$\left\{ \begin{array}{ll} H^1(\pi, K_{i-1}k \otimes \bar F^\bullet) = 0 & i > 0 \\ H^2(\pi, \bar F^\bullet) = 0 & \text{Tsen that } Br(F) = 0. \end{array} \right.$$

$$H^1(\pi, \bar F^\bullet) = 0$$

Thus E_2-term

$$\begin{array}{|ccccc}
\hline
\mathbb{Z} & 0 & \mathrm{Hom}(\pi, \mathbb{Q}/\mathbb{Z}) & 0 & 0 \\
F^\bullet & 0 & 0 & 0 & 0 \\
(k\otimes F^\bullet)^\pi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
(K_3 k\otimes F^\bullet)^\pi & 0 & 0 & 0 & 0
\end{array}$$

and what works is the following.

~~$k\otimes F^\bullet$~~

$$\begin{tikzcd}
(\mathbb{Q}\otimes F^\bullet)^\pi \arrow[r] \arrow[d, phantom, "\shortparallel"] & (\mathbb{Q}/\mathbb{Z}\otimes F^\bullet)^\pi \arrow[r] \arrow[d, phantom, "\shortparallel"] & H^1(\pi, F^\bullet) \arrow[r] \arrow[d, phantom, "\shortparallel"] & {} \\
{} & {} & 0 &
\end{tikzcd}$$

$\to \mathbb{Q}\otimes F^\bullet$

so the spectral sequence seems to work. Anyhow now suppose that $F^\bullet$ ~~is non-zero~~

Next want to take a curve over a finite fields. It would appear reasonable that things would work. More or less have to

---

$$k_1 = \overline{k_0[T]} \qquad k_0 = \overline{\mathbb{F}_p}.$$

and we "know" ~~that~~

$$K_i(k_1) = \begin{cases} \mathbb{Z} & i=0 \\ k_1^\bullet & i=1 \\ 0 & i=2,4,6,\dots \\ K_i(k_0) & i=3,5,7,\dots \end{cases}$$

9. Try taking a curve over k_1, Unlike k, now $k^{\cdot}$ is non-torsion. And the same is true for the $\mathrm{Pic}(X)$. So again

$$\begin{tikzcd}
 & 0 & & & & \\
 & A \arrow[u, "a"'] \arrow[r] & B & & & \\
0 \arrow[r] & \tilde{R}_i(X)/F_0 \arrow[u] \arrow[r] & \tilde{K}_{i+1}F^{\cdot} \arrow[u] \arrow[r] & K_ik\otimes D \arrow[r] & F_1K_iX \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Tor}_1(K_ik, \tilde{K}_0) \arrow[u, hook] \arrow[r] & K_ik\otimes(F^{\cdot}/k^{\cdot}) \arrow[u] \arrow[r] & K_ik\otimes D \arrow[u, equal] \arrow[r] & K_ik\otimes \tilde{K}_0X \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

So what we ~~do~~ might have is a map

$$\mathrm{Tor}_1(K_ik, \tilde{K}_0) \hookrightarrow \tilde{K}_i(X)/F_0K_i(X)$$

should be injective by Chern class consider-ations.

$$0 \leftarrow F_1K_iX \twoheadleftarrow K_ik\otimes\tilde{K}_0$$

If you are in the good range where K_ik torsion then $K_ik\otimes\tilde{K}_0X = 0$, so $F_1K_i(X)=0$, and the conjecture is that you have an isom.

---

general conjecture is that one has an exact sequence

$$0 \to \mathrm{Tor}_1(K_ik, \tilde{K}_0) \to \tilde{K}_i(X)/F_0 \to K_ik\otimes\tilde{K}_0(X)$$

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$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K}_{i+1}(X)/F_0 \arrow[r] & \widetilde{K}_{i+1}(F) \arrow[r] & K_i k \otimes D \arrow[r] & F_1 K_i(X) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Tor}_1(K_i k, \widetilde{K}_0) \arrow[r] \arrow[u] & K_i k \otimes (F^\bullet/k^\bullet) \arrow[r] \arrow[u] & K_i k \otimes D \arrow[r] \arrow[u, equal] & K_i k \otimes \widetilde{K}_0 X \arrow[r] \arrow[u] & 0 \\
 & \cdot & \cdot & Q & Q \arrow[u, hook] & \\
0 \arrow[r] & \widetilde{K}_{i+1}(X)/F_0 \arrow[r] & \widetilde{K}_{i+1}(F) \arrow[r] \arrow[uu] & \mathrm{Im}\,\partial \arrow[r] \arrow[u] & 0 & \\
0 \arrow[r] & \mathrm{Tor}_1 \arrow[r] \arrow[u] & K_i k \otimes F^\bullet/k^\bullet \arrow[r] \arrow[u] & I \arrow[r] \arrow[u] & 0 & \\
 & \cdot & 0 & 0 & &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}\,\partial \arrow[r] & K_i k \otimes D \arrow[r] & F_1 K_i \arrow[r] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u, hook] & K_i k \otimes D \arrow[r] \arrow[u, equal] & K_i k \otimes \widetilde{K}_0 \arrow[r] \arrow[u] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If $Q = \mathrm{Ker}\,(K_i k \otimes \widetilde{K}_0 \longrightarrow K_i(X))$.

then I have

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & A \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & Q \arrow[r] & 0 \\
0 \arrow[r] & K_{i+1}(X)/F_0 \arrow[r] \arrow[u] & \widetilde{K}_{i+1}(F) \arrow[r] \arrow[u] & \mathrm{Im}\,\partial \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{Tor}_1(K_i k, \widetilde{K}_0) \arrow[r] \arrow[u] & K_i k \otimes (F^\bullet/k^\bullet) \arrow[r] \arrow[u] & I \arrow[r] \arrow[u] & 0 \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

" Is it reasonable to conjecture the A is zero $\boxed{}$

Conjecture (?) :

$$\begin{tikzcd} \mathrm{Tor}_1(K_i k, \tilde{K}_0) \arrow[r, "\sim"] & H^0(X, \mathcal{K}_{i+1}) \arrow[d, phantom, "\cap"] \\ & K_{i+1} \end{tikzcd}$$

Assume spectral sequence degenerates canonically $\Rightarrow$.

might there be a chance that

$$0 \to \mathrm{Tor}_1(K_i k, \tilde{K}_0) \longrightarrow \tilde{K}_{i+1}F \longrightarrow D \otimes K_i k \longrightarrow$$

be exact? Probably not because for K_2 one expects that $K_1 k \otimes (F^\bullet/k^\bullet) \longrightarrow \tilde{K}_2 F$ certainly would be onto the torsion elements

$$\mathrm{Im}\left\{ K_{i+1}(X) \longrightarrow K_{i+1}(F) \right\}$$

Is there a chance that this might consist of torsion elements?

$$K_{i+1}(F) = \varinjlim K_{i+1}(X-S)$$

thus I have an element of $K_{i+1}(X)$ which vanishes in $K_{i+1}(X-S)$ for S finite, hence which is in the image of the transfer with respect to S.

$$\mathbb{Z}S \otimes K_{i+1}k \longrightarrow K_i$$

i odd

$$\begin{tikzcd}
0 \arrow[r] & K_{i+1}X \arrow[r, hook] & \tilde{K}_{i+1}F \arrow[r] & K_i k\otimes \bar{D}_0 \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Tor}_1(\quad) \arrow[r] \arrow[u] & K_i k\otimes(F^\bullet/k^\bullet) \arrow[r] \arrow[u] & K_i k\otimes D_0 \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

so what we want is consistent.

so what you want to prove maybe is that if all tame symbols vanish then the element is torsion

Thus one might try to see if $\tilde{K}_{i+1}(F) \to K_i k\otimes D$ has to be injective modulo torsion in the complete case

$$\begin{tikzcd}
K_{i+1}k\otimes D \arrow[r] & \tilde{K}_{i+1}(X) \arrow[r] \arrow[d] & \tilde{K}_{i+1}(F) \\
 & \tilde{K}_{i+1}(X-\infty) \arrow[r] & \tilde{K}_{i+1}(F)
\end{tikzcd}$$

affine

and you want to show something about char. classes of bundles. Anyway.

Deligne's [illegible] construction. L line bundle trivialized at a set of points S. View as divisors D strange to S modulo lin. equivalence by $g\equiv 1$ on S.

$f\in H^0(X-S, \mathcal{O})$

$$\cancel{\textstyle\sum' f(P)}$$

$$\sum' f(P)^{\mathrm{ord}_P(D)} \cdot P \quad \in K_1 k\otimes D$$

13. $\sum' f(P)^{\mathrm{ord}_P(D)} \otimes P$ if $D = \sum_i \mathrm{ord}_p(D) \cdot P$

then to check that

$$\sum' f(P)^{\mathrm{ord}_P(g)} \otimes P = 0 \quad \text{if} \quad g \equiv 1 \text{ on } S$$

~~$Q \in S$ then~~

$$\partial\{f,g\} = \sum{}' (-1)^{v_P(f) v_P(g)} \frac{f^{v_P(g)}}{g^{v_P(f)}}(P) \otimes P$$

and if you are ~~[illegible]~~ at ~~[illegible]~~ $Q \in S$ then

$$v_Q(g) = 0$$

so we have $\dfrac{f^0}{g^{v_P(f)}(P)} = 1.$

The significance is not too clear

$$\begin{tikzcd} H^0(X-S, \mathcal{O}_X^{\cdot}) \otimes \left\{\begin{array}{l}\text{line bundle}\\ \text{trivialized at points}\\ \text{of } S\end{array}\right\} \arrow[r] & H^1(X, \mathcal{K}_2) \arrow[d, phantom, "\cap"] \\ & K_1(X) \arrow[d] \\ & K_1(F) \end{tikzcd}$$

$$0 \longrightarrow H^0(X-S, \mathcal{O}_X^{\cdot}) \longrightarrow K_1F \longrightarrow \coprod_{P \in S} \mathbb{Z}$$

$$K_1F \longrightarrow \coprod_{P \notin S} \mathbb{Z} \longrightarrow$$

14. Generalized Jacobians:

1.) rational map of a curve to an alg. group comm.

2) Picard scheme of a singular curve.

---

first point is this: If $f: X-S \longrightarrow G$ is given, one can extend f to divisors prime to S and then show the existence of a module, i.e. an $\mathfrak{m}$ such that $g \equiv 1 \pmod{\mathfrak{m}} \Rightarrow f((g)) = 0$.

Then we have universal such thing $J_{\mathfrak{m}}$

---

somehow the singular curve arises from the equivalence relation defined by the module.

Better approach would be to start stability thm.

X $\qquad \mathcal{P}(X)$ cat of v.b. on X.

have to filter by rank

$$0 = F_0\mathcal{P}(X) \subset F_1\mathcal{P}(X) \subset \cdots$$

and I consider corresp. filt. of Q

$$pt = F_0(Q\mathcal{P}(X)) \subset F_1(Q\mathcal{P}(X)) \subset \cdots$$

$$pt \subset \bigvee_{L \in Pic(X)} \Sigma\, Aut(L) \subset$$

and I consider the assoc. filtration.

15

$$\begin{tikzcd} F_{p-1}Q \arrow[r, "j"] & F_pQ \\ & M \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

hom.

$f : \mathcal{C} \to \mathcal{C}'$

[redacted]

$Y \mapsto H_i(f/Y)$ covariant in Y.

To compute $L_q j$

homology. Have a spectral sequence

$$E^2_{pq} = H_p(\mathcal{C}', Y \mapsto H_q(f/Y)) \Rightarrow H_{p+q}(\mathcal{C}).$$

So I want to consider the setup, which goes as follows

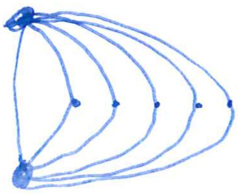
f/Y is the susp. of the building of Y_η.

so $$F_{p-1}Q \xrightarrow{j} F_pQ \xleftarrow{i} \coprod_{E \in Vect_p(X)} Aut(E)$$.

$p=1$

$$H_q(j/E) = H_q(\text{susp. of } X(E_\eta))$$

$$= \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \neq 0, p-1 \\ St(E_\eta) & q = p-1 \end{cases}$$



$p \geq 2$

if rank $E = p$

and $= \mathbb{Z}$ in deg. 0 if rank $E < p$.

Thus we have an exact seq.

$$0 \to \coprod_{rg(E)=p} i_{E!}\, St(E_\eta)^{[p-1]} \longrightarrow L_q j_!(\mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

should say triangle

16.

$$(i_{!}F)(\cancel{Y}) = H_0(i/\cancel{Y}, F)$$

$$(i_{!}F)(\cancel{Y}) = H_0(i/\cancel{Y}, F)$$

now if $\boxtimes, i\boxtimes \longrightarrow Y$ then is empty if ~~that~~ $Y \in F_{p-1}Q$. And if $Y \in$ [illegible] U_p

---

$\therefore$ $i_!$ is exact and extends by zero.

---

so get long exact sequence

$$\longrightarrow H_{q+1}(F_pQ) \longrightarrow \coprod_E H_q(\overset{\mathrm{Aut}\,E}{,}St(E_\eta)[p-1]) \longrightarrow H_q(F_{p-1}Q) \longrightarrow H_p(F_pQ)$$

$$\|$$

$$H_{q-p}$$

$$\longrightarrow \coprod_E H_{q-n+1}(\mathrm{Aut}\,E, St(E_\eta)) \longrightarrow H_q(F_{n-1}Q) \longrightarrow H_q(F_nQ) \longrightarrow \cdots H_{q-n}$$

$$rg(E) = p.$$

~~[illegible]~~

~~so now I want to consider the fibre of the map~~

$$\cdots\cdots\longrightarrow F_{n-1}Q \longrightarrow F_nQ \longrightarrow F_{n+1}Q \longrightarrow \cdots\cdots$$

I am interested in $\pi_1 Q. = H_1 Q.$

$$\boxed{H_2(F_1) \xrightarrow{\sim} \coprod_L H_1(\mathrm{Aut}\,L, St(L_\eta))} \longrightarrow \underset{\overset{\shortparallel}{0}}{H_1(F_0)} \longrightarrow H_1(F_1) \longrightarrow 0$$

so $H_1(F_1)$

$$0 \to H_1(F_1) \to \coprod_{L \in Pic} H_0(Aut(L), St(L_\eta)) \to H_0(F_0) \xrightarrow{\sim} H_0(F_1)$$

$$\left[\begin{array}{lr} H_i(F_1) \simeq \coprod_{L \in Pic} H_{i-1}(Aut(L)) & i > 0 \\ \qquad\qquad \mathbb{Z} & i = 0. \end{array}\right.$$

$$H_2(F_1) \simeq \coprod_{L \in Pic} H_1(Aut(L)) \qquad \boxed{}$$

$$\begin{tikzcd}
 & k^* \otimes \mathbb{Z}[Pic] \arrow[d, phantom, "\shortparallel"] & & \\
\coprod_{E \in V_2} H_1(Aut\, E, St) \arrow[r] & H_2(F_1) \arrow[r] & H_2(F_2) \arrow[r] & \overset{0}{\coprod_{E \in V_2} H_0(Aut(E), St(E_\eta))}
\end{tikzcd}$$

$$H_2(F_2) \to H_2(F_3) \to \overset{\overset{0}{\shortparallel}}{H_{2-3}}$$

~~and in any case it shouldn't be so bad.~~

$$\begin{tikzcd}
H_1(F_1) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_1(F_2) \arrow[r] & \cancel{\coprod_{E \in V_2} H} \; 0 \\
\mathbb{Z}[Pic] \arrow[r] & \mathbb{Z} \oplus Pic &
\end{tikzcd}$$

$$H_{1-2} \to H_1(F_2) \xrightarrow{\sim} H_1(F_3) \to \coprod H_{1-3}$$

in any case what happens?

the point unfortunately is that this leads nowhere because you have no way to get at the Steinberg homology.

e.g. take a field. ^{or a P.I.D.} Then $H_1(F_1) = \mathbb{Z} = H_1(F_2)$.

$$H_0(Aut(E), St(E_\eta)) = 0$$ since $Aut(E)$ transitive on lines

$$St(E_\eta) = \mathbb{Z}[P^1]/\mathbb{Z}.$$

Thus

$$\begin{tikzcd} & H_2(F_1) \arrow[r, two heads] & H_2(F_2) \arrow[r, two heads] & H_2(F_3) \arrow[r, equal] & H_2(Q) \\ H_1(\mathrm{Aut}\, E_2, St(E_2)) \arrow[ur, bend left] & & & & \end{tikzcd}$$

~~$H_1(\mathbb{Z}[G/B]/\mathbb{Z}) \rightarrow H_0(\mathbb{Z}) \rightarrow H$~~

~~$H \longrightarrow H_1(B)$~~

$$0 \rightarrow \bar{\mathbb{Z}}[G/B] \longrightarrow \mathbb{Z}[G/B] \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$\begin{tikzcd} H_2(B) \arrow[r] & H_2(G) \arrow[r] & H_1(G, \bar{\mathbb{Z}}[G/B]) \arrow[dll, bend left] \\ H_1(B) \arrow[r] & H_1(G) \arrow[r] & {} \end{tikzcd}$$

~~in any case there is~~

This approach doesn't lead anywhere even in the case of a field. Except it must give some stability result. Suppose I want to know about $\pi_2(F_n)$.

$$\longrightarrow H_2(F_1) \longrightarrow H_2(F_2) \longrightarrow \coprod_{\eta} H_0(\mathrm{Aut}\, E, St(E_\eta))$$

thus $H_2(F_2) \twoheadrightarrow H_2(F_3) \xrightarrow{\sim} H_2(F_4) \xrightarrow{\sim} \cdots$

Thus this tells me that the K_1 of a curve is determined by bundles of rank ≤ 3. I would like it to be determined by bundles of rank ≤ 2.

19. Basic stability yoga should demand that on something of dim d K_i should be gen. by bundles of rank $\leq i+d$ and inj for bundles of rank $\leq i+d+1$.

$$\text{rank } E > d \implies E = \underline{E'} \oplus 1.$$

A local ring. Can do the same thing, namely, can ~~consider schemes~~ ~~Tits~~ building. So consider a free A-module E and flags inside of it ~~$E_1 \subset E_2 \subset E_3$~~.

A local ring E free f.g. A-module rank n to consider X(E) ~~categ. of~~ ordered set of subquotients of rank $\leq n$. To show X(E) begins in dim n-1. Suspension of building in some sense.

$$X(E) \longrightarrow X(\bar{E}) \qquad \bar{E} = k \otimes_A E$$

I want

Aut(E), X(E) to have a certain

connectivity. NOT clear.

20. Suppose A is a local ring and I filter Q-cat according to the rank. What gives? So I have to worry about $\mathrm{Aut}(E)$ acting on the reduced chains of $X(E)$.

The point should have something to do with the way ~~[illegible]~~ the map

$$X(E) \longrightarrow X(\bar{E})$$

identifies things together. So consider the situation in The situation $X(E) \longrightarrow X(\bar{E})$ E/mE. ~~YES~~ anyway. ~~So suppose now that things vary~~

~~Let J be compact complete etc.~~

E being given consider all possible ways of extending $\bar{F} \subset \bar{E}$ to $F \subset E$. Thus I have ~~for each of~~ E is given

Question: The problem has to do with the discrepancy between the K-theory of X and the integral of the local K-theory.

$$\boxed{K \longrightarrow \mathrm{ho}\Gamma(\mathcal{K})}$$

Now ~~from a spatial viewpoint~~ one has a map of spectra ~~$\mathrm{ho}\Gamma(\mathcal{K}$~~

$$K \longrightarrow Pic$$

given by the determinant. As spaces this map splits so that on the spatial level one has

$$K = SK \times Pic$$

$$\mathrm{ho}\Gamma(\mathcal{K}) = \mathrm{ho}\Gamma(S\mathcal{K}) \times \mathrm{ho}\Gamma(\mathbb{G}_m)$$

Thus the discrepancy on the K should be at least ~~that~~ as big as the discrepancy on $\mathbb{G}_m$.

$\mathbb{G}_m$ is a sheaf of abelian groups in fact it is $\bar{F}^\bullet$ in the Galois situation. Thus one has no group completion problem. ~~$K = SK \times Pic$~~ One has geometrically

$$H^*(\pi, \bar{F}^\bullet)$$

F
0
$Br(F)$
$?$

$$0 \to \mu \to \bar{F}^\bullet \to V \to 0$$

$$H^q(\mu) = H^q(\bar{F}^\bullet) \qquad q \geq 2$$

$$\to H^1(\mu) \to \underset{\overset{\|}{0}}{H^1(\bar{F}^\bullet)} \to 0 .$$

~~Compute~~ Know $H^i(X, \mathcal{O}_X^*) = \begin{cases} \Gamma(X, \mathcal{O}_X^*) & \\ \mathrm{Pic}(X) & \\ 0 & i \geq 2 \end{cases}$

so that in fact there is no **Zariski** discrepancy.

Situation: If I look at the $\mathbb{G}_m$ situation, then I do not have a cohomological descent setup. That is the stack of line bundles is not ~~trivial as~~ ~~these are~~ its homotopy integral since $\mathbb{G}_m$ has higher cohomology.

And I know that the situation is **no good** from the spectral viewpoint, because even $\mathbb{Z}$ has higher cohomology, not to mention the Brauer group.

GL situation is as follows

~~Suppose~~

You have a mechanism for understanding rational K-groups at least. Thus X scheme has

$$0 \to K_n(\mathcal{O}_X) \to \coprod_{x \in X_0} K_n(k(x)) \to \cdots$$

this is a resolution of sorts. But perhaps it is a resolution in the etale topology provided you forget torsion.

X regular scheme

for any $U \to X$ etale I want to consider

$$\coprod_{x \in U_p} K_q(k(x)) \otimes \mathbb{Q}$$

which is a contravariant functor in U.

Conjecture: This is a sheaf ~~for the etale topology.~~ and it is flask.

~~And what can one hope to understand.~~

$$U \times_X U \rightrightarrows U \xrightarrow[\text{surj}]{} X$$

Probably true, so what. tells me that

$$H^*_{et}(X, \mathcal{K}_q \otimes \mathbb{Q}) = H^*_{Zar}(X, \mathcal{K}_q \otimes \mathbb{Q}).$$

$$i : \operatorname{Spec} K \to X$$

$$0 \to \mathbb{G}_{m,X} \to i_* \mathbb{G}_{m,K} \to \coprod_{x \in X_0} (i_x)_* \mathbb{Z} \to 0$$

$$H^1(X, \mathbb{G}_m) \to H^1(X, i_* \mathbb{G}_{m,K}) \to \coprod_{x \in X_0} H^1(X, (i_x)_* \mathbb{Z})$$

$$\begin{tikzcd} H^1(X, (i_x)_* \mathbb{Z}) \arrow[d, phantom, "\shortparallel"] \\ H^1(\operatorname{Spec}(k(x)), \mathbb{Z}) \arrow[d, phantom, "\shortparallel"] \\ 0 \end{tikzcd}$$

curve

$$H^2(X, \mathbb{G}_m) \to H^2(X, i_* \mathbb{G}_{m,K})$$

$$R^q i_*(\mathbb{G}_{m,K}) \qquad H^q(K \times_X U, \mathbb{G}_m) = \begin{cases} 0 & q = 1 \\ Br(\ldots) & q = 2. \end{cases}$$

$X_0, X_1, X_2, \ldots$

next point might be to consider

$$\coprod_{x\in X_0} K_2\, k(x) \longrightarrow \coprod_{x\in X_1} k(x)^{*} \longrightarrow \coprod_{x\in X_2} \mathbb{Z}$$

problem - under what conditions might I get an etale sheaf?

Is their some arrangement?

$$U \longmapsto \coprod_{x\in U_0} K_2\, k(x)$$ presheaf.

what is the associated sheaf. Conj. would be that it is the ~~set of the~~ $(i_x)_* K$

Choose ~~Ω sep. alg closure of $x \in X_0$~~

~~then have Spec Ω~~

Ω sep closure of $k(x)$ so have Galois module $K_2\Omega$

certainly the stalk ought to be $K_2\,\overline{k(x)}$

25. ~~elliptic curves~~

~~elliptic curves~~

[illegible] basic problems

you must understand buildings

need upper bound methods. The place to start must be with curves over finite fields.

Milgram theory.

F.S.S. review $I\delta$ theory!