November 1, 1971

Let $C$ be a small category and let $X$ be a polyhedron (= a space endowed with an equivalence class of triangulations, two triangulations being considered equivalent if they have a common subdivision.) Recall that a sheaf on a polyhedron is compatible with the PL structure if there exists a triangulation (admissible) such that the sheaf is constant over each open simplex of the triangulation. Recall also that a $C$-torsor is a sheaf $P$ over $X$ endowed with a left action of $C$ such that

$$F \longmapsto F \times^{C} P$$

is a morphism of topoi from $C^\wedge$ (contrav. functors or sets over $Ob\text{-}C$ with right $C$-action) to $Top(X)$. Equivalently the stalk of $P$ over each $x \in X$ is an ind-representable functor from $C$ to Sets.

Definition: A $C$-torsor over $X$ is compatible with the PL structure if the sheaf $P$ is and if each stalk is representable.

The PL $C$-torsors on $X$ form a category $\underline{\text{Tors}}^{PL}(X, C)$.

Prop: Two PL $C$-torsors $P$ and $P'$ are in the same component of $\underline{\text{Tors}}^{PL}(X, C)$ iff they are homotopic i.e. restrictions of a PL $C$-torsor on $X \times I$.

Proof. If $u : P \longrightarrow P'$ is a morphism, then over $X \times I$ we construct a torsor which is $P$ over $X \times 0$ and $P' \times (0,1]$ on $X \times (0,1]$.

On the other hand given a torsor $Q$ over $X \times I$ joining $P$ and $P'$ choose a triangulation of $X \times I$ over which $Q$ is constant. Projecting the vertices to $I$, we can subdivide $I$ and so reduce to the case where all vertices of the triangulation are either on $X \times 0$ or on $X \times 1$. Then there are maps $P = Q_0 \longrightarrow Q_{1/2} \longleftarrow Q_1 = P'$, so done. ?

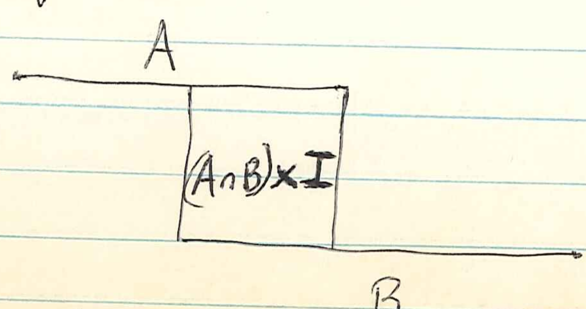
Denote by $\pi\, \underline{\mathrm{Tors}}^{PL}(X, \mathcal{C})$ the components of the category $\underline{\mathrm{Tors}}^{PL}(X, \mathcal{C})$. Preceding prop. shows that

$$X \longmapsto \pi\, \underline{\mathrm{Tors}}^{PL}(X, \mathcal{C})$$

satisfies the homotopy axiom. On the other hand if $X = A \cup B$ where $A, B$ are two subcomplexes then

$$\pi\, \underline{\mathrm{Tors}}^{PL}(X, \mathcal{C}) \longrightarrow \pi\, \underline{\mathrm{Tors}}^{PL}(A, \mathcal{C}) \underset{\pi\, \underline{\mathrm{Tors}}^{PL}(A \cap B, \mathcal{C})}{\times} \pi\, \underline{\mathrm{Tors}}^{PL}(B, \mathcal{C})$$

is surjective. Indeed given $P$ over $A$, $Q$ over $B$, and a homotopy from $P$ to $Q$ over $A \cap B$ we get a torsor on




which is homotopy equivalent to $X$.

Here is a basic construction with torsors. Given a torsor $P$ over $X$, an open subset $U \subset X$, and a map $P_U \longrightarrow Q$ of torsors over $U$, form cocartesian squares

$$\begin{tikzcd} P_U \arrow[r] \arrow[d, hook] & Q \arrow[d] \\ P \arrow[r] & P' \end{tikzcd} \qquad \left( \begin{tikzcd} j_! j^* P_U \arrow[r] \arrow[d] & j_! Q \arrow[d] \\ P \arrow[r] & P' \end{tikzcd} \right)$$

of sheaves over $X$. Claim $P \longrightarrow P'$ is a map of torsors. Indeed everything is clear by looking at the stalks.

Suppose now that $Z \subset X$ is closed, $P$ is a torsor over $X$ and $Q \longrightarrow P_Z$ is a map of torsors. Then we can form a cartesian square

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P \arrow[d] \\ i_* Q \arrow[r] & i_* i^* P \end{tikzcd} \qquad i : Z \longrightarrow X$$

and ~~[illegible]~~ $P' \longrightarrow P$ will be a map of torsors. Again the condition on the stalks has not changed.

November 2, 1971

Suppose $E$ is a flat complex vector bundle over a manifold $X$ of ~~[illegible]~~ dimension $n$. I claim that there are canonical classes in

$$H^{2i-1}(X, \mathbb{R}) \qquad 0 < i \leq n$$

associated to $E$. Here are two approaches:

~~1) (that of Borel). Let $P$ be the $GL_n(\mathbb{C})$ principal bundle defined by $E$, and form [illegible]~~

~~$P/U_n \simeq$ bundle of [illegible] hermitian inner products on the fibres of $E$~~

1) (that of Borel). Let $\tilde{X}$ be a universal covering of $X$ and $\pi$ the deck translation group, so that

$$E \simeq \tilde{X} \times^{\pi} V$$

where $V$ is a complex repn. of $\pi$ of dim. $n$. Let $S$ be the symmetric space of hermitian inner products on $V$ and form the bundle

$$\tilde{X} \times^{\pi} S.$$

Let $\omega$ be an invariant differential form (real coeff.) on $S$; Then $\omega$ gives rise to a form on $\tilde{X} \times^{\pi} S$, namely pull up to $\tilde{X} \times S$ and descend. Thus we have a map of complexes

$$I^{\cdot}(S) \longrightarrow \text{DeRham complex of } \tilde{X} \times^{\pi} S.$$

hence a map of cohomology

$$H^i(I^{\cdot}(S)) \longrightarrow H^i(\tilde{X} \times^{\pi} S, \mathbb{R})$$
$$\wr|$$
$$H^i(X, \mathbb{R}).$$

One knows that $I^{\cdot}(S)$ has zero differentials and is an exterior algebra with generators in degrees $1, 3, \ldots, 2n-1$.

2) (that of Bott). Choose a unitary reduction of $E$ and a connection $D$ compatible with the unitary structure. Then for any invariant polynomial $\varphi$ on the Lie algebra of $\mathrm{Aut}(V)$ we know that there is a canon. formula

$$\varphi(K(D)) = d\lambda$$

where $\lambda$ comes from the homotopy of $D$ to the given flat connection in $E$. As $D$ preserves the unitary structure $\varphi(K(D))$ is a real form, hence

$$d\,\mathrm{Im}(\lambda) = 0$$

defining classes in $H^{2i-1}(X, \mathbb{R})$ corresponding to the Chern classes.

(Argument showing why the classes are well-defined. The choice of a unitary reduction + a unitary connection is a section of a bundle $Y \xrightarrow{f} X$ with contractible fibre where over $Y$ one has a canonical such choice. Thus the two classes on $X$ obtained from the two choices of sections are restrictions of a class on $Y$ and hence the classes coincide.)

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Problems:

1) Show these methods coincide

2) Produce a proof that the Chern classes of a flat bundle are torsion, which is independent of the Weil-Chern definition.

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Line bundles: $n = 1$ $V = \mathbb{C}$

Why the Chern classes are torsion:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp(2\pi i)} \mathbb{C}^* \longrightarrow 0$$

Chern class is image under $\delta$ in long exact sequence

$$\longrightarrow H^1(X, \mathbb{C}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{C}).$$

Thus $c_1(L)$ goes to zero in $H^2(X, \mathbb{C})$, so is a torsion-class. Notice that we have a converse $c_1(L)$ torsion $\Leftrightarrow$ $L$ has a flat reduction.

The class in $H^1(X, \mathbb{R})$ belong to a flat bundle should be the image under the homomorphism

$$\mathbb{C}^* \longrightarrow \mathbb{R}$$
$$z \longmapsto \log|z|.$$

Notice that this is a real characteristic class. For example, consider over $S^1$ the flat bundle in which the generator goes to multiplication by $\alpha \in \mathbb{C}^*$. Then the class in $H^1(S^1, \mathbb{R})$ is $\ln|\alpha| \cdot$ fdl. class.

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Real flat bundles: Here the invariant polynomials $\varphi$ which are of odd degree (like those giving the 2·odd degree Chern classes) will vanish for a connection compatible with an orthogonal reduction of the bundle. Thus if $\lambda$ is the homotopy ~~form~~ $\ni$

$$0 = \varphi(K(D)) - \varphi(K(D^{flat})) = d\lambda$$

$\lambda$ defines a cohomology class on $X$. Thus from the odd-style Chern classes we get elements

$$H^{4j+1}(X, \mathbb{R})$$

such as $\log|\det|$. These real classes should be restrictions of complex classes.

$SL_2(\mathbb{R})$: symmetric space is $S = SL_2(\mathbb{R})/SO_2$ = complex upper half plane

$$z \longmapsto \frac{az+b}{cz+d}$$

(stabilizer of $i$: $i(ci+d) = ai+b$
$\iff b=-c$ and $a=d$
$\iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ orthogonal )

Now we know the volume element of $S$ is an invariant real 2-form, hence defines an element $\lambda$ of $H^2(SL_2(\mathbb{R}), \mathbb{R})$ (discrete topology). I claim this element is non-trivial. It suffices to show there is a subgroup $\Gamma$ of $SL_2(\mathbb{R})$ discrete with compact quotient, for then the image of $\lambda$ in $H^2(\Gamma, \mathbb{R})$ will be the fundamental class times the volume of $S/\Gamma$. But if one takes a Riemann surface of genus $g>1$, it is of the form $S/\Gamma$. Incidentally, conjugating $\Gamma$ by the matrix

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \qquad a>0$$

changes the volume of $S/\Gamma$ by $a$, hence $\lambda \in H^2(SL_2(\mathbb{R}), \mathbb{R})$ is not an integral class, in fact the action of $GL_2(\mathbb{R})/SL_2(\mathbb{R}) = \mathbb{R}^*$ is multiplication.

## November 3, 1971:

Let $k$ be a field. I think I can prove the homotopy axiom:

$$K_*(k) \xrightarrow{\sim} K_*(k[z]).$$

It is necessary to show that

$$BGL(k)^+ \longrightarrow BGL(k[z])^+$$

~~induces isomorphisms~~ is a homotopy equivalence. There is a map backwards, namely, the one induced by $k[z] \to k$, $z \mapsto 0$. We have to show the composite

$$k[z] \longrightarrow k \longrightarrow k[z]$$

~~induces a map homotopic to the identity on the~~ induces the identity on K-groups. Using universal property, it is enough to show [redacted] homotopy commutativity in

$$\begin{tikzcd} BGL_n(k[z]) \arrow[r] \arrow[d, "in"'] & BGL_n(k) \arrow[d] \\ BGL(k[z])^+ & BGL_n(k[z]) \arrow[l, "in"] \end{tikzcd}$$

in fact, for any finite subcomplex $X$ of $BGL_n(k[z])$. Can suppose [redacted] is a finite simplicial complex endowed with a map $X \longrightarrow BGL_n(k[z])$, i.e. with a [redacted] locally constant sheaf $E$ of $k[z]$-modules with fibres $\simeq k[z]^n$.

At each vertex $v$ of $X$, choose a good $L_v \subset E_v$. Then for each 1-simplex $(v_0, v_1)$ of $X$ choose a good $L_{v_0 v_1} \subset E_{v_0 v_1}$ containing $L_{v_i}$. Continue in this fashion to define $L_\sigma \subset E_\sigma$ for simplices $\sigma$ of $X$ such that for $\sigma \subset \tau$, we have

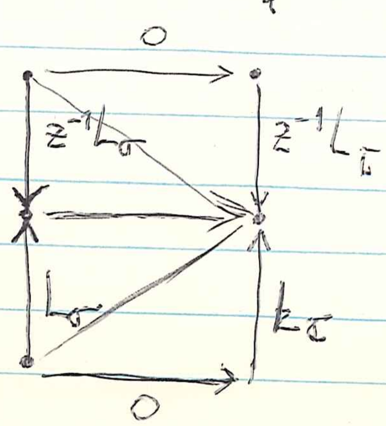
$$\begin{tikzcd} L_\sigma \arrow[r, hook] \arrow[d] & E_\sigma \arrow[d, "\cong"] \\ L_\tau \arrow[r, hook] & E_\tau \end{tikzcd}.$$

Next recall that there is a canonical isomorphism

$$L_\tau / L_\sigma \xleftarrow{\ \sim\ } z^{-1} L_\tau / z^{-1} L_\sigma$$

given by multiplication by $z$. (Injectivity clear. For surjectivity, use that $L_\tau \subset L_\sigma + zL_\sigma + \dots + z^m L_\sigma$. hence given $x \in L_\tau$ we have $\exists y \in L_\sigma$ with $x - y = z \cdot u$, with $u \in z^{-1} L_\tau$ as $x - y \in L_\tau$.)

Let $X'$ be the barycentric subdivision of $X$, and let $I'$ be the subdivision of $I = \Delta(1)$. Then $X' \times I'$ is a simplicial complex, and we have constructed a Roos style bundle over $X' \times I$



November 8, 1971.

I want to generalize Grothendieck's result that the K for vector bundles and coherent sheaves coincides for regular schemes. It is necessary to understand the process of choosing a resolution of a sheaf F by locally-free sheaves

$$0 \to L_n \to \cdots\cdots \to L_1 \to L_0 \to F \to 0.$$

Let K be a finite simplex complex, and let $\sigma \mapsto F_\sigma$ be a local coefficient system on K where each $F_\sigma$ is a coherent sheaf on a regular noetherian scheme S. Then we can write F as a quotient of a functor

$$P_\sigma = \bigoplus E_{\tau_i}\left[\mathrm{Hom}(\tau_i, \sigma)\right]$$

where $\tau_i$ runs over various simplices of K and $E_{\tau_i}$ is some locally free coherent sheaf on S. Note that

$$\sigma \subset \tau \implies P_\sigma \subset P_\tau$$

and that this property persists to subfunctors.

Now construct a resolution of F

$$0 \to P_n \to P_{n-1} \cdots\cdots \to P_1 \to P_0 \to F \to 0$$

where $P_n$ is defined as the kernel of the next map. Then $P_{n,\sigma}$ is loc. free over $S$ for each $\sigma$ provided $n \geq$ homological dim of $S$. Moreover

$$\sigma \subset \tau \Longrightarrow P_\sigma \longrightarrow P_\tau$$

injective with $P_\tau / P_\sigma$ locally-free and acyclic.

November 10, 1971

Let $E$ be a f.t. $k[z]$-module. Recall we say that a f.t. $k$-submodule $L \subset E$ is good if i) $L$ generates $E$, ii) $z^{-1}L \subset L$.

<u>Lemma:</u> Suppose $E' \subset E$ is a $k[z]$-submodule, and that $E'' = E/E'$. Given $L \subset E$, set $L' = L \cap E'$ and $L'' = L + E'/E'$. Then

$$\left.\begin{array}{l} L \text{ good for } E \\ + \; L' \text{ generates } E' \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} L' \text{ good for } E' \\ L'' \text{ good for } E'' \end{array}\right. .$$

<u>Proof:</u> $z^{-1}L' = z^{-1}L \cap E' \subset L \cap E' = L'$; thus $L'$ is good for $E'$. Clearly $L''$ generates $E''$. To show $z^{-1}L'' \subset L''$, we must show that if

$$ze = l + e' \qquad l \in L, \quad e \in E, \; e' \in E'$$

then $e \in L + E'$. As $L'$ generates $E'$ we have $e' \in L' + \dots + z^n L'$ for some $n$, hence

$$e' = l' + ze'_1$$

for some $l' \in L'$, $e'_1 \in E'$. Then

$$z(e - e'_1) = l + l' \in L$$

so $e - e'_1 \in z^{-1}L \subset L$; so $e \in L + E'$. Thus $L''$ is good for $E''$.

~~$\Leftarrow$) Given $e$, since $L''$ generates $E''$, we know that $e$ [illegible] thus $L$ generates $E$.~~
~~Suppose $ze \in L$; as $L''$ good, $e \in L + E'$. Assume~~

Defn: Given $E' \subset E$, call $L$ good for $E' \subset E$ if $L, L', L''$ good for $E, E', E''$ respectively.

Corollary: If $L$ good for $E' \subset E$, and if $L \subset L_1$ and $L_1$ is good for $E$, then $L_1$ is good for $E' \subset E$.

Proof. Have only to check that $L_1'$ generates $E'$; this is clear as $L_1' \supset L'$.

Defn: Given a $q$-filtered object

$$(*) \qquad 0 \subset F_1 E \subset \cdots \subset F_q E = E$$

say $L$ good for this filtered object if $F_j L / F_i L$ is good for $F_j E / F_i E$ for all $0 \leq i < j \leq n$.

It suffices for $L$ to be good for $E$ and $F_i L = L \cap F_i E$ to generate $F_i E$ for each $i = 1, \dots, n-1$.

Cor: If $L$ good for $(*)$ and $L'$ good for $E$ with $L \subset L'$, then $L'$ good for $(*)$.

November 12, 1971

Problem: Homotopy axiom $K_*^{coh}(A) \simeq K_*^{coh}(A[z])$, for a noetherian ring $A$.

Given an $A[z]$-module $E$ (f.t. always) we try to resolve it by

$$0 \longrightarrow A[z]\otimes_A z^{-1}L \longrightarrow A[z]\otimes_A L \longrightarrow E \longrightarrow 0$$

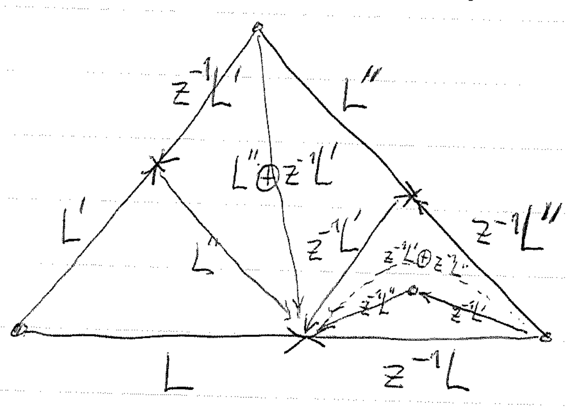
where $L$ is a good $A$-submodule of $E$. Thus I want to pass from $E$ to the pair $(L, z^{-1}L)$. I have seen already that $A$-modules + their isoms. can be effectively replaced by good $L$ and inclusions. It is now necessary to understand exact sequences, and more generally filtered $A$-modules.

Given $E' \subset E$ an $L \subset E$ is good for the filtration if $L$ is good for $E$ and if $L \cap E'$ generates $E'$. In this case I have established that $L' = L \cap E'$ is good for $E'$ and that $L'' = \operatorname{Im}\{L \to E'' = E/E'\}$ is good for $E''$.

Now if I think of a Roos system $\sigma\tau \mapsto E_{\sigma\tau}$ on a simplicial complex $K$, then an exact sequence appears as a 2-simplex of $K'$

$$\begin{tikzcd}
& \sigma \arrow[dr, dash, "E_{\sigma\tau}"] & \\
\rho \arrow[ur, dash, "E_{\rho\sigma}"] \arrow[rr, "E_{\rho\tau}"'] & & \tau
\end{tikzcd}$$

I am thinking of replacing $E$ by pairs $L, z^{-1}L$. It therefore appears necessary to see what an exact sequence should be replaced by. The best way I could find is as follows.




Terribly ugly. Transition steps: To relate $(L, z^{-1}L)$ to $(L', z^{-1}L') + (L'', z^{-1}L'')$, one has the following steps

$$\cdot \xrightarrow{L'} \cdot \xleftarrow{z^{-1}L'} \cdot \xrightarrow{L''} \cdot \xleftarrow{z^{-1}L''} \cdot$$

$$\cdot \xrightarrow{L'} \cdot \xrightarrow{L''} \cdot \xrightarrow{z^{-1}L'} \cdot \xleftarrow{z^{-1}L''} \cdot$$ change of order

$$\cdot \xrightarrow{L'} \cdot \xrightarrow{L''} \cdot \xleftarrow{z^{-1}L''} \cdot \xleftarrow{z^{-1}L'} \cdot$$ "

$$\cdot \xrightarrow{L} \cdot \xleftarrow{z^{-1}L'} \cdot$$ exact sequences

Other possibility: suppose instead of a Roos system, we use a $E^{\pm}_{\sigma}$ system. Then what we can do is to choose $L^{\pm}_{\sigma} \subset E^{\pm}_{\sigma}$ such that

i) $L^{\pm}_{\tau}$ good relative to all $E^{\pm}_{\sigma} \subset E^{+}_{\tau}$ for $\sigma \subset \tau$.

ii) $\sigma \subset \tau \implies L^{\pm}_{\sigma} \subset L^{\pm}_{\tau}$

iii) $\operatorname{Im}\{L^{+}_{\tau} \longrightarrow E_{\sigma\tau}\} \supset \operatorname{Im}\{L^{-}_{\tau} \longrightarrow E_{\sigma\tau}\}$.

In effect one constructs $L^{\pm}_{\sigma}$ by induction on $\dim(\sigma)$. Thus if this has been done for all $\sigma < \tau$, one chooses $L^{-}_{\tau}$ sufficiently big to contains all $L^{-}_{\sigma}$ and to be good relative to all $E^{-}_{\sigma}$. Then one chooses $L^{+}_{\tau}$ sufficiently big so that i) ii) and iii) hold.

Now we associate to $\sigma$ the element

$$[L^{+}_{\sigma}] - [z^{-1}L^{+}_{\sigma}] - [L^{-}_{\sigma}] + [z^{-1}L^{-}_{\sigma}]$$

and when $\sigma \subset \tau$ we have the following transitions

$$
\begin{array}{ccccccc}
L^{+}_{\sigma} & - & z^{-1}L^{+}_{\sigma} & - & L^{-}_{\sigma} & + & z^{-1}L^{-}_{\sigma} \\[1em]
E^{+}_{\sigma} \cap L^{+}_{\tau} & - & z^{-1}(E^{+}_{\sigma} \cap L^{+}_{\tau}) & - & E^{-}_{\sigma} \cap L^{-}_{\tau} & + & z^{-1}(E^{-}_{\sigma} \cap L^{-}_{\tau}) \\[1em]
A & - & z^{-1}A & - & L^{-}_{\tau} & + & z^{-1}L^{-}_{\tau} \\[1em]
L^{+}_{\tau} & - & z^{-1}L^{+}_{\tau} & - & L^{-}_{\tau} & + & z^{-1}L^{-}_{\tau}
\end{array}
$$

where $A =$ inverse image in $L^{+}_{\tau}$ of $\operatorname{Im}\{L^{-}_{\tau} \to E_{\sigma\tau}\}$ (still need to check $A$-good.)

# November 14, 1971:

Trying to get the $\pm$ systems to work.

Notation: $E = (E^+, E^-)$, morphism $i : E \longrightarrow F$ consists of $i^\pm : E^\pm \longrightarrow F^\pm$; oriented morphism

$$(i, \xi) : E \longrightarrow F$$

where $i : E \longrightarrow F$ is a morphism and

$$\xi : \ Cok(i^+) \xrightarrow{\sim} Cok(i^-)$$

is the orientation.

Note that we have cobase change

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "j"'] & F \arrow[d, "j'"] \\ V \arrow[r, "i'"'] & W \end{tikzcd}$$

and that

$$Cok(i) \xrightarrow{\sim} Cok(i')$$
$$Cok(j) \xrightarrow{\sim} Cok(j')$$
$$Cok(j'i) = Cok(i) \oplus Cok(j)$$

thus an orientation for $i$ (resp. $j$, resp. $i$ and $j$) induces one for $i'$ (resp. $j'$, resp. $j'i$).

Unfortunately orientations do not compose. Thus given

$$E \xrightarrow{(i,\xi)} E' \xrightarrow{(j,\eta)} E''$$

we have an exact sequence of pairs

$$0 \longrightarrow \mathrm{Cok}(i) \longrightarrow \mathrm{Cok}(j) \longrightarrow \mathrm{Cok}(ji) \longrightarrow 0$$

but the trivializations $\xi, \eta$ need not extend to $\mathrm{Cok}\, j$, and if they do the extension is not usually unique.
Two simplex of oriented maps

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & E' \arrow[dl] \\ & E'' & \end{tikzcd}$$

More generally $n$-simplex. Observe a cocartesian square is built up of 2-simplices

$$\begin{tikzcd} E \arrow[r, "{(i,\xi)}"] \arrow[d, "{(j,\eta)}"'] \arrow[dr, "{(j'i,\zeta)}"] & F \arrow[d, "{(j',\eta')}"] \\ V \arrow[r, "{(i',\xi')}"'] & W \end{tikzcd}.$$

$\zeta$ direct sum of $\xi$ and $\eta$ relative to isom.

$$\mathrm{Cok}(j'i) = \mathrm{Cok}(j') \oplus \mathrm{Cok}(i)$$

<u>Problem</u>: Show the path

$$E \xrightarrow{(i,\xi)} V \xleftarrow{(i,\eta)} E$$

is equivalent to the path

$$E \xrightarrow{(in_1,\xi)} E \oplus C \xleftarrow{(in_1,\eta)} E$$

where $C = \mathrm{Coker}(i)$.

<u>Solution</u>: Trivial if $V = E \oplus C$. So reduce to this case by splitting the extension by a map $j : E \longrightarrow E'$

$$\begin{tikzcd}
E \arrow[r, "i"] \arrow[d, "j"] \arrow[dr, dashed] & V \arrow[d, "j"] & E \arrow[l, "i"'] \arrow[d, "j"] \arrow[dl, dashed] \\
E' \arrow[r, "i'"] & V' & E' \arrow[l, "i'"'] \\
E \arrow[u, "j"] \arrow[r, "in_1"] \arrow[ur, dashed] & E \oplus C \arrow[u] & E \arrow[u, "j"'] \arrow[l, "in_1"'] \arrow[ul, dashed]
\end{tikzcd}$$

All squares cocartesian. Let $\alpha$ be any orientation for $j$ (can take $j = i$ and $\alpha = \xi$). Then all squares break into triangles, so done.

<u>Problem:</u> $I$ ideal in $A \ni I^2 = 0$. Let $\sigma \mapsto E_\sigma = (E_\sigma^+, E_\sigma^-)$ be a $K$-system of $A$-modules over $X$. To decompose it into $A/I$-modules.

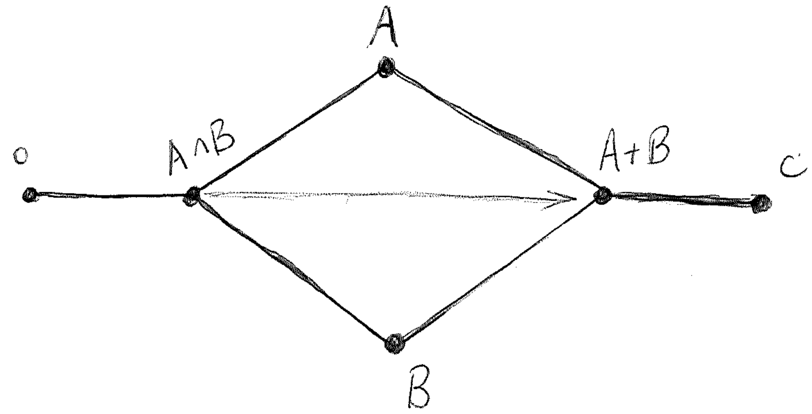
First attempt: For any $A$-module $M$ let
$$\mathfrak{a}(M) = \{m \in M \mid Im = 0\}.$$
Then both $\mathfrak{a}(M)$ and $M/\mathfrak{a}(M)$ are killed by $I$ so we might try
$$\sigma \longmapsto \left(\mathfrak{a}E_\sigma^+ \oplus E_\sigma^+/\mathfrak{a}E_\sigma^+,\ \mathfrak{a}E_\sigma^- \oplus E_\sigma^-/\mathfrak{a}E_\sigma^-\right)$$

Suppose $\sigma \subset \tau$. Then
$$E_\sigma \cap \mathfrak{a}(E_\tau) = \mathfrak{a}(E_\sigma)$$
so
$$\mathfrak{a}(E_\sigma) \hookrightarrow \mathfrak{a}(E_\tau)$$
$$E_\sigma/\mathfrak{a}(E_\sigma) \hookrightarrow E_\tau/\mathfrak{a}(E_\tau).$$

Cokernel of first inclusion is
$$\mathfrak{a}E_\tau/\mathfrak{a}E_\sigma \simeq \mathfrak{a}E_\tau + E_\sigma/E_\sigma \subset E_\tau/E_\sigma$$
and of second inclusion is
$$E_\tau/\mathfrak{a}E_\tau + E_\sigma.$$

$$\begin{tikzcd}[column sep=small]
 & & \mathfrak{a}E_\tau^+ + E_\sigma^+/E_\sigma^+ \arrow[dr, phantom, "\subset"] \arrow[drr, bend left, "2"] & & \\
 & I\cdot E_\tau^+/E_\sigma^+ \arrow[ur, phantom, "\subset"] \arrow[urr, bend left, "1"] \arrow[d, phantom, "\cong"] & & \mathfrak{a}(E_\tau^+/E_\sigma^+) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cong"] & E_\tau^+/E_\sigma^+ \arrow[d, equal] \\
0 \arrow[ur] \arrow[r] \arrow[dr] & I\cdot E_\tau^-/E_\sigma^- \arrow[dr, phantom, "\subset"] \arrow[drr, bend right, "3"'] & & \mathfrak{a}(E_\tau^-/E_\sigma^-) \arrow[r, phantom, "\subset"] & E_\tau^-/E_\sigma^- \\
 & & \mathfrak{a}E_\tau^- + E_\sigma^-/E_\sigma^- \arrow[ur, phantom, "\subset"] \arrow[urr, bend right, "4"'] & &
\end{tikzcd}$$

Now the problem is that ① ⊕ ② and ③ ⊕ ④ need not be isomorphic. But quite generally, given two filtrations we refine them:



Ultimately it will be necessary to understand trivializations of K-systems. Simplest examples:

$$E \xrightarrow{(i,\beta)} V \xleftarrow{(j,\alpha)} O$$

where

$$\alpha : V^+ \xrightarrow{\sim} V^-$$
$$\beta : C^+ \xrightarrow{\sim} C^-$$

$C = \mathrm{Coker}(i)$. Two cases:

I.) $E^+ \oplus \alpha^{-1}(E^-) \xrightarrow{\sim} V^+$. To simplify suppose that $\alpha = id$, $V^+ = V^-$. Then $E^+ \oplus E^- \xrightarrow{\sim} V$, so we have canonical isomorphisms

$$E^+ \simeq C^-$$
$$E^- \simeq C^+$$

and in particular a map

$$O \xrightarrow{(0,-\beta^{-1})} E .$$

The composite map

$$O \xrightarrow{(0,-\beta^{-1})} E \xrightarrow{(i,\beta)} V$$

$$(E^+, E^-) \xrightarrow{(in_1, in_2, \beta)} (E^+ \oplus E^-, E^+ \oplus E^-)$$

is the automorphism $\theta : V \longrightarrow V$ given by

$$\theta(x,y) = (\beta y, -\beta^{-1} x)$$

Strictly speaking we have a triangle

$$\begin{tikzcd} & E \arrow[dr, "(i,\beta)"] & \\ 0 \arrow[ur, "(0,-\beta^{-1})"] \arrow[rr, "(0,\Theta)"'] & & V \end{tikzcd}$$

Now $\Theta$ is a product of elementary matrices

$$\begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\beta^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\beta^{-1} & 1 \end{pmatrix}$$

hence the path $(0,\Theta)$ will be homotopic to the identity.

II.) $E^+ = \alpha^{-1}(E^-)$. Then the path

$$E \xrightarrow{(i,\beta)} V \xleftarrow{(0,\alpha)} 0$$

is homotopic to

$$E \xrightarrow{(i,\beta)} V \xleftarrow{(i,\alpha'')} E \xleftarrow{(0,\alpha')} 0$$

where $\alpha' : E^+ \longrightarrow E^-$, $\alpha'' : C^+ \longrightarrow C^-$ are the maps induced by $\alpha$. On page 3 we saw this was equivalent to

$$\begin{tikzcd} E \arrow[r, "(in_1,\beta)"] & E \oplus C & E \arrow[l, "(in_1,\alpha'')"'] & 0 \arrow[l, "(0,\alpha')"'] \\ 0 \arrow[u, "(0,\alpha')"] \arrow[r, "(0,\beta)"'] & C \arrow[u, "(in_2,\alpha')"] & 0 \arrow[l, "(0,\alpha'')"] \arrow[u, "(0,\alpha')"] & \end{tikzcd}$$

Consequently the original path

$$E \xrightarrow{(i,\beta)} V \xleftarrow{(0,\alpha)} O$$

is homotopic to

$$E \xleftarrow{(0,\alpha')} O \xrightarrow{(0,\beta)} C \xleftarrow{(0,\alpha'')} O$$

<u>Better proof.</u> Let $p: V \to C$ be the natural surjection. Then we have three bicart. squares and a triangle

$$\begin{tikzcd}
E \arrow[r, "{(i,\beta)}"] \arrow[d, "{(i,\alpha'')}"] & V \arrow[d, "{(in_1,\alpha'')}"] & O \arrow[l, "{(0,\alpha)}"'] \arrow[ddl, "{(0,\alpha'')}"] \\
V \arrow[r, "{(in_1+p,\beta)}"'] & V\oplus C & \\
O \arrow[u, "{(0,\alpha)}"] \arrow[r, "{(0,\beta)}"'] \arrow[uu, bend left, "{(0,\alpha')}"] & C \arrow[u, "{(in_2,\alpha)}"'] &
\end{tikzcd}$$

---

Why elementary autos are homotopic to the identity. Given

$$O \xrightarrow{\alpha} V \xleftarrow{\beta} O$$

such that $\beta^{-1}\alpha : V^+ \to V^+$ is elementary, i.e. $\exists$

$$O \to V'^{+} \to V^+ \to V''^{+} \to O$$

such that $\beta^{-1}\alpha$ induces identity on $V'^+$ and $V'^-$. Then we have

$$\begin{tikzcd}
& V & \\
o \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & & O \arrow[ul, "\beta"'] \arrow[dl, "\beta'"] \\
& V' \arrow[uu, "\alpha'' = \beta''"] &
\end{tikzcd}$$

so it's clear.

November 15, 1971: Substitute for Tits' building.

Let $K$ be a local field and $V$ a $K$-vector space of finite dimension. Let $X$ be the simplicial complex whose $q$-simplices are chains of lattices in $V$

$$L_0 \subset L_1 \subset \cdots \subset L_q$$

such that $L_q/L_0$ is a $k$-module. Claim $X$ is contractible.

Proof: Let $K$ be a finite subcomplex of $X$; it suffices to contract $K$ within $X$. $K$ is contained in the subcomplex $X(L_a, L_b)$ of lattices between $L_a$ and $L_b$ for some $L_a$ and $L_b$. It suffices to show $X(L_a, L_b)$ is contractible. If $L_a \subset L_b \subset L_c$ and $L_c/L_b$ is a $k$-module, we will show that

$$X(L_a, L_b) \longrightarrow X(L_a, L_c)$$

$$L \longmapsto L \cap L_c$$

is a collapsing map. The point is that given a simplex

$$\begin{array}{ccccccc} L_a \subset & L_0 & \subset \cdots \subset & L_q & \subset L_b \\ & \cup & & \cup & \\ & L_0 \cap L_c & \subset \cdots \subset & L_q \cap L_c & \subset L_c \end{array}$$

we have that $L_q / L_0 \cap L_c \subset L_q/L_0 \times L_b/L_c$

is a $k$-module so we have a map

$$\Delta(q) \times \Delta(1) \longrightarrow X(L_a, L_b)$$

furnishing the required homotopy.

---

Stabilizer of a simplex $\sigma$ under the $\mathrm{Aut}_K(V)$ action is compact.

$$\dim X = \dim_K V.$$

---

Key point is that if

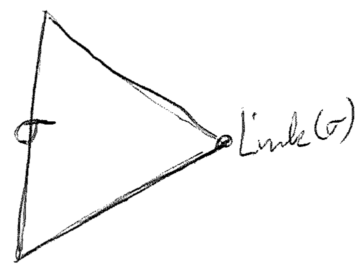
$$\begin{tikzcd} & B \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & D \\ & C \arrow[ur] & \end{tikzcd}$$

is cartesian, then

$$D/A \subset D/B \times D/C$$

so that if $D/B$ and $D/C$ are killed by $m$ so is $D/A$.

Recall that in a simplicial complex the link of a simplex $\sigma$ describes the normal structure




i.e. $$U_\sigma \overset{\text{homeo.}}{\cong} \operatorname{Int}\operatorname{Cone}\{\operatorname{Link}(\sigma)\} \times \operatorname{Int}\sigma$$

hence

$$H^i_c(U_\sigma) = H^{i-d}(\operatorname{Cone}\operatorname{Link}(\sigma), \operatorname{Link}(\sigma)), \quad d = \dim \sigma$$

$$= \tilde{H}^{i-d-1}(\operatorname{Link}(\sigma))$$

(with convention that when $\operatorname{Link}(\sigma)$ is empty, then

$$\tilde{H}^j(\phi) = \begin{cases} \mathbb{Z} & j = -1 \\ 0 & j \neq -1. \end{cases}$$

Thus the reduced cohomology of the link is the same except for dimension-shift as the local cohomology of the space.

In the Tits immeuble the link is essentially the immeuble for the residue field. It would be nice if the local cohomology of the simplicial complex $X$ were concentrated in 1-dimension $n = \dim V$.

Example 1: Take a simplex

$$\sigma = L_0 \subset \cdots \subset L_8$$

with $L_0 = \pi L_q$, $(\pi) = \mathfrak{m}$. Then a simplex $\tau$ in $\mathrm{Link}(\sigma)$ is the same thing as a family $\tau_i$ where $(L_{i-1}, \tau_i, L_i)$ is a simplex. $\tau_i \neq \phi$ for some $i$. Thus

$$\mathrm{Link}(\sigma) = \operatorname*{Join}_{i=1,\dots,q} \mathrm{Imm}(L_i / L_{i-1}).$$

Recall that

$$\tilde{H}^{*+1}(X * Y) = \tilde{H}^{*+1}(X) \otimes \tilde{H}^{*+1}(Y)$$

Now by Tit's theorem

$$\mathrm{Imm}(W) \sim \bigvee S^{d-2} \qquad d = \dim W$$

in number $q^{\frac{d(d-1)}{2}}$ = order of Sylow group. It follows that

$$\mathrm{Link}(\sigma) \sim \bigvee S^{n-q-1} \qquad \sum_{i=1}^{q}(d_i - 1) - 1$$

in number

$$q^{\sum \frac{d_i(d_i-1)}{2}}$$

Sylow subgroup of stabilizer of $L_1/L_0 \subset \cdots \subset L_q/L_0$ in $GL(\pi^{-1}L_0/L_0)$

Example 2: Take a single $L$. Remove vertex $\pi L$ from Link$(\sigma)$. Claim that

$$\mathrm{Link}(\sigma) - L \ \overset{\text{collapses}}{} \ \text{to} \ \overset{\text{Cone}}{\mathrm{Imm}}(\pi^{-1}L/L)$$

Send $\pi L < V < L$ into $\pi^{-1}V$, and the homotopy for $\pi L < V_0 < \cdots < V_k < L$ is

$$\begin{tikzcd}
\pi^{-1}V_0 & \cdots & \pi^{-1}V_k \\
V_0 \arrow[u] \arrow[r] & \cdots \arrow[ul] \arrow[r] & V_k \arrow[u] \arrow[ul]
\end{tikzcd}$$

~~What I am saying is that we can map~~

Given a vertex $V$ two possibilities:

(i) $\pi L < V < L$. Send $V \mapsto \pi^{-1}V$

(ii) $L < V \subset \pi^{-1}L$. Send $V \longmapsto V$.

This map is simplicial: a simplex

$$V_0 < \cdots < V_r \overset{<L<}{<} V_{r+1} \cdots < V_s \qquad (\text{note } V_s \subset \pi^{-1}V_0)$$

goes to

$$V_{r+1} < \cdots < V_s \subset \pi^{-1}V_0 < \cdots < \pi^{-1}V_r .$$

This map is a collapsing. Order the vertices by $f(V) = \dim L/V$ if $V < L$ and $f(V) = 0$ if $V > L$. Then we have various subcomplexes $K_j = \{V \mid f(V) \le j\}$, and we collapse $K_j$ to $K_{j-1}$ by sending

$$V_0 < \cdots V_r < L < V_{r+1} < \cdots < V_s$$

to $\qquad V_1 \qquad\qquad\qquad V_s \subset \pi^{-1} V_0$

Note that we are sliding through the simplex

$$V_0 < \cdots \qquad\qquad \cdots \pi^{-1} V_0$$

so this is a collapsing map.

So the moral of this calculation is that

$$\mathrm{Link}(\sigma) \sim \underset{\mathrm{Susp}}{\overset{\shortparallel}{\Sigma}} \mathrm{Imm}(\pi^{-1}L/L).$$

hence it is a bouquet of $(n-1)$-spheres, in number

$$q^{\frac{n(n-1)}{2}}$$

= order of Sylow group of $GL(\pi^{-1}L/L)$.

Example 3: $L_0 \subset \cdots \subset L_g$ where $L_g < \pi^{-1}L_0$. Then we will move $\pi^{-1}L_g < V < L_0$ to $\pi^{-1}V > L_g$.

$$\mathrm{Link}(\sigma) = pt * \mathrm{Im}(L_1/L_0) * \cdots * \mathrm{Im}(L_g/L_{g-1}) * \mathrm{Im}(\pi^{-1}L_0/L_g) * pt$$

is $\sim$ bouquet of $S^{n-g-1}$ in number

$$q^{\sum_i \frac{d_i(d_i-1)}{2} + \frac{(n-\sum d_i)(n-\sum_i d_i - 1)}{2}}$$

= order of Sylow group of $\mathrm{Aut}(L_1/L_0 \subset \cdots \subset L_g/L_0 \subset \pi^{-1}L_0/L_0)$

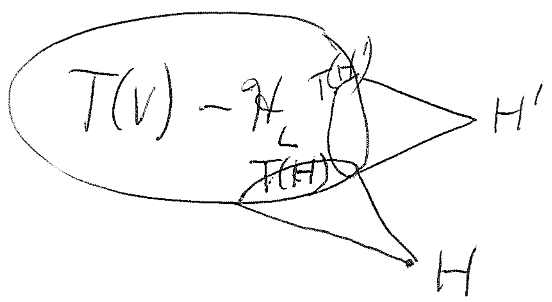
$V$ vector space of dim $n$ over a field $K$. Let $T$ be the simplicial complex whose simplices are chains of proper subspaces of $V$. Claim $T$ is $\sim$ a bouquet of $(n-2)$-spheres. Enough to show that $T$ is $(n-3)$-connected. Use following argument from Folkman (J. Math. Mech. 15 (1966) 631–636). May assume $n \geq 3$.

Let $C$ be lines in $V$ and make $C$ a simplicial complex by calling simplices subsets which do not span $V$. Then any subset of $C$ of card $(q+1)$ with $q \leq n-2$ is a $q$-simplex. Thus $C$ has the same $(n-2)$-skeleton as a big simplex, so it is $(n-3)$-connected. It remains to show that $C$ and $T$ have the same homotopy type. But for each line $x \in C$ let $L_x \subset T$ be subcomplex of subspaces containing $x$. Then if $(x_0, \ldots, x_q)$ is a subset of $C$

$$L_{x_0} \cap \cdots \cap L_{x_q} = \begin{cases} \phi & \text{if } (x_0, \ldots, x_q) \text{ spans } V \text{ i.e. not a simplex} \\ \text{subspaces containing } x_0 + \cdots + x_q. \end{cases}$$

Thus the nerve of the covering $\{L_x\}$ is the simplicial complex $C$ and since each intersection is contractible we are done.

$V$ vector space of dimension $n$ over $k$, $T(V)$ the simplicial complex whose simplices are chains of proper subspaces. Let $L$ be a line in $V$, and $\mathcal{H}_L$ the set of complementary hyperplanes. Then mapping $W \longmapsto W+L$ is a defm. retraction of $T(V) - \mathcal{H}_L$ onto the subcomplex of $W$ containing $L$, i.e. the closed star of $L$. Thus $T(V) - \mathcal{H}_L$ is contractible. On the other hand $\mathcal{H}_L$ contains only vertices and the link of $H \in \mathcal{H}_L$ is simply $T(H)$. Thus




so $T(V)$ has the homotopy type of

$$\bigvee_{H \in \mathcal{H}_L} \Sigma\, T(H)$$

showing by induction that $T(V) \sim \bigvee S^{n-2}$ in number $q^{\frac{n(n-1)}{2}}$ if $q = \operatorname{card}(k)$.

<u>November 17, 1971.</u> more on K-systems

Suppose we have the standard d.v.r. situation:

$$k \longleftarrow A \longrightarrow K$$

Let $E = (E_\sigma^+, E_\sigma^-)$ be a K-system of A-modules over a simplicial complex X. Assume that $E \otimes K$ is trivial. Then we want to prove $E$ comes from a K-system of A-modules.

<u>Special case</u>: suppose that $E_\sigma^\pm$, $E_\tau^\pm / E_\sigma^\pm$ are all torsion-free over A. Suppose that $E \otimes K$ is trivialized by an isomorphism

$$\alpha : E_\sigma^+ \otimes K \xrightarrow{\sim} E_\sigma^- \otimes K$$

such that $\alpha(E_\sigma^+) \supset E_\sigma^-$. By compatibility of $\alpha$ we have

$$\begin{tikzcd}
E_\sigma^+ \otimes K \arrow[r] \arrow[d, "\alpha"] & E_\tau^+ \otimes K \arrow[r] \arrow[d, "\alpha"] & (E_\tau^+ / E_\sigma^+) \otimes K \arrow[d, "\mathrm{can} \otimes K"] \\
E_\sigma^- \otimes K \arrow[r] & E_\tau^- \otimes K \arrow[r] & (E_\tau^- / E_\sigma^-) \otimes K
\end{tikzcd}$$

commutes. Thus as things are torsion-free

$$\begin{tikzcd}
0 \arrow[r] & E_\sigma^- \arrow[r] \arrow[d, "\alpha^{-1}"] & E_\tau^- \arrow[r] \arrow[d, "\alpha^{-1}"] & E_\tau^- / E_\sigma^- \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E_\sigma^+ \arrow[r] & E_\tau^+ \arrow[r] & E_\tau^+ / E_\sigma^+ \arrow[r] & 0
\end{tikzcd}$$

commutes, hence

if we set $\quad W_\sigma = E_\sigma^+/\alpha^{-1}(E_\sigma^-)\quad$ we have canon. isomorphism

$$W_\sigma \xrightarrow{\sim} W_\tau$$

showing that $W$ is a local coefficient system of fin. length $A$-modules on $X$, and that $E$ is the $K$-system obtained by projective resolution of $W$.

Now drop the condition $\alpha(E_\sigma^+) \supset E_\sigma^-$. Then there exists an integer $N$ such that

$$\alpha(\pi^N E_\sigma^+) \subset E_\sigma^-$$

for all $\sigma$. Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & E_\sigma^- \arrow[r] & E_\tau^- \arrow[r] & E_{\sigma\tau} \arrow[r] & 0 \\
0 \arrow[r] & \pi^N E_\sigma^+ \arrow[r] \arrow[u, "\alpha"] \arrow[d, hook] & \pi^N E_\tau^+ \arrow[r] \arrow[u, "\alpha"] \arrow[d, hook] & \pi^N(E_{\sigma\tau}) \arrow[r] \arrow[u, hook] \arrow[d, hook] & 0 \\
0 \arrow[r] & E_\sigma^+ \arrow[r] & E_\tau^+ \arrow[r] & E_{\sigma\tau} \arrow[r] & 0
\end{tikzcd}$$

commutes, because things are torsion-free and it commutes after $\otimes K$. Set

$$W_\sigma^{-} = \mathrm{Coker}\,\{\alpha : \pi^{+N} E_\sigma^+ \to E_\sigma^-\}$$

$$W_\sigma^{+} = E_\sigma^+/\pi^N E_\sigma^+$$

Then from serpent considerations we have for $\sigma \subset \tau$

$$\begin{array}{ccccccccc} 0 \to & W_\sigma^+ & \hookrightarrow & W_\tau^+ & \longrightarrow & E_{\sigma\tau}/\pi^N E_{\sigma\tau} & \to 0 \\ & & & & & \wr\| & \\ 0 \to & W_\sigma^- & \hookrightarrow & W_\tau^- & \longrightarrow & E_{\sigma\tau}/\pi^N E_{\sigma\tau} & \to 0 \end{array}$$

so that $W = (W^+, W^-)$ is a K-system of finite length $A$-modules.

<u>Critical problem</u>: It seems that I construct not K-systems but pairs $M = (M^+, M^-)$ such that $M_\sigma \hookrightarrow M_\tau$ for $\sigma \subset \tau$, but such that $M_\tau^+/M_\sigma^+$ and $M_\tau^-/M_\sigma^-$ are graded modules associated to two filtrations of the same module.

<u>Example 1</u>: Consider the situation of p. 4 where we have a K-system of $A$-modules $E$ which I want to break up into $A/I$-modules. Then I get

$$M = \alpha(E) \oplus E/\alpha(E)$$

and the cokernel is

$$M_\tau/M_\sigma = \left(E_\sigma + \alpha(E_\tau)/E_\sigma\right) \oplus \left(E_\tau/E_\sigma + \alpha(E_\tau)\right).$$

Thus we get the graded modules of $E_\tau^+/E_\sigma^+ \simeq E_\tau^-/E_\sigma^-$ with respect to the submodules

$$E_\sigma^+ + \alpha(E_\tau^+)/E_\sigma^+ \quad \text{and} \quad E_\sigma^- + \alpha(E_\tau^-)/E_\sigma^- .$$

*Example 2*: Suppose $A$ is Dedekind and we wish to replace a $K$-system $W = (W^+, W^-)$ by projective modules. We can choose resolutions

$$0 \longrightarrow B_\sigma \longrightarrow E_\sigma \longrightarrow W_\sigma \longrightarrow 0$$

where $E_\sigma$ and $B_\sigma$ are projective over $A$ such that $E_\sigma \hookrightarrow E_\tau$ and $B_\sigma \hookrightarrow B_\tau$ when $\sigma \subset \tau$. Then if

$$(M^+, M^-) = (E^+ \oplus B^-, E^- \oplus B^+)$$

the cokernel is

$$E^+_\tau / E^+_\sigma \oplus B^-_\tau / B^-_\sigma \;,\; E^-_\tau / E^-_\sigma \oplus B^+_\tau / B^+_\sigma .$$

Since we have exact diagram

$$\begin{tikzcd}
 & 0 & \cdot \arrow[d] \arrow[r, phantom, "\simeq"] & B^+_\tau / B^+_\sigma \arrow[d] & \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d, "\wr"] & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & E^+_\tau / E^+_\sigma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^-_\tau / B^-_\sigma \arrow[r] \arrow[d] & E^-_\tau / E^-_\sigma \arrow[r] \arrow[d] & W_\tau / W_\sigma \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

these are two graded modules associated to $Z$.

<u>Example 3</u>: Consider the homotopy situation. Given $E = (E^+, E^-)$ a K-system of $A[z]$-modules, and we choose $L_\sigma \subset E_\sigma$ $\ni$ i) good, ii) $\sigma \subset \tau \Longrightarrow L_\sigma \subset L_\tau$ and that $L_\tau$ is good for $E_\sigma \subset E_\tau$. (can be done – see Nov. 12). Set

$$M_\sigma = \left(L_\sigma^+ \oplus z^{-1}L_\sigma^-,\ L_\sigma^- \oplus z^{-1}L_\sigma^+\right).$$

Then

$$M_\tau^\bullet / M_\sigma^\bullet = \left(L_\tau^+/L_\sigma^+ \oplus z^{-1}L_\tau^-/z^{-1}L_\sigma^-,\ L_\tau^-/L_\sigma^- \oplus z^{-1}L_\tau^+/z^{-1}L_\sigma^+\right)$$

Suppose that $E_{\sigma\tau}$ is $z$-torsion-free and denote by a bar $\otimes_{A[z]} A$. Then

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & z^{-1}L_\sigma \arrow[r, "z"] \arrow[d] & L_\sigma \arrow[r] \arrow[d] & \bar{E}_\sigma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & z^{-1}L_\tau \arrow[r, "z"] \arrow[d] & L_\tau \arrow[r] \arrow[d] & \bar{E}_\tau \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & z^{-1}L_\tau/z^{-1}L_\sigma \arrow[r, "z"] \arrow[d] & L_\tau/L_\sigma \arrow[r] \arrow[d] & \bar{E}_{\sigma\tau} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

– z torsion-free used here

Thus we have

$$\begin{tikzcd}
0 \arrow[r] & z^{-1}L_\tau^+/z^{-1}L_\sigma^+ \arrow[r, ".z"] & L_\tau^+/L_\sigma^+ \arrow[r] & \bar{E}^+_{\sigma\tau} \arrow[r] \arrow[d, phantom, "\wr|"] & 0 \\
0 \arrow[r] & z^{-1}L_\tau^-/z^{-1}L_\sigma^- \arrow[r, ".z"] & L_\tau^-/L_\sigma^- \arrow[r] & \bar{E}^-_{\sigma\tau} \arrow[r] & 0
\end{tikzcd}$$

and so we are in the same situation as the preceding examples. Without the hypothesis that $E_{\sigma\tau}$ be $z$-torsion-free we have exact sequences

$$0 \to \mathrm{Tor}_1^{A[z]}(E_{\sigma\tau}, A) \longrightarrow z^{-1}L_\tau / z^{-1}L_\sigma \xrightarrow{\cdot z} L_\tau / L_\sigma \longrightarrow E_{\sigma\tau} \otimes_{A[z]} A \to 0$$

which again constitute the trivialization of $M_\tau / M_\sigma$ we are after.

November 19, 1971

<u>Problem</u>: Let $E$ be a $K$-system of $A$-modules. Why is $(E^-, E^+)$ the "negative" of $E$?

Intuitively $(E^+, E^-) + (E^-, E^+) = (E^+ \oplus E^-, E^- \oplus E^+)$ is trivial because of the isomorphism

$$T : E^+_\sigma \oplus E^-_\sigma \xrightarrow{\sim} E^-_\sigma \oplus E^+_\sigma$$

$$T(x, y) = (y, x).$$

Unfortunately this is not compatible with inclusions:

$$\begin{tikzcd}
E^+_\sigma \oplus E^-_\sigma \arrow[d] & E^-_\sigma \oplus E^+_\sigma \arrow[d] \\
E^+_\tau \oplus E^-_\sigma \arrow[d] & E^-_\tau \oplus E^+_\tau \arrow[d] \\
E^+_{\sigma\tau} \oplus E^-_{\sigma\tau} \arrow[r, "\gamma+\gamma^{-1}"] & E^-_{\sigma\tau} \oplus E^+_{\sigma\tau}
\end{tikzcd}$$

and $T \neq \gamma + \gamma^{-1}$ as isom. $E^+_{\sigma\tau} \oplus E^-_{\sigma\tau} \xrightarrow{\sim} E^-_{\sigma\tau} \oplus E^+_{\sigma\tau}$

Actually it becomes clear now that perhaps $T$ should become $(x, y) \longmapsto (y, -x)$ in order that

$$T^{-1}(\gamma + \gamma^{-1})(x, y) = T^{-1}(\gamma x, \gamma^{-1} y) = (-\gamma^{-1} y, \gamma x)$$

should be equivalent to the identity automorphism.

Let $I$ be a nilpotent ideal in a ring $A$. If $M$ is an $A$-module, let $X(M)$ be the simplicial complex whose $q$-simplices are chains of submodules

$$N_0 < N_1 < \cdots < N_q$$

of $M$ such that $I \cdot N_q \subset N_0$. I recall that $X(M)$ is contractible, in fact that if $M' \subset M$ is such that $I(M/M') = 0$, then

$$X(M) \longrightarrow X(M')$$
$$N \longmapsto N \cap M'$$

is a deformation retraction.

Let $\underline{M}$ be a filtered $A$-module of length $p$:

$$0 \subset M_1 \subset \cdots \subset M_p$$

and let $X(\underline{M})$ be the full subcomplex of $X(M_p)$ whose vertices $N$ are submodules such that $M_{j-1} \subseteq N \subset M_j$ for some $j = 1, \ldots, p$. I claim that $X(\underline{M})$ is contractible. Indeed let $M_{p-1} \subset M' \subset M_p$ be such that $IM_p \subset M'$. Then the retraction $N \mapsto N \cap M'$ is a deformation retraction from $X(M_1 \subset \cdots \subset M_p)$ to $X(M_1 \subset \cdots \subset M_p \subset M')$. Thus it is clear that $X(M_1 \subset \cdots \subset M_p)$ def-retracts to $X(M_1 \subset \cdots \subset M_{p-1})$ and so by induction, it is contractible.

As usual, let $\Delta$ be the category of ~~simplexes~~ $[n] = \{0, \ldots, n\}$ with monotone maps, and let $R(A)$ be the category whose objects are pairs $(n, \underline{M})$ where $n \geq 0$ and $\underline{M}$ is a filtered $A$-module of finite type of length $n$:

$$M_1 \subset \cdots\cdots \subset M_n.$$

By a morphism $(n, \underline{M}) \longrightarrow (n', \underline{M}')$ we mean a monotone map $\varphi : [n] \longrightarrow [n']$ together with an isomorphism

$$\underline{M} \simeq \varphi^* \underline{M}'$$

where $\varphi^* \underline{M}'$ denotes the filtered object

$$M'_{\varphi(1)} / M'_{\varphi(0)} \subset \cdots\cdots \subset M'_{\varphi(n)} / M'_{\varphi(0)}.$$

Thus $R(A) \longrightarrow \Delta$ is a *fibred category* whose fibre over $[n]$ is the groupoid of filtered f.g. $A$-modules of length $n$ and their isomorphisms.

Now given an object $(n, \underline{M})$ of $R(A)$ we have associated a simplicial complex $X(n, \underline{M})$ and given

$$(n, \underline{M}) \longrightarrow (n', \underline{M}')$$

we will define a map

$$X(n, \underline{M}) \longleftarrow X(n, \underline{M}')$$

as follows. Given a $N \subset M'_{n'}$ we send it to

$$M'_{\varphi(0)} + N \cap M'_{\varphi(n)} / M'_{\varphi(0)} = (N + M'_{\varphi(0)}) \cap M'_{\varphi(n)} / M'_{\varphi(0)}$$

Thus we collapse $N$ into the smaller interval

$$M'_{\varphi(n)} / M'_{\varphi(0)} \simeq M_n$$

So only the $N \subset M'_{\varphi(0)}$ and $N \supset M'_{\varphi(n)}$ really get altered.

(Must check transitivity of collapsing. Given

$$0 \subset A' \subset A'' \subset B'' \subset B' \subset B$$

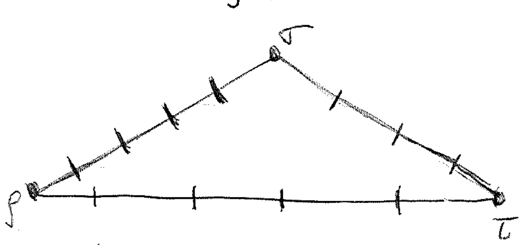
and $N \subset B$ we have to compute

$$A''/A' + \left[\left(A' + (N \cap B')/A'\right) \cap B''/A'\right] \Big/ A''/A'.$$

$$[A' + (N \cap B')] \cap B'' = A' + N \cap B'' \quad \text{modular law.}$$

So we do get $(A'' + N \cap B'')/A''$ as expected.)

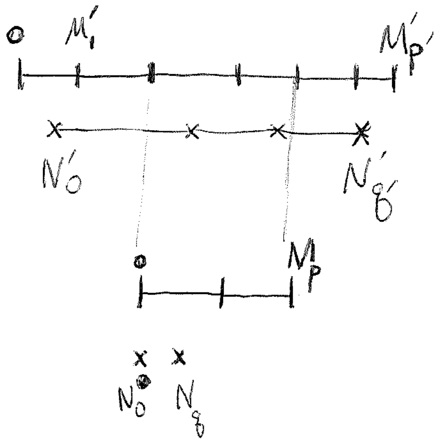
There is also the possibility of assigning ~~to~~ $(n, \underline{M})$ the complex $X(M_n)$. This seems better geometrically. over the 1-simplex $(\sigma, \tau)$ we have $X(M_{\sigma\tau})$ and so the 1-simplex lifts to a path joining $0$ to $M_{\sigma\tau}$. Then over a 2-simplex $(\rho, \sigma, \tau)$ we have to connect the $\rho\sigma + \sigma\tau$-path to the $\rho\tau$-path by a subdivided 2-simplex in $X(M_{\rho\tau})$




Clearly OKAY.

November 21, 1971

Let $X(R(A))$ be the category whose objects are pairs $(\underline{M}, \underline{N})$ where $\underline{M} = (M_1 \subset \cdots \subset M_p)$ is an object of $R(A)$ and $\underline{N}$ is a simplex of $X(\underline{M})$. By a morphism $(\underline{M}, \underline{N}) \longrightarrow (\underline{M}', \underline{N}')$ we mean a pair $(\xi, \eta)$ where $\xi : \underline{M} \longrightarrow \underline{M}'$ and $\eta : \underline{N} \longrightarrow \xi^*(\underline{N}')$.




Thus we think of a map $(\underline{M}, \underline{N}) \longrightarrow (\underline{M}', \underline{N}')$ as an isomorphism of the former with a face of the latter.

Observation: Let $R(A)^{nd}$ be the full subcategory of $R(A)$ consisting of non-degenerate $\underline{M}$, that is,

$$0 < M_1 < M_2 < \cdots < M_p$$

Then $R(A)^{nd}$ is fibred over $\Delta^{inj}$ (injective monotone maps). $R(A)^{nd} \hookrightarrow R(A)$ is a homotopy equivalence, since the map which associates to a $\underline{M}$ the unique non-degenerate thing of which it is a degeneracy is a functor.

I feel uncomfortable about degenerate objects. So work with $R(A)^{nd}$ whose objects are f.t. A-modules $M$ endowed with a chain of submodules containing 0 and $M$. A map $M' \longrightarrow M$ is constituted by two submodules $P \subset Q$ in the structural filtration of $M$ together with an isomorphism

$$M' \xleftarrow{\sim} P/Q$$

carrying the structural filtration of $M'$ into a subset of the structural filtration of $M$. $X(R(A)^{nd})$ is then the category whose objects are pairs $(M, N)$ where $M$ is an object of $R(A)^{nd}$ and $N$ is a simplex of $X(M)$.

Denote by $M * N$ the object of $R(A)^{nd}$ given by $M$ with the filtration which is the union of that of $M$ and $N$. Then $(M, N) \longmapsto M * N$ is a functor from $X(R(A)^{nd})$ to $R(A)^{nd}$.

More precisely $R(A)^{nd}$ is the category whose objects $\underline{M}$ are f.t. A-modules $M$ endowed with finite linearly ordered set of submodules including 0 and $M$. A morphism $\underline{M}' \longrightarrow \underline{M}$ is an isomorphism $\alpha$ of $M'$ with a subquotient of $M$ such that the map from submodules of $M'$ to submodules of $M$ induced by $\alpha$ carries the distinguished submodules of $M'$ into distinguished submodules of $M$. Call an object of $R(A)^{nd}$ a filtered A-module, and if

$$0 < M_1 < \cdots < M_p = M$$

are the distinguished submodules of $M$ arranged in order, say that $\underline{M}$ has dimension $p$.

As before, let $X(\underline{M})$ be the subcomplex of $X(M)$ consisting of simplices $\underline{N}=(N_0 < \cdots < N_q)$ compatible with the filtration of $M$, i.e. such that the union of the set of distinguished submodules of $M$ and the $\{N_j\}$ is linearly ordered. Now denote by $X(R(A)^{nd})$, the category of pairs $(\underline{M},\underline{N})$ where $\underline{M}$ is a filtered $A$-module and $\underline{N}$ is a simplex of $X(\underline{M})$. Given an arrow $\theta: \underline{M}' \longrightarrow \underline{M}$ it induces an embedding

$$X(\underline{M}') \hookrightarrow X(\underline{M})$$

and we define a map $(\underline{M}',\underline{N}') \longrightarrow (\underline{M},\underline{N})$ ~~to~~ be a map $\theta: \underline{M}' \longrightarrow \underline{M}$ such that $\theta(\underline{N}') \subset \underline{N}$. It is clear that $X(R(A)^{nd})$ is cofibred over $R(A)^{nd}$, and we have seen already that the fibres are contractible, so $X(R(A)^{nd}) \longrightarrow R(A)^{nd}$ is a homotopy equivalence.

(Observe $X(R(A)^{nd})$ is not fibred over $R(A)^{nd}$, namely given $\theta: \underline{M}' \to \underline{M}$ and $\underline{N}$ let $\theta^*(\underline{N})$ be the largest face of $\underline{N}$ which lies in $\underline{M}'$. Then $\theta_*(\underline{N}') \subset \underline{N} \Leftrightarrow \underline{N}' \subset \theta^*(\underline{N})$. Unfortunately $\theta^*(\underline{N})$ might be empty.)

Suppose given a pair $(\underline{M},\underline{N})$ let $\underline{M} * \underline{N}$ denote the object of $R(A)^{nd}$ defined by the submodule $M$ whose distinguished submodules are those of $\underline{M}$ or of $\underline{N}$. Then

$$(\underline{M},\underline{N}) \longmapsto \underline{M} * \underline{N}$$

is a functor from $X(\mathcal{R}(A)^{nd})$ to $\mathcal{R}(A)^{nd}$.

Let $\underline{N}/N_0$ be the object of $\mathcal{R}(A/I)^{nd}$ given by

$$0 < N_1/N_0 < \cdots < N_q/N_0$$

if $\underline{N} = (N_0 < N_1 < \cdots < N_q)$ is a simplex of $X(\underline{M})$. Then

$$(\underline{M}, \underline{N}) \longmapsto \underline{N}/N_0$$

is a functor from $X(\mathcal{R}(A)^{nd})$ to $\mathcal{R}(A/I)^{nd}$. Moreover there are natural transformations

$$\underline{N}/N_0 \longrightarrow \underline{M} * \underline{N} \longleftarrow \underline{M}$$

which shows that any Roos system of A-modules is homotopic to a Roos system of A-modules.

Suppose we start with a Roos $A/I$-system. Define

$$s : \mathcal{R}(A/I)^{nd} \longrightarrow X(\mathcal{R}(A)^{nd})$$
$$\underline{N} \qquad\qquad (\underline{N}, \underline{N})$$

this is a functor so we get this diagram:

$$\begin{tikzcd} & X(\mathcal{R}(A)^{nd}) \arrow[dl, shift left, "pr_1"] \arrow[dr, "pr_2"] & \\ \mathcal{R}(A/I)^{nd} \arrow[ur, shift left, "s"] \arrow[rr, hook, "i"] & & \mathcal{R}(A) \end{tikzcd}$$

$pr_1 s = id$, $pr_2 s = i$, $\quad i\, pr_1 \sim pr_2$. It follows that in the homotopy category $i$ is a retract of $pr_2$, hence $i$ is a homotopy equivalence as $pr_2$ is. Thus we have proved

<u>Theorem:</u> $R(A/I)^{nd} \hookrightarrow R(A)^{nd}$ is a homotopy equivalence. Hence

$$R(A/I) \hookrightarrow R(A)$$

is a homotopy equivalence.

Notice formal similarity between diagram on the preceding page and

$$\begin{tikzcd} & X \times_Y Y^I \arrow[dl, shift left, "pr_1"] \arrow[dr, "p"] & \\ X \arrow[ur, shift left, "i"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

$p(x, \lambda) = \lambda(1)$

$i(x) = (x, \text{constant path at } f(x))$

$\lambda$ joins $f(x)$ to $\lambda(1)$.

In fact, there is strong similarity because a map from a filtered $A/I$-module $P$ to $M$ in its simplest form will consist of arrows

$$\underline{P} \longrightarrow \underline{Z} \longleftarrow \underline{M}$$

which is pretty close to having $P = \underline{N}/N_0$ where $\underline{N}$ is a simplex in $X(\underline{M})$. Not same because $Z \neq M$ is possible in above diagram.

November 22, 1971.

Given an $A$-module $M$ consider the category $\mathcal{C}$ whose objects are pairs $(P, u)$, $u : P \to M$ surjective, $P$ projective (more generally locally free in the case of schemes). A morphism $(P, u) \longrightarrow (P', u')$ is an injection $P \to P'$ with $P'/P$ locally free and $\ni$

$$\begin{tikzcd} P \arrow[r] \arrow[d, hook] & M \\ P' \arrow[ur] & \end{tikzcd}$$

commutes. Claim this category is contractible. Indeed fixing an object $P_0 \xrightarrow{u_0} M$ we have natural transformations

$$\begin{tikzcd} P_0 \arrow[r, "in_1"] \arrow[dr, "u_0"'] & P_0 \oplus P \arrow[d, "u_0 + u"] & P \arrow[l, "in_2"'] \arrow[dl, "u"] \\ & M & \end{tikzcd}$$

from the identity functor of $\mathcal{C}$ to the constant functor $(P, u) \longmapsto (P_0, u_0)$.

Now suppose we have an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and we have chosen $P_0' \twoheadrightarrow M'$, $P_0 \twoheadrightarrow M$, $P_0'' \twoheadrightarrow M''$. Then we can find an exact sequence

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

mapping onto the above and maps $P_0 \hookrightarrow P$, $P_0' \hookrightarrow P'$ etc.

~~[illegible]~~ Indeed we have

$$\begin{tikzcd} P_0' \arrow[r, "in_1"] \arrow[d] & P_0' \oplus P_0 \arrow[r, "pr_2"] \arrow[d] & P_0 \arrow[d] \\ M' \arrow[r] & M \arrow[r] & M'' \end{tikzcd}$$

now choose $P \twoheadrightarrow M \times_{M''} P_0''$ and add the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & Ker \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

The point is that to the exact sequence $(M', M, M'')$ we assign the category of projective exact sequences which cover it.

Now suppose we are given a filtered $A$-module

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_p = M$$

then we can consider ~~[illegible]~~ locally-free filtered $A$-modules

$$0 \subset P_1 \subset P_2 \subset \cdots \subset P_p = P$$

(each $P_i/P_j$ locally free) ~~together~~ with a surjection $u: P \to M$ carrying $P_i$ ~~onto~~ $M_i$. We can make such $(P, u)$ into a category using injections $P \to P'$ such that ~~[illegible]~~ $P_i = P_i' \cap P$ for all $i$ and such that the quotient filtered module $P'/P$ is locally-free. Again we have direct sums $P \oplus P'$ so the category is contractible.

November 23, 1971.

Suppose we have a module with two filtrations

$$(*) \qquad \begin{aligned} &0 \subset F_1 \subset \cdots \subset F_p = M \\ &0 \subset M_1 \subset \cdots \subset M_q = M \end{aligned}$$

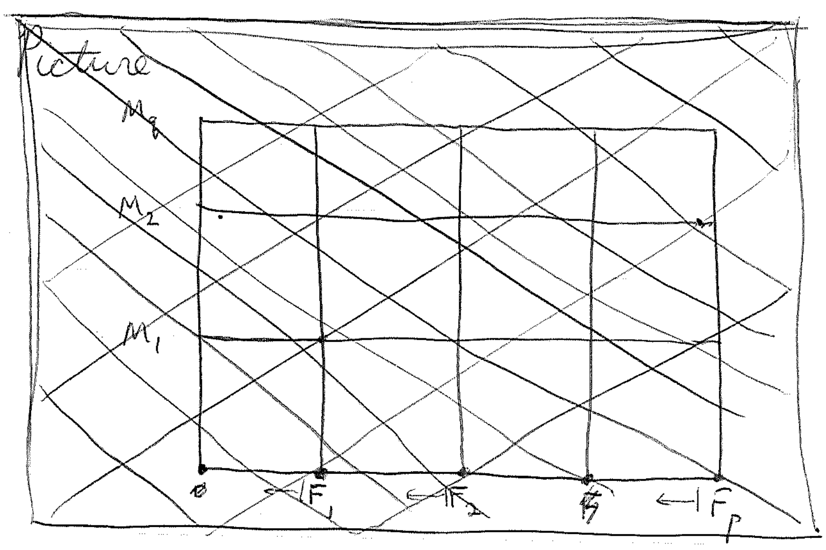
Shreier's lemma tells us that

$$gr''_j(gr'_i M) = gr''_j(F_i/F_{i-1}) = (F_i \cap V_j) + F_{i-1} \big/ (F_i \cap V_{j-1}) + F_{i-1}$$

$$\simeq \frac{F_i \cap V_j}{(F_i \cap V_j) \cap [(F_i \cap V_{j-1}) + F_{i-1}]}$$

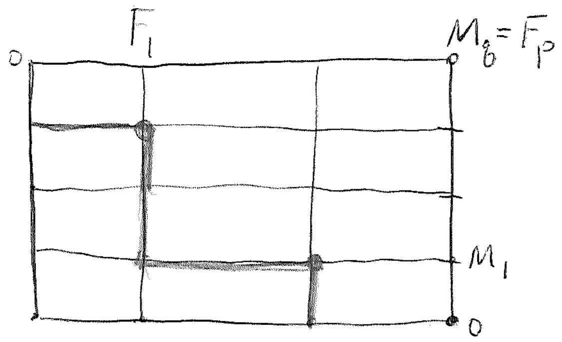
$$= \frac{F_i \cap V_j}{F_i \cap V_{j-1} + F_{i-1} \cap V_j}$$

so $\qquad gr''_j(gr'_i M) = gr'_i(gr''_j M) \qquad$ by symmetry.



Consider the lattice of submodules generated by the two chains (*). One knows that it is a distributive lattice with join-irreducibles $F_i \cap M_j$ $i, j \geq 1$. (Every element of a finite distributive lattice is uniquely representable as an irredundant join of join-irreducibles, hence may identify elements of the lattice with subsets closed under specialization of the set of join irreducibles.)

In the non-degenerate case where all $gr_j''(gr_i' M) \neq 0$ we therefore may identify elements of the lattice with specialization-closed subsets of $\{1, \ldots, p\} \times \{1, \ldots, q\}$ for the order topology. Picture of a typical element of the lattice

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On Bruhat's decomposition: suppose $V_i$ is a full flag in a vector space $V$. Given another full flag $F_i$ consider pairs $(i, j)$ such that $gr_j'' gr_i' V \neq 0$. Since $V_i/V_{i-1}$ is one-dimensional $\exists$ exactly one $j \ni gr_j(V_i/V_{i-1}) \neq 0$ and similarly given $j$ $\exists$ exactly one $i$. Hence get a permutation, which describes the double coset to which $F$ belongs in the Bruhat decomposition.

<u>November 28, 1971.</u> Resolution problem (continued)

Suppose we consider coherent sheaves over a regular noetherian scheme. Call this category $\mathcal{A}$ and denote by $\mathcal{A}_j$ the subcategory (full) of sheaves of Tor dim $\leq j$ so that $\mathcal{A}_0$ consists of the locally free coherent sheaves. What we want to prove is that the inclusions

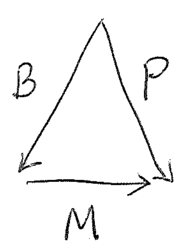
$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots\cdots \subset \mathcal{A}_d = \mathcal{A}$$

induce homotopy equivalences for the Roos style categories.

Given a coh. sheaf $M$ we know it can be written as a quotient of a locally free sheaf $P$:

$$0 \longrightarrow B \longrightarrow P \longrightarrow M \longrightarrow 0$$

and that if $M$ is in $\mathcal{A}_j$, then $B$ is in $\mathcal{A}_{j-1}$. What this means is that we are going to try to replace $M$ in $\mathcal{A}_j$ by pairs $(P,B)$ of objects in $\mathcal{A}_{j-1}$.



There is a basic problem how to define morphisms. Either (i) define $(P \twoheadrightarrow M) \longrightarrow (P' \twoheadrightarrow M)$ to be an injection $P \to P'$ with cokernel in $\mathcal{A}_{j-1}$ such that the triangle

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & P' \arrow[d] \\ & M \end{tikzcd}$$

commutes

(ii) define $(P \twoheadrightarrow M) \longrightarrow (P' \twoheadrightarrow M)$ to be a surjection $P \to P'$ such that the triangle commutes.

In either case one obtains a contractible category because there are maps

$$\begin{tikzcd} P \arrow[r, "in_1"] \arrow[dr] & P \oplus P_0 \arrow[d] & P_0 \arrow[l, "in_2"'] \arrow[dl] \\ & M & \end{tikzcd}$$

in case (i) and maps

$$\begin{tikzcd} P \arrow[dr] & P \times_M P_0 \arrow[l, "pr_1"'] \arrow[r, "pr_2"] \arrow[d] & P_0 \arrow[dl] \\ & M & \end{tikzcd}$$

in case (ii). (Note that $P \times_M P_0 \longrightarrow P_0$ has kernel $B = \mathrm{Ker}(P \to M)$ which is in $\mathcal{a}_{j-1}$. Thus $P \times_M P_0$ is in $\mathcal{a}_{j-1}$.)

Next consider the problem of replacing an $\mathcal{a}_j$-Roos system over a complex $X$ by an equivalent $\mathcal{a}_{j-1}$-system. One begins by choosing for each inclusion $\sigma \subset \tau$ (one simplex in $X'$) a resolution

$$0 \longrightarrow B_{\sigma\tau} \longrightarrow P_{\sigma\tau} \longrightarrow M_{\sigma\tau} \longrightarrow 0$$

Then given $\rho \subset \sigma \subset \tau$ one has

$$0 \longrightarrow M_{\rho\sigma} \longrightarrow M_{\rho\tau} \longrightarrow M_{\sigma\tau} \longrightarrow 0$$

and we have to relate $P_{\rho\sigma}$, $P_{\rho\tau}$, $P_{\sigma\tau}$, etc.

November 29, 1971:

What seems necessary is to find an exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bar{B}_{\rho\sigma} \arrow[r] \arrow[d] & \bar{B}_{\rho\tau} \arrow[r] \arrow[d] & \bar{B}_{\sigma\tau} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{P}_{\rho\sigma} \arrow[r] \arrow[d] & \bar{P}_{\rho\tau} \arrow[r] \arrow[d] & \bar{P}_{\sigma\tau} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{\rho\sigma} \arrow[r] \arrow[d] & M_{\rho\tau} \arrow[r] \arrow[d] & M_{\sigma\tau} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

together with arrows $\bar{P}_{\rho\sigma} \longrightarrow P_{\rho\sigma}$ etc. This amounts to considering the fibred category over $\mathcal{K}(\mathcal{A}_j)$ whose fibre over $M=(M_1,\ldots,M_q)$ has for its objects $P=(P_1\subset\cdots\subset P_q)$ mapping $P_i$ onto $M_i$ and $\ni$ $P_j/P_i \in \mathcal{A}_{j-1}$ for each $0 \le i \le j \le q$. The morphisms $P \longrightarrow P'$ should be strictly compatible with filtration hence injective with $P_i' \cap P = P_i$ and $P_i'/P_i \in \mathcal{A}_{j-1}$ in case (i) and $P_i \twoheadrightarrow P_i'$ in case (ii).

Thus we get the following gadget: For each simplex $\alpha$ of the barycentric subdivision $X'$ we have a filtered projective $P^\alpha = (P_1^\alpha \subset \cdots \subset P_q^\alpha)$ mapping ~~onto~~ $M^\alpha = (M_{\sigma_0\sigma_1} \subset M_{\sigma_0\sigma_2} \subset \cdots \subset M_{\sigma_0\sigma_q})$ if $\alpha = (\sigma_0,\ldots,\sigma_q)$. If $\beta$ is a face of $\alpha$ we have surjective maps

$$\partial^\alpha_\beta P^\alpha \twoheadrightarrow P^\beta \qquad \text{compatible with } \partial^\alpha_\beta M^\alpha \simeq M^\beta$$

where $\partial^\alpha_\beta$ means we take the face of $P^\alpha$ corresponding to $\beta$.

To such a gadget I must associate a category mapping to $R(\mathrm{Proj})$.

## Bisimplicial gadgets:

If we have a functor from simplices to modules and inclusions, say $\sigma \mapsto M_\sigma$, then we obtain a Roos-system

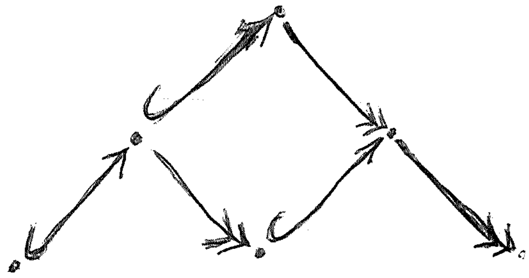
$$\sigma \subset \tau \longmapsto M_\tau / M_\sigma$$

In some situation we also want to consider surjections $M \twoheadrightarrow M''$ as associated to a one simplex. For examples, in the resolution problem, I want to associate to a filtered module $M$ the category of all objects $N \to M$ with distinguished images. In order to connect $0$ to $M$ in this category using projective steps it is necessary to allow both injections with nice cokernel and surjections with nice kernel. Let's carefully investigate ~~[illegible]~~ exactly what we obtain in this situation.

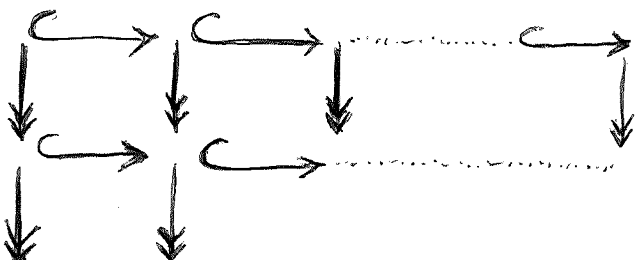
$M$ is a ~~coherent~~ sheaf of Tor dim $\leq j$ (in $\mathcal{a}_j$) 0-simplices will be objects of the category of maps $N \to M$ with image either $0$ or $M$. 1-simplices will be diagrams (up to isomorphism)

$$N_1 \hookrightarrow N_2 \twoheadrightarrow N_3$$

where the cokernel of the first and the kernel of the second are in $\mathcal{a}_{j-1}$. Picture of a 2-simplex



where the square commutes. Thus it seems that we have taken the bisimplicial category whose $(p,q)$-objects are commutative diagrams



and formed the Artin–Mazur "total" simplicial category.

If we wish to pass to the assoc. Roos-style system, ~~[illegible]~~ $(p,0)$-simplices are increasing filtered objects

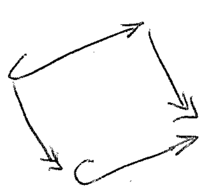
$$0 \subset M_1 \subset \cdots \subset M_p$$

~~while $(0,q)$ simplices are decreasing filtered object~~

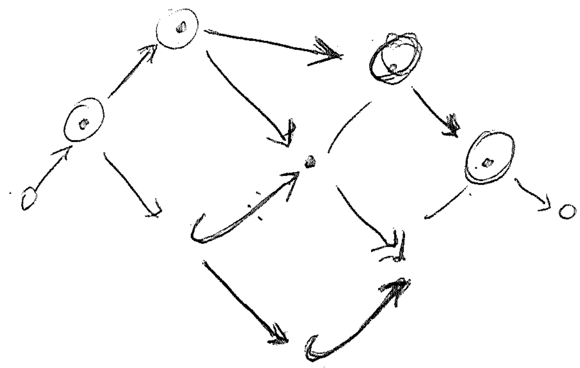
$$\cancel{0 = M_q \subset \cdots \subset M_0 = M}$$

~~better~~ $\cancel{0 \leftarrow M/M_0 \leftarrow M/M_1 \leftarrow \cdots \leftarrow}$

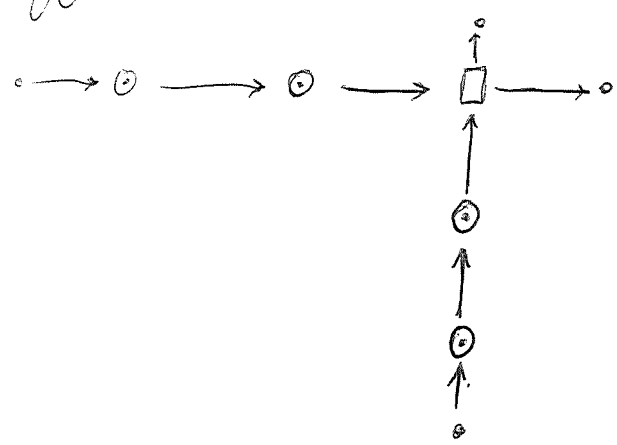
the $(1,1)$-simplices will consist ~~four things~~ of an exact sequence of length 4. The idea is that a square



gives rise to



Similarly we can fit the old problem (pg 12 ff) into a bisimplicial setup where the squares ~~[illegible]~~ correspond to different resolutions



What is needed next is a ~~[illegible]~~ way to reduce these bisimplicial gadgets back to ordinary Roos systems.

~~E A-bundle over X. Then~~

[redacted]

~~$St(X,A) = \lim [X, BG$~~

E A-bundle over X
gives rise to an element of
$\rho_E \in [X, BGL(A)]$.

and $\quad c_*\rho_E = \phi(E) \in \tilde{K}(X,A)$

Similarly if

$$\mathcal{E} : 0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is an exact sequence get

$$\rho_{\mathcal{E}} \in [X, BGL^{(2)}A] \ni$$

$$p_{1*}\,\rho_{\mathcal{E}} = \rho_{E'} \qquad p'$$

$$p_{2*}\,\rho_{\mathcal{E}} = \rho_{E''} \qquad p''$$

$$\theta_*\,\rho_{\mathcal{E}} = \rho_E . \qquad p$$

Also in

$$\theta_* j_* (\rho_{E'}, \rho_{E''}) = \rho_{E' \oplus E''}.$$

$$\pi''_*\,\rho_{\mathcal{E}}$$

so have to show $\quad j_*\pi_* = id.$

inducing an injection

$$\theta : GL^{(2)}(A) \longrightarrow GL(A)$$

Then composite

$$GL(A)^2 \xrightarrow{\iota} GL^{(2)}(A) \xrightarrow{\theta} GL(A)$$

is map induces by $\nu$.

---

$E$ an $A$-bundle over $X$ defines

$$\rho_E \in [X, BGL(A)].$$

~~If X connected, then an isom. class~~

choose map $\varepsilon : \mathbb{N}' \amalg \mathbb{N}' \hookrightarrow \mathbb{N}'$ so as to obtain

$$\mu : GL(A) \times GL(A) \longrightarrow GL(A)$$

and point is that

$$\mu_*(\rho_E \times \rho_{E'}) = \rho_{E \oplus E'}.$$

Outline of §. Canonical map $R(X, A) \longrightarrow K(X, A)$.

~~Defn: A-bundle over X is a (locally trivial) fibre bundle E over X, endowed with an A module structure on the fibres locally isomorphic to X×P where P~~

Defn: A-bundle over X is a locally constant sheaf of A-modules over X such that each stalk is a finitely-gen. proj. A-module.

X connected — simply a representation of fundamental group over A.

To E over X is associated a canonical element.

$$\boxed{\rho_E \in [X, BGL(A)]}$$

(depending only on stable isom. class of E.)

Moreover ~~given $\mathbb{N}' \cup \mathbb{N}'' \to \mathbb{N}'$ an embedding~~ if

$$\mu : GL(A) \times GL(A) \longrightarrow GL(A)$$

is the embedding assoc. to $\mathbb{N}' \cup \mathbb{N}'' \hookrightarrow \mathbb{N}'$, then

$$\boxed{\mu_*(\rho_E * \rho_{E'}) = \rho_{E \oplus E'}}$$

$$X \xrightarrow{(\rho_E, \rho_{E'})} BGL(A) \times BGL(A) \xrightarrow{\mu} BGL(A).$$

~~Theorem:~~ ~~$\rho_E \rho_{E'}\ \rho_{E \oplus E'}$~~

Def: $\phi_E = c_* \rho_E \in [X, BGL(A)^+]$

**Defn:** A-bundle over X = locally free sheaf of A-modules whose stalks are f.g. proj. A-modules.

~~When X connected such a thing may be identified with a~~

It is thus a loc. trivial fibre bundle ~~with discrete~~ whose fibres are f.g. proj. A-modules endowed with discrete topology. ~~When~~ X is connected ~~then~~ and pointed, then a A-bundle E is determined up to isomorphism by the action of $\pi_1 X$ on the fibre over the basepoint. In this way one obtains an equivalence between the category of A-bundles over X and the category of representations of $\pi_1 X$ over A, where by a representation of a group G over A we mean a fin. gen. proj. A-module endowed with a linear action of G.

~~To them~~ To an A-bundle E over X, we ~~define~~ associate a canonical element

$$\rho_E \in [X, BGL(A)]$$

in the following manner. Can assume X connected.

Prop: $\exists!\ \rho : St(X, A) \longrightarrow [X, BGL(A)] \ni$
$\rho(E_n) =$ class of inclusion.

~~daughter Becky~~ ~~Yes~~

$$\rho_E \in [X, BGL(A)]$$

~~suppose next I consider~~

~~$\exists!\ \rho$~~ Suppose $X$ connected,

$Vect(X, A)$ abelian monoid, $S$ submonoid

$$St(X, A) = Vect(X, A)/S.$$

$$= \varinjlim_{(S,S)} [X, B\,Aut(P_s)]$$

But $(\mathbb{N}, \mathbb{N}) \longrightarrow (S, S)$ cofinal, hence

$$St(X, A) = \varinjlim_{n} [X, BGL_n(A)]$$

and this maps in an obvious way to $[X, BGL(A)]$.

Down to earth terms: Given $E$, ~~suppose~~ $\exists\ \varepsilon \ni$

$$(E \oplus \varepsilon)_x \simeq A^n$$

hence $E \oplus \varepsilon$ det. an element of $[X, BGL_n(A)]$.

But suppose

$$E \oplus \varepsilon \simeq A^n$$

$$E \oplus \varepsilon' \simeq A^{n'}$$

then

then

$$\boxed{\begin{tikzcd} & n \arrow[dr] & \\ s \arrow[ur] \arrow[dr] & & n'' \\ & n' \arrow[ur] & \end{tikzcd}}$$

then

$$\boxed{\begin{aligned} E \oplus \varepsilon \oplus A^{n'} &\simeq A^{n+n'} \\ E \oplus \varepsilon' \oplus A^{n} &\simeq A^{n+n'} \end{aligned}}$$

and