September 3, 1971: cancellation:

Suppose $P \oplus A \simeq Q \oplus A$ and we want to prove $P \simeq Q$. Look at this as follows. We have a bundle E and two non-vanishing (everywhere) sections s_1 and s_2 with [redacted] $E/As_1 = P$, $E/As_2 = Q$. Suppose [redacted] s_1, s_2 are everywhere independent, i.e. $(s_1, s_2): A^2 \longrightarrow E$ is a direct injection. Then we have exact sequences

$$0 \longrightarrow A \xrightarrow{\bar{s}_2} E/As_1 \longrightarrow E/As_1 + As_2 \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{\bar{s}_1} E/As_2 \longrightarrow E/As_1 + As_2 \longrightarrow 0$$

so

$$P \simeq A \oplus R \simeq Q$$

where $R = E/As_1 + As_2$. In fact this isom. of P and Q is unique up to an automorphism of P inducing the identity on $A\bar{s}_2$ and R. (θ is an elementary automorphism of P.)

Thus if s_1 and s_2 are connected in the unimodular complex of E, i.e. in the same component, then ~~[illegible]~~ by choosing a path one obtains an isomorphism of E/As_1 and E/As_2 unique up to elementary autos. of E/As_1.

$GL_2(\mathbb{F}_2)$ begins dim 1

$GL_3(\mathbb{F}_2)$ begins dim 2

~~[illegible]~~

Σ_2

0	1		
1	σ		
2	σ^2	a_1	a_2
3			
		a_1^2	

$$H_2(\Sigma_3) \leftarrow H_2(\Sigma_4)$$

$$H_n(\Sigma_{2n-1}) < H_n(\Sigma_{2n})$$

fibre has dimension $n-1$

$GL_4(\mathbb{F}_2)$ Σ_{15}

1 $*$

1 $*$

1 $*$ regular repn of A

thus have ~~[illegible]~~ acting on $\mathbb{P}^3$ $\mathbb{P}^2$ $\mathbb{P}^1$ — $\mathbb{P}^0$

and have reg. repn on K^3 $\boxed{8}$ ~~8~~ 4 2 1

$H^4(\Sigma_{15})$ $H^5(\Sigma_{31})$ $H^6(\Sigma_{63})$

~~has class of~~ degree 4

16 $\boxed{\text{class degree}}$ 8.

~~2~~ GL_1 . GL_2 3, 4

$2(n-1)$ GL_{n-1} GL_n $2n-1$

September 4, 1971: Serre's stability theorem.

E projective A-module of rank larger than the dimension of the maximum spectrum X of A. To show E has an everywhere non-vanishing section.

rank $E \geq 1$. Let $X = \bigcup X_i$ be the irreducible components and choose $x_i \in X_i - \bigcup_{j \neq i} X_j$. Then by Chinese remainder theorem can find a section s of E with prescribed values at x_i. Since $E(x_i) \neq 0$ for all i, can find $s \ni s(x_i) \neq 0$ for all i. Then the dependency set $D(s)$ is closed in X and doesn't contain any x_i, hence is of codim ≥ 1.

rank $E \geq 2$. Let s_1 be such that $D(s_1)$ has codim ≥ 1. Choose s_2 independent of s_1 at the points x_i and at a similarly selected set of points in $D(s_1)$. Then $D(s_1, s_2)$ has codim ≥ 1. Choose g to be non-zero on the irreducible components of $D(s_1)$ and to be zero somewhere on each of the irred. components of $D(s_1, s_2)$ not in $D(s_1)$. Then $D(s_1 + g s_2)$ has codim ≥ 2. Indeed OKAY over $D(s_1, s_2)$ and on $D(s_1, s_2) - D(s_1)$ OKAY by vanishing of g, while in $D(s_1)$ OKAY as $g, s_2 \neq 0$.

rank $E \geq 3$. Suppose s_1, s_2 chosen so that $D(s_1, s_2)$ has cod. ≥ 1, $D(s_1)$ codim ≥ 2 (possible by preceding.) Choose s_3 ind of (s_1, s_2) at ~~each~~ some interior point of the irred. components of X, ind of (s_1, s_2) at some int point of each irred components of $D(s_1, s_2)$, and indep of s_1 at some interior point of each ~~[illegible]~~ irred. component of $D(s_1)$. Choose g to be ~~[illegible]~~ gen. non-zero on $D(s_1, s_2)$ and $D(s_1)$ and zero at some interior point of irred. components of $D(s_1, s_2, s_3)$ outside of $D(s_1, s_2)$. Then

$$\text{cod } D(s_1, s_2 + g s_3) \geq 2.$$

~~[illegible]~~ off $D(s_1, s_2, s_3)$ OKAY

$D(s_1, s_2, s_3) - D(s_1, s_2)$ " vanishing of g

$D(s_1, s_2)$ " non-vanish. of g, s_1

and s_1, $s_2 + g s_3$ never vanish simultaneously, because $s_1 = 0 \Rightarrow g s_3 \neq 0$ ind. of s_2.

So can assume $D(s_1, s_2)$ codim ≥ 2 and s_1, s_2 ~~[illegible]~~ never simultaneously vanish. ~~[illegible]~~ Now choose $g = 0$ at "gen" points of $D(s_1, s_2)$ where $s_1 \neq 0$ and $g = 1$ at ———————— $s_2 \neq 0$ and consider $s_1 + g s_2$. Thus

$$\text{cod } D(s_1 + g s_2). \geq 3$$

September 5, 1971:

Recall that a map is called acyclic if its homotopy-theoretic fibres are acyclic, hence acyclic maps are closed under composition and base change. Moreover given

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

then gf, f acyclic $\Longrightarrow$ g acyclic.

Question: In the homotopy category, do acyclic maps permit calculus by right fractions. NO:

So we want to consider all maps $Y \longrightarrow X$ with X fixed. For the equalization axiom we would need to know that $Y \longrightarrow Y \times_X Y$ was acyclic, however, when $X = pt$ this isn't the case as the map on fundamental groups is not surjective.

killing perfect subgroup of fundamental gp. 1

pointed spaces

$[X, Y]$ homotopy classes of basepoint-preserving maps

X (pointed) CW complex

$\pi_0 X = 0$

$\pi_1 X$ perfect $\qquad H_1 X = 0$

$$\bigvee_I S^1 \xrightarrow{f = \Sigma f_i} X \longrightarrow X'$$

class f_i generate $\pi_1 X$

van Kampen $\Longrightarrow$ $\pi_1 X' = 0$

$$H_3 X \longrightarrow H_3 X'$$
$$\hookrightarrow 0 \longrightarrow H_2 X \longrightarrow H_2 X'$$
$$\hookrightarrow \bigoplus_I \mathbb{Z} \longrightarrow 0$$

$\therefore$ $H_q X \simeq H_q X' \qquad q \neq 2$

and

$$\begin{tikzcd}
0 \arrow[r] & H_2 X \arrow[r] & H_2 X' \arrow[r, "\partial"] & \bigoplus_I \mathbb{Z} \arrow[r] \arrow[dl, dashed, "\lambda"] & 0 \\
 & & \pi_2 X' \arrow[u, "\cong"] & &
\end{tikzcd}$$

Choose a splitting and maps $g_i : S^2 \longrightarrow X' \ni \partial(\text{class } g_i) = e_i$

$$\bigvee_I S^2 \longrightarrow X' \longrightarrow X''$$

van Kampen $\Rightarrow$ $\pi_1 X'' = 0$

$$\begin{tikzcd}
0 \arrow[r] & H_3 X' \arrow[r] & H_3 X'' \arrow[dll, out=-10, in=170] \\
\bigoplus_I \mathbb{Z} \arrow[r] \arrow[dr, dotted] & H_2 X' \arrow[r, two heads] \arrow[d, equal] & H_2 X'' \\
& \pi_2 X' &
\end{tikzcd}$$

Thus $$H_2 X' = \operatorname{Im} H_2 X + \operatorname{Im} \lambda$$

and $$\operatorname{Ker}\{H_2 X' \to H_2 X''\} = \operatorname{Im} \lambda$$

$\therefore$ $H_2 X \xrightarrow{\sim} H_2 X'$ and we obtain

<u>Proposition 1</u>: *If* $\pi_0 X = 0$, $\pi_1 X$ perfect, then there is an embedding $i : X \longrightarrow X^+$, with X^+ obtained by attaching 2 + 3 cells to X, such that $\pi_1(X^+) = 0$ and

$$H_* X \xrightarrow{\sim} H_* X^+.$$

<u>Proposition 2</u>: (Universal property of X^+) Given f

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & Y \\
X^+ \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

$\ni$ $\pi_1(f) = 0$ $\exists g$. Moreover g unique up to homotopy.

$\left\{\begin{array}{l}\text{Can assume } Y \text{ 1-connected} \\ \text{Obstructions lie in } H^*(X^+, X;\ \pi_* Y) = 0.\end{array}\right.$

<u>Remark A</u>: $X^+ = \mathbb{Z}_\infty(X)$ because

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{Z}_\infty(X) \arrow[d] \\ X^+ \arrow[r, "\sim", "\text{as } \pi_1 X^+ = 0"'] & \mathbb{Z}_\infty(X^+) \end{tikzcd}$$

(ask Kan why $X \to \mathbb{Z}_\infty(X)$ induces isos. on homology.)

<u>Remark B</u>: Let AX = Fibre of $X \to X^+$

$$(*) \qquad AX \longrightarrow X \longrightarrow X^+ \qquad \begin{array}{l}\pi_0 AX = 0 \\ \pi_1 AX = \pi_1 X\end{array}$$

spectral sequence

$$E^2_{pq} = H_p(X^+, H_q AX) \Rightarrow H_{p+q}(X)$$
$$\qquad\qquad \| $$
$$H_p X^+ \otimes H_q AX \qquad \text{if field coeffs.}$$

one sees from s.s. that $H_+(AX, k) = 0$ all fields k

hence AX is acyclic.

$$\begin{tikzcd} AX \arrow[r, "j"] & X \arrow[r] \arrow[dr] & \mathrm{Cone}(j) \arrow[d] \\ & & X^+ \end{tikzcd}$$

$\pi_1 \mathrm{Cone}(j) = 0$ van Kampen

$H_* \mathrm{Cone}(j) \xrightarrow{\sim} H_* X^+ \Longrightarrow \mathrm{Cone}(j) \xrightarrow{\sim} X^+$.

so (*) is also a cofibration sequence.

Suppose $E \subset \pi_1 X$ is a perfect subgroup. Let $p : X' \longrightarrow X$ be the covering space with $\pi_1 X = E$

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "p"'] & X'^{+} \arrow[d, "p^{+}"] \\ X \arrow[r, "i"'] & X^{+} \end{tikzcd}$$

cocartesian

Thus X^+ obtained by attaching 2 + 3 cells to X.

van Kampen $\Longrightarrow$ $\pi_1 X^+ = \pi_1 X / N$
where N is the normal subgroup of $\pi_1 X$ generated by E.

If L is any $\pi_1 X$-module, it defines local coefficients systems on the above four spaces and have

$$\begin{tikzcd} \cdots H_q(X', L) \arrow[r, "i'_*"] \arrow[d] & H_q(X'^{+}, L) \arrow[r] \arrow[d] & H_q(X'^{+}, X'; L) \arrow[r, "\partial"] \arrow[d, "\cong"] & {} \\ \cdots H_q(X, L) \arrow[r] & H_q(X^{+}, L) \arrow[r] & H_q(X^{+}, X; L) \arrow[r] & {} \end{tikzcd}$$

i'_* is an isom, hence

$$i_* : H_*(X, L) \xrightarrow{\sim} H_*(X^+, L).$$

for all $\pi_1 X^+$ - modules L.

<u>Proposition 1</u>: *Let* $E \subset \pi_1 X$ be perfect and N the normal subgroup gen. by E. Then $\exists$ embedding $i : X \longrightarrow X^+$ such that

a) $\pi_1 X^+ = \pi_1 X / N$

b) $H_*(X; L) \xrightarrow{\sim} H_*(X^+; L)$ for all $\pi_1 X^+$ - mods L.

<u>Corollary</u>: Given $f : X \longrightarrow Y$ $\ni$ $\pi_1 f\,(E) = 0$ $\exists$ $g : X^+ \longrightarrow Y$ $\ni$ $gi = f$. Moreover ~~g is unique~~ ~~up to homotopy~~

$$i^* : [X^+, Y] \xrightarrow{\sim} \{ \alpha \in [X, Y] \mid \pi_1(\alpha) \text{ kills } E \}$$

Obstructions lie in $H^*(X^+, X;\ \pi_r Y)$ $\quad r \geq 2$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X^+ \arrow[r] & K(\pi_1 Y, 1)
\end{tikzcd}$$

$\pi_r Y$ is a $\pi_1 X^+$ module by hypothesis.

(Corollary implies that any two ~~obstruction~~ spaces X^+ ~~same~~ are homotopy equivalent)

Notation: X/E for the space of Prop. 1. Reason:

$$\underline{\mathrm{Hom}}(Z,X) \longrightarrow \underline{\mathrm{Hom}}(Z \cup pt, X) \longrightarrow X$$

$\pi_1 X$ acts on $[Z,X]$ for any Z.

Hence $\pi_1 X$ acts on X as an object of ptd. homotopy category. denote action by $\tilde{\gamma}$ ~~Claim~~ Given

$$X \xrightarrow{f} Y$$

claim $\pi_1(f)$ kills $E \iff f\tilde{\gamma} = f$ all $\gamma \in E$

$\Rightarrow$ because $f\cdot\tilde{\gamma} = \widetilde{\pi_1(f)\gamma}\cdot f = f$

$\Leftarrow$ because $\pi_1(\tilde{\gamma})$ ~~acts on~~ $(\alpha) = \gamma\alpha\gamma^{-1}$ so

$f\tilde{\gamma} = f \Rightarrow \pi_1(f)(\gamma\alpha\gamma^{-1}) = \pi_1(f)$

for all $\alpha \Rightarrow \pi_1(f)$ kills $[E,E] = E$.

Remark: $X/E \longrightarrow X/N$ is a heq.

Corollary: X_1, X_2 pointed connected $E_i \subset \pi_1 X_i$ perfect. Then $\exists$ canonical heq.

$$(X_1 \times X_2)/(E_1 \times E_2) \longrightarrow (X_1/E_1)\times(X_2/E_2) \quad ?$$

stability 1

<u>September 14, 1971.</u>

Let V be a vector space over a field k. Given a subspace W of V and a subset S of V, let $L(S,W)$ be the simplicial complex whose simplices are finite subsets $\{s_1,\dots,s_m\}$ of S which are independent of W, i.e. $\dim(W+ks_1+\dots+ks_m) = \dim W + m$. Observe that if $\sigma = \{s_1,\dots,s_m\}$ is a simplex of $L(S,W)$, then

$$\mathrm{Link}(L(S,W),\sigma) = L(S, \underbrace{W+ks_1+\dots+ks_m}_{W+k\sigma}).$$

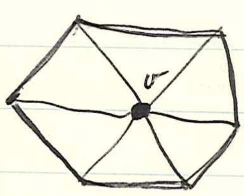
($\tau = \{t_1,\dots,t_n\}$ belongs to the link iff τ,σ,W independent, i.e. iff τ independent of $W+k\sigma$.)

Observe also that if $v \in L(S,W)$, then

$$L(S,W) = L(S-\{v\},W) \underset{L(S,W+kv)}{\cup} \mathrm{Cone}\, L(S,W+kv)$$

(In general, given a simplicial complex K, we have

$$K = (K - \underbrace{U_v}_{\text{open star of } v}) \underset{\mathrm{Link}(K,v)}{\cup} \mathrm{Cone}(\mathrm{Link}(K,v)).$$



)

More generally have for σ a simplex in $L(S,W)$

$$L(S,W) = L(S-\sigma, W) \underset{L(S,W+k\sigma)}{\cup} \left(L(S,W+k\sigma) \underset{\text{join}}{*} \sigma \right)$$

<u>Proposition</u> (R. Reid): Assume that $\dim L(S,W) \geq n \geq 0$. Then $L(S,W)$ is $(n-1)$-connected (meaning $\neq \emptyset$ for $n-1=-1$).

~~Proof: We can suppose S finite. Can argue by induction on n starting from n = 1, so can assume n > 1 and the result true for smaller values of n. Suppose S minimal counterexample~~

<u>Proof</u>: Arguing by induction on n, starting from $n=0$ which is trivial, we can assume $n>0$ and that the result is true for all (S,W) and smaller values of n. We can assume S finite. Let (S,W) be a counterexample with S having fewest elements.

<u>Case 1</u>: There is more than one n-simplex in $L(S,W)$. If this is so, we can find a vertex v belonging to one n-simplex, so that $L(S, W+kv)$ has $\dim \geq n-1$, and also not belonging to some n-simplex, so that $L(S-\{v\}, W)$ has $\dim \geq n$. By induction hypothesis

$$L(S, W+kv) \text{ is } (n-2)\text{-connected}$$

and by minimality of S

$$L(S-\{v\}, W) \text{ is } (n-1)\text{-connected}$$

so it follows from the formula

$$L(S,W) = L(S-\{v\},W) \underset{L(S,W+kv)}{\cup} \operatorname{Cone} L(S,W+kv)$$

that $L(S,W)$ is $(n-1)$-connected.

<u>Case 2</u>: There is only one n-simplex in $L(S,W)$. Then we will show that $L(S,W)$ is this simplex. Denote the simplex by $\sigma = \{s_0, \ldots, s_n\}$, and v be another vertex of $L(S,W)$. Then $v \in W + k\sigma$, so let j be least such that $v \in W + ks_0 + \cdots + ks_j$. Note $j \geq 0$ as $v \notin W$. By exchange condition

$$W + ks_0 + \cdots + ks_j = W + ks_0 + \cdots + ks_{j-1} + kv$$

hence $\{s_0, \ldots, s_{j-1}, v, s_{j+1}, \ldots, s_n\}$ is an n-simplex. By uniqueness of the n-simplex, we have $v = s_j \in \sigma$ as claimed. In this case it is clear that $L(S,W)$ is ∞-connected. q.e.d.

(Observe: lemma at stake here is that if $\{s_0, \ldots, s_n\}$ is independent ~~[illegible]~~ in a vector space (V/W here) and if $v \neq 0$ and $v \in \operatorname{span}\{s_0, \ldots, s_n\}$, then for some j $\{s_0, \ldots, s_{j-1}, v, s_{j+1}, \ldots, s_n\}$ is independent.)

Consider now the situation where V is a projective A-module, ~~when~~ W is a direct summand of V, and A is a local ring with residue field k. ~~[illegible]~~ For any subset S of V define $L(S,W)$ to be the simplicial complex whose simplices are subsets $\sigma = \{s_0, \ldots, s_m\}$ of S such that the images of $s_0, \ldots, s_m$ in $(V/W) \otimes_A k$ are independent. Then

$$\mathrm{Link}\,(L(S,W), \sigma) = L(S, W + A\sigma);$$

in effect τ is in the link iff τ independent of $W + A\sigma$.

I claim the preceding proposition holds. Case 1 clear, so check case 2. Suppose $\sigma = \{s_0, \ldots, s_n\}$ is the only n-simplex of $L(S,W)$. However given v in $L(S,W)$ then there is a j such that

$$s_0, \ldots, \hat{s}_j, \ldots, s_n, v$$

is again an n-simplex in $L(S,W)$. The point is that independence is measured within $(V/W) \otimes_A k$, so that it is enough to note that given an independent set $\{s_0, \ldots, s_n\}$ over a field, and something $v \neq 0$ in its span, then for some j $\{s_0, \ldots, \hat{s}_j, \ldots, s_n, v\}$ is independent. So in the case that $L(S,W)$ has a unique n-simplex, we see $L(S,W)$ is an n-simplex.

$\therefore$ Reid's proposition holds for a local ring.

September 16, 1971: Serre's theorem

Mike wants to prove Serre's theorem as follows: Let E be a projective A-module which is a quotient of A^n. Then to split off a trivial bundle of E means we must find a section of

$$\operatorname{Spec} A[X_1, \ldots, X_n] \rightleftarrows \operatorname{Spec} A$$
$$(X_i) \longmapsto a_i$$

which does not meet the closed subscheme Z of affine n-space defined by the kernel of $A^n \to E$. His idea consists in choosing $a_1, a_2, \ldots\ldots$ inductively ~~so that~~ the dimension of

$$Z \cap \{X_1 = a_1, \ldots, X_\nu = a_\nu\}$$

goes down each time. If $d = \dim(A)$, and $r = \operatorname{rank}(E) > d$, then Z has dimension $d + (n - r)$. So if one can do this $\nu = r - d + 1$ steps one is done for the intersection is empty. Now by Serre's theorem, Mike's program has to work.

__Mike's problem:__ Let A be a noetherian ring and C a closed subset of $\operatorname{Spec} A[X_1, \dots, X_n]$. Assume that for every closed point $\{\mathfrak{m}\}$ of $\operatorname{Spec} A$ one can find a rational point of the affine space over the fibre $\operatorname{Spec} A/\mathfrak{m}\,[X_1, \dots, X_n]$ which does not lie in C. Assume also that

(?) [redacted] $\dim C < n$

Then one can find a section of the affine space $X_i = a_i$ **not** meeting C.

__Special case:__ Suppose A semi-local and let

$$J = (f_1(\underline{X}), \dots, f_m(\underline{X}))$$

be the ideal of polynomials vanishing on C. If $\mathfrak{m}$ is a maximal ideal of A, there [redacted] exists, by hypothesis, [redacted] $\underline{\lambda} \in$ [redacted] $(A/\mathfrak{m})^n$ such that some $f_j(\underline{\lambda})$ [redacted] $\neq 0$ in $A/\mathfrak{m}$. Since A has finitely many maximal ideals, Chinese R.T. says we can find $\underline{a} \in A^n$ such that for each $\mathfrak{m}$ there exists a j with $f_j(\underline{a}) \notin \mathfrak{m}$. But

$$f_j(\underline{a}) = -\sum_{i=1}^{n} g_{ji}(\underline{X})(X_i - a_i) + f_j(\underline{X})$$

$$\in (\underline{X} - \underline{a}) + (f_1(\underline{X}), \dots, f_m(\underline{X})) \subset A[\underline{X}]$$

Thus

$$\left[J + (\underline{X} - \underline{a})\right] \cap A \supset \left(f_1(\underline{a}), \dots, f_m(\underline{a})\right)$$

and the latter $= A$ because it does so at each closed point of Spec(A). Thus $J + (\underline{X} - \underline{a})$ is the unit ideal, so we have a section not meeting C.

~~Remark: The preceding shows that once we produce a section which does not~~

Remark: The preceding amounts to the fact that given a section s, its bad set is closed in Spec(A), hence it meets Max(A). When A is semi-local, CRT guarantees we can find a section good on Max(A), hence good everywhere.

Special case: Suppose all of the residue fields of A are infinite. We claim $\exists\, a_1 \in A$ such that $X_1 = a_1$ doesn't contain any irreducible component of C. Indeed if $\mathfrak{p}_i$ are the prime ideals belonging to the irreducible components of C, then we want a_1 such that

$$X_1 - a_1 \notin \bigcup_{i=1}^{m} \mathfrak{p}_i$$

But $X_1 - a_1 \in \mathfrak{p}_i \iff a_1 \in X_1 + \mathfrak{p}_i$ so if a_1 doesn't exist we have

$$A = \bigcup_{i=1}^{m} (X_1 + \mathfrak{p}_i) \cap A$$

But $(X_1 + \mathfrak{p}) \cap A$ is a torsor for $\mathfrak{p} \cap A$, so we have

$$(*) \qquad A = \bigcup_{i=1}^{m} (a_i + \mathfrak{q}_i)$$

for $a_i \in A$ and $\mathfrak{q}_i$ prime ideals in A. Claim $(*)$ impossible: Can assume $\mathfrak{q}_i$ maximal. If $\mathfrak{q} \in \{\mathfrak{q}_i\}$ then $A/\mathfrak{q}$ is infinite by hypothesis, so $\exists\, a_{\mathfrak{q}}$ distinct mod $\mathfrak{q}$ from all a_i with $\mathfrak{q}_i = \mathfrak{q}$. By CRT $\exists\, a \equiv a_{\mathfrak{q}} \mod \mathfrak{q}$, then $a \notin (a_i + \mathfrak{q}_i)$ for any i.

Thus by induction we can find $a_1, \ldots, a_n$ such that

$$\dim C \cap \{X_1 = a_1, \ldots, X_n = a_n\} \leq \dim C - n$$

and hence a section not meeting C if
$\dim C < n$.

__Cor:__ Let G be a nilpotent group. Then the group of automorphisms of G inducing the identity on G^{ab} is nilpotent.

Proof. If θ induces id on G^{ab}, then it induces the identity on $gr(G)$ which is generated as a Lie algebra by G^{ab}. Then θ stabilizes the lower central series of G, and the group of these is nilpotent by the Kaloujnine theorem.

__Remark:__

$$[[A,B],C] \subset \text{normal subgroup generated by } [[A,C],B] \text{ and } [A,[B,C]].$$

<u>September 20, 1971:</u> $\quad VF = FV = p^d$.

Suppose A ring of characteristic p such that $F : A \longrightarrow A$ is a finite free map of rank p^d. For example, if A is an imperfect field such that $[A : A^{(p)}]$ is finite. Then on the K-groups $K_n A$ we have maps

$$K_n A \underset{F}{\overset{V}{\rightleftarrows}} K_n A$$

where V is the transfer or trace with respect to F. Now

$$\boxed{VF = p^d}$$

because as an A-module $A_F \simeq A^{p^d}$. On the other hand, FV is determined by the map

$$E \longmapsto A \otimes_{A^{(p)}} E = (A \otimes_{A^{(p)}} A) \otimes_A E .$$

What should be true is that

$$gr(A \otimes_{A^{(p)}} A) = Sym^A(\Omega^1_{A/\mathbb{F}_p}) / (x^p = 0)$$

(restricted symmetric algebra). Thus if I filter

$$A \otimes_{A^{(p)}} A \supset I \supset I^2 \supset \cdots\cdots \supset I^{p^d+1} = 0$$

this will be a filtration by $(A \otimes A)$- modules and

$$gr(A \otimes_{A^{(p)}} E) = \bigoplus_\nu I^\nu \otimes_A E / I^{\nu+1} \otimes_A E = gr(A \otimes_{A^{(p)}} A) \otimes E$$

will be multiplication by the element

$$p^d = [A \otimes_{A^{(\varphi)}} A] \in K_0 A$$

(Note tensoring with an $A \otimes A$-bimodule as an operation from $K_* A$ to $K_* A$ is not usually the same as multiplying by an element of $K_0 A$, e.g. in the case of a Galois extension it amounts to taking the sum of the conjugates). Thus have

$$\boxed{FV = p^d}$$

Now the idea I have is to try to use this together with the conjecture that $K_n A$ should have high γ-filtration for n large. So consider the essential case:

<u>Lemma:</u> Let M be an abelian group endowed with two endos. F, V such that

$$FV = VF = p^d$$

and

$$F(x) = p^{d+r} x \qquad r > 0.$$

Then

$$M = M' \oplus M''$$

where

$$M' = p^d M \quad \text{is the largest } p\text{-divisible subgroup}$$

$$M'' = {}_{(p^d)}M \quad \text{is the } p\text{-torsion subgroup.}$$

Thus M is the direct sum of a $\mathbb{Z}[p^{-1}]$-module and a group of exp. p^d.

<u>Proof</u>: $$p^d x = FVx = p^{d+r} Vx$$
so $$p^d(x - p^r Vx) = 0.$$
Thus $$\mathrm{Im}\,(1-p^r V) \subset {}_{(p^d)}M.$$
But $1-p^r V$ is an automorphism of ${}_{(p^d)}M$, hence
$$M = \mathrm{Ker}(1-p^r V) \oplus {}_{(p^d)}M.$$

Clearly $$p^d M \subset \mathrm{Ker}(1-p^r V)$$
But $x = p^r Vx = p^{2r} V^2 x = \cdots \in p^d M$,
so they are equal and $p^d M = p^s M$ for all $s > d$.
Thus $p^d M$ is the p-divisible subgroup. Also
$$p^{d+r} x = 0 \Rightarrow VFx = 0 \Rightarrow p^d x = 0$$
so ${}_{(p^d)}M$ is the p-torsion subgroup. Thus the lemma is proved.

So if I suppose that $K_n A$ is of γ-filtration $> d$, so that it admits a filtration
$$0 \subset \mathrm{Ker}\,(F - p^{d+1}) \subset \mathrm{Ker}\,(F-p^{d+2})(F-p^{d+1}) \subset \cdots \subset K_n A$$
it follows that $K_n A$ is direct sum of a $\mathbb{Z}[p^{-1}]$-module and group of exponent p^d.

Serre's thm. Sept. 25, 1971

I: If rank$(P) \geq r$, there exist $\alpha_1, \dots, \alpha_r \in P \ni$

$$\text{Codim } D_j(\alpha_1, \dots, \alpha_r) \geq r-j$$

where $D_j(\alpha_1, \dots, \alpha_r) = \{x \mid \text{rank}\{\alpha_1(x), \dots, \alpha_r(x)\} \leq j\}$.

Proof: Assume true for $r-1$, whence $\exists\ \alpha_1, \dots, \alpha_{r-1}$

$$\text{codim } D_j(\alpha_1, \dots, \alpha_{r-1}) \geq r-1-j \qquad \forall j \geq 0$$

Let C be irred. comp of $D_j(\alpha_1, \dots, \alpha_{r-1})$ ~~of~~ codim $r-1-j$. Then $C \not\subset D_{j-1}(\alpha_1, \dots, \alpha_{r-1})$ so can find a finite set $S_j \subset D_j(\alpha_1, \dots, \alpha_{r-1})$ not meeting $D_{j-1}(\alpha_1, \dots, \alpha_{r-1})$ and meeting each irred comp of codim $r-1-j$. Now arrange α_r to be independent of $\alpha_1, \dots, \alpha_{r-1}$ at each point of $\cup S_j$. Then possible because rank $P \geq r$

$$D_j(\alpha_1, \dots, \alpha_r) \subset D_j(\alpha_1, \dots, \alpha_{r-1}) - S_j$$

because at a point of S_j the rank of $\alpha_1, \dots, \alpha_{r-1}$ is j, hence $\alpha_1, \dots, \alpha_r$ has rank $j+1$ there. Thus

$$\text{cod } D_j(\alpha_1, \dots, \alpha_r) \geq r-j$$

II Suppose $\text{Codim } D_j(\alpha_1, \dots, \alpha_r) \geq k-j \quad \forall j, 0 \leq j < r$

$\Rightarrow \exists\ \beta_i = \alpha_i + a_i \alpha_r \quad 1 \leq i < r \ni$

$$\text{Codim } D_j(\beta_1, \dots, \beta_{r-1}) \geq k-j \quad \forall j,\ 0 \leq j < r-1.$$

Proof: Assume $0 \leq j < r$.

$$\underset{\text{codim} \geq k-j+1}{D_{j-1}(\alpha_1, \dots, \alpha_r)} \subset \underset{\text{codim} \geq k-j}{D_j(\alpha_1, \dots, \alpha_r)}$$

so no irred comp. C of $D_j(\alpha_1, \dots, \alpha_r)$ of codim $k-j$ is contained in D_{j-1}, hence $\exists$ S_j meeting each C not meeting D_j. At each point m of S_j, $\alpha_1, \dots, \alpha_r$ have rank $j < r$ hence can find a_i at m so that

$$\beta_i = \alpha_i + a_i \alpha_r \qquad 1 \leq i < r$$

have rank j at m. Then

$$D_{j-1}(\beta_1, \dots, \beta_{r-1}) \subset D_j(\alpha_1, \dots, \alpha_r)$$

and doesn't meet S_j because the β's have rank j there. Thus

$$\text{Codim } D_{j-1}(\beta_1, \dots, \beta_{r-1}) \geq k-j+1 \qquad 0 \leq j < r$$

$$\text{Codim } D_j(\beta_1, \dots, \beta_{r-1}) \geq k-j \qquad 0 \leq j < r-1.$$

September 25, 1971: Burnside ring

If G is a finite group, then the Burnside ring $B(G)$ is the Grothendieck group of finite G-sets. This is the naive K-functor associated to the family of symmetric groups.

$B(G)$ is a free $\mathbb{Z}$-module with basis the iso. classes of transitive G-sets, which may be identified with conjugacy classes of subgroups of G.

Given a subgroup H of G, the map

$$X \longmapsto \mathrm{card}(X^H)$$

transforms sums to sums and products to products, hence it induces a ring homomorphism

$$\varphi_H : B(G) \longrightarrow \mathbb{Z},$$

which clearly depends only on the conjugacy class of H.

~~Let k be a field and consider the composite homomorphism~~

$$\cancel{B(G) \longrightarrow \mathbb{Z} \longrightarrow k}$$

~~in fact, take k to~~

Let ℓ be a prime number not dividing $|G|$, and suppose H, H' are two subgroups such that

$$\varphi_H \equiv \varphi_{H'} \pmod{\ell}.$$

Then as

$$(G/H)^H = N/H \qquad N = \text{normalizer of } H \text{ in } G$$

has cardinality prime to l, we have

$$\varphi_H(G/H) \not\equiv 0 \pmod{l}$$

so

$$(G/H)^{H'} \neq \emptyset \qquad H'xH = xH$$

i.e.

$$H' \subset xHx^{-1} \quad \text{for some } x \in G.$$

Similarly H is conjugate to a subgroup of H' and so H and H' are conjugate subgroups.

This shows that the homomorphisms φ_H mod l are distinct, and so by comm. algebra we know that

$$(\varphi_H)_{H \in I} : B(G) \longrightarrow \prod_{H \in I} \mathbb{Z}$$

(I = reps. for conjugacy classes of subgroups) becomes an isomorphism after inverting order of G.

Next suppose G is a p-group. In this case $\operatorname{card}(X) \equiv \operatorname{card}(X^H) \pmod{p}$ for all subgroups, so all the φ_H mod p coincide with the mod p augmentation. I claim the augmentation ideal of $B(G) \otimes (\mathbb{Z}/p\mathbb{Z})$ is nilpotent. This ideal is generated by $[G/H]$ with $H < G$. Assume we know $[G/H']$ belongs to the nilideal for all H' with $|H'| < |H|$. Then by double coset formula

$$[G/H] \times [G/H] = [N:H][G/H] + \sum_i [G/H_i]$$

where $H_i = H \cap g_i H g_i^{-1} < H$. In a p-group $N > H$ so this formula shows that $[G/H]^2$ belongs to the nil-ideal, and so have

<u>Prop</u>: If G is a p-group, then the augmentation ideal of $B(G) \otimes \mathbb{Z}/p\mathbb{Z}$ is nilpotent.

So this shows that $B(G)$ is split etale off p and totally ramified at p. It also implies that

$$I^N \subset pI$$

$I =$ aug. ideal of $B(G)$. Note quite generally that

$$[G] \cdot [S] = [G] \cdot \operatorname{card}(S)$$

whence $[G] \cdot I = 0$ so

$$|G| \cdot I = (|G| - [G]) I \subset I^2$$

Thus for a p-group, the I-adic topology on I and the p-adic topology on I coincide implying that

$$\varprojlim_n I/I^n = I \otimes_{\mathbb{Z}} \mathbb{Z}_p .$$

Suppose now that p divides the order of G. If $H \triangleleft H'$ and H'/H is a p-group, then

$$\mathrm{card}\, X^{H'} = \mathrm{card}\, (X^H)^{H'/H} \equiv \mathrm{card}\, X^H \pmod p$$

so $\varphi_H \equiv \varphi_{H'} \bmod p$. Now starting with H we can form groups

$$H_0 \subset H \subset H_1$$

where H_0 is the char. subgroup of H gen. by the p'-elements and where H_1 is a Sylow p-subgroup of the normalizer of H_1. The indices are powers of p and H_1 is of index prime to p in its normalizer. Any subgroup H' of G conjugate to a subgroup between H_0 and H_1 satisfies

$$\varphi_{H'} \equiv \varphi_{H_0} \equiv \varphi_H. \pmod p$$

On the other hand, if H is already of p'-index in its normalizer, then

$$\varphi_H \equiv \varphi_{H'} \pmod p$$

$$\Longrightarrow \qquad \varphi_{H'}(G/H) \equiv \mathrm{card}\, N/H \not\equiv 0 \pmod p$$

so $(G/H)^{H'} \neq \phi$ and $H' \longrightarrow H$. Similarly if $H'_0 = H'_1$, then we have $H' \simeq H$. It follows that we get all the **different** φ_H, when we let H runs over a set J of representatives for subgroups $\ni$ $H^1(H) = 0$ up to conjugacy (or of p'-index in their normalizers).

Conjecture:

$$(\varphi_H) \; : \; B(G) \otimes \mathbb{Z}/p\mathbb{Z} \,/\, \mathrm{rad} \xrightarrow{\sim} \prod_{H \in J} \mathbb{Z}/p\mathbb{Z}$$

We know this map is onto. It only remains to produce enough nilpotent elements.

Lemma: If the index of H in its normalizer is divisible by p, then $[G/H]$ is nilpotent

First we show for any subgroup K that $\varphi_K([G/H]) \equiv 0 \mod p$. We may assume that K is generated by its p'-elements. Then consider the principal H/H_0-bundle

$$G/H_0 \longrightarrow G/H$$

where H/H_0 acts on the right, hence commutes with the action of K on the left. If $xH \in (G/H)^K$ then $x^{-1}Kx \subset H$ and as K is gen. by its p'-elements $x^{-1}Kx \subset H_0$. Thus

$$(G/H_0)^K \longrightarrow (G/H)^K$$

is a principal H/H_0-bundle, so

$$\varphi_K[G/H_0] = [H : H_0]\, \varphi_K[G/H]$$

Taking $H = H_1$ gives

$$\varphi_K[G/H_0] = [H_1 : H_0]\, \varphi_K[G/H_1]$$

so

$$\varphi_K[G/H] = [H_1 : H]\,\varphi_K[G/H_1]$$

for these special K. Therefore if $H < H_1$, $\varphi_K[G/H] \equiv 0$ mod p for all subgroups K.

It follows that for any subgroup K, the number of elements in G/H with isotropy groups equal to K is $\equiv 0$ (mod p). This is a Mobius inversion type formula. Specifically one has

$$X^K = \coprod_{K' \supset K} X^{\{K'\}} \qquad X^{\{K'\}} \text{ points with isot gp. } K'$$

and so if one knows that X^K, $X^{\{K'\}}$ for $K' \supsetneq K$ have card $\equiv 0$ (mod p), it follows $X^{\{K\}}$ has card $\equiv 0$ (p).

Now

$$[G/H] \cdot [G/H] = \sum_{HxH} [G/H \cap xHx^{-1}]$$

The double cosets are the orbits of H on G/H. We break these orbits up into orbit types. Suppose that any orbit H/K occurs d times. Then the number of elements of G/H with isotropy group K is

$$d.\ \text{card}\,(H/K)^K = d.\,|N/K|$$

where N is the normalizer of K in H. Since this is $\equiv 0$ (mod p), either $d \equiv 0$ mod p whence the orbit type H/K contribution is 0, or $|N/K| \equiv 0$ mod p whence we know by an induction hypothesis that $[G/K]$ is nilpotent.

This proves the lemma, and with it the conjecture at the top of page 5.

September 29, 1971: Theorem of Kaloujnine

Thm. Let $G=G_0 \supset G_1 \supset G_2 \supset \cdots$ be a sequence of normal subgroups of a group G, and let A_r, $r \geq 0$ be the group of automorphisms of G which normalize A_i and induce the identity on G_i/G_{i+r} for each $i \geq 0$. Then $A_0 \supset A_1 \supset A_2 \supset \cdots$ is a filtration of the group A_0, i.e. $[A_r, A_s] \subset A_{r+s}$.

~~Proof: In the semi direct product~~ $\cancel{A_0 \tilde{\times} G}$, ~~the~~ $\cancel{G_i}$ ~~are normal, hence~~

~~Proof: Work in the group~~ $\cancel{A_0 \tilde{\times} (G/G_{i+s})}$. ~~Then we can apply the three subgroup lemma:~~

$$\cancel{[[A_r, G_i], A_s] \subset G_i}$$

Proof:

$$[[A_s, G_i], A_r] \subset G_{i+r+s}$$

$$[[G_i, A_r], A_s] \subset G_{i+r+s}$$

so working in the group $A_0 \tilde{\times} G/G_{i+r+s}$ and applying the three subgroup lemma, we have

$$[[A_r, A_s], G_i] \subset G_{i+r+s} \qquad \text{for all } i$$

hence $[A_r, A_s] \subset A_{r+s}$. q.e.d.

~~September 1, 1971~~

<u>Tate's theorem</u>: Let G be a finite group such that

$$H^1(G,\mathbb{Z}/p) \xrightarrow{\sim} H^1(P,\mathbb{Z}/p)$$

where P is a Sylow p-subgroup. Then G is p-nilpotent.

<u>Proof</u>: Let $G^\wedge$ be the p-completion of G. Then P maps onto $G^\wedge$, as Sylow groups map onto Sylow groups for surjective homomorphisms. One has in general

$$H^1(\hat{G}) \xrightarrow{\sim} H^1(G)$$

$$H^2(\hat{G}) \hookrightarrow H^2(G).$$

Indeed if an extension E of $\hat{G}$ by $\mathbb{Z}/p$ lifts to a trivial extension over G, then one has a homo $G \longrightarrow E$ which factors through $\hat{G}$ as E is a p-group. Let $N = \operatorname{Ker}\{P \longrightarrow G^\wedge\}$, so that we have an exact sequence

$$0 \longrightarrow H^1(G^\wedge) \longrightarrow H^1(P) \longrightarrow H^1(N)^{G^\wedge} \longrightarrow H^2(\hat{G}) \longrightarrow H^2(P).$$

Since $H^2(G) \hookrightarrow H^2(P)$ by transfer theory, the last map is injective, so we conclude

$$H^1(N)^{G^\wedge} = 0.$$

Thus $H^1(N) = 0$ and $N = 0$, so G is p-nilpotent.

October 3, 1971: cohomology theories and Σ_n

I want to understand why

$$\Omega B\left(\coprod_{n\geq 0} P\Sigma_n \times^{\Sigma_n} X^n\right) \simeq \Omega^\infty S^\infty(X\cup\infty).$$

The important thing seems to be to understand why the ~~homotopy object~~ functor on the left transforms cofibrations to fibrations.

Work semi-simplicially. Given a set X form simplicial monoid

$$M(X) = \coprod_{n\geq 0} P\Sigma_n \times^{\Sigma_n} X^n$$

which is the nerve of the category whose objects are finite sequences $(x_1, \ldots, x_n)$ and permutations for morphisms. Basic fact:

$$P\Sigma_n \times^{\Sigma_n} (A \amalg B)^n \sim \coprod_{0\leq i\leq n} \left(P\Sigma_i \times^{\Sigma_i} A^i\right) \times \left(P\Sigma_{n-i} \times^{\Sigma_{n-i}} B^{n-i}\right)$$

because the category defined by Σ_n acting on $(A\amalg B)^n$ is equivalent to the disjoint union for $0\leq i\leq n$ of the full subcategory with objects $A^i \times B^{n-i}$, and the latter is the category defined by $\Sigma_i \times \Sigma_{n-i}$ acting on $A^i \times B^{n-i}$. Consequence:

$$\begin{array}{ccc} M(A\amalg B) & \longleftarrow & M(A)\times M(B) \\ in_1(\alpha)\cdot in_2(\beta) & & (\alpha, \beta) \end{array}$$

is a weak equivalence (note: it is not a monoid homomorphism)

Now I denote by $\Gamma(X)$ an intelligent group completion of $M(X)$. Thus we can choose a functorial free monoid resolution

$$P(M(X)) \longrightarrow M(X)$$

using the sets-monoid triple and set

$$\Gamma(X) = \overline{P(M(X))} \qquad (\text{— group completion})$$

Then we have a commutative diagram

$$\begin{tikzcd} M(A) \times M(B) \arrow[r] & M(A \amalg B) \\ PM(A) \times PM(B) \arrow[r] \arrow[u] \arrow[d] & PM(A \amalg B) \arrow[u] \arrow[d] \\ \Gamma(A) \times \Gamma(B) \arrow[r] & \Gamma(A \amalg B) \end{tikzcd}$$

Because $M(A)$ is homotopy commutative, it follows that the group completion theorem applies to it, hence the map

$$H_*(M(A)) \xleftarrow{\sim} H_*(PM(A)) \longrightarrow H_*(\Gamma(A))$$

is localization wrt $\pi_0 M(A) = \coprod_{n \geq 0} SP^n(A) =$ free commutative monoid gen. by A. Consequently one sees that

$$\Gamma(A) \times \Gamma(B) \longrightarrow \Gamma(A \amalg B)$$

is a homotopy equivalence of s. sets by Whitehead thm, because both s.sets are simple.

More generally, suppose we have a functor $\Gamma : (\text{Sets}) \longrightarrow \text{s.groups}$ $\ni$

i) $$\Gamma(A) \times \Gamma(B) \longrightarrow \Gamma(A \amalg B) \qquad \text{weq}$$

ii) Γ commutes with filtered lim. ind.

Then given a simplicial set X I get a bisimplicial group $\Gamma(X)$ and the claim is that

$$\pi_*(\Delta \Gamma(X)) = h_*(X; \Gamma)$$

is a generalized homology theory. To prove this we prove the Mayer-Vietoris axiom:

<u>Lemma</u>: Let $A, B \subset C$ be sets, and let $\Gamma(A \cap B)$ act on the right of $\Gamma(A) \times \Gamma(B)$ by

$$(\alpha, \beta) . \gamma = (\alpha . \gamma, \gamma^{-1} \beta)$$

(Observe that $A \subset A' \Rightarrow \Gamma(A) \subset \Gamma(A')$ because $\exists\ r : A' \to A$ retraction.) Then

$$\Gamma(A) \times \Gamma(B) / \Gamma(A \cap B) \longrightarrow \Gamma(A \cup B)$$
$$(\alpha, \beta) \longmapsto \alpha\beta$$

is a homotopy equivalence of simplicial sets.

(This lemma shows that if A, B are simplicial subsets of X ~~[illegible]~~ then we have a principal-fibring

$$\Gamma(A\cap B) \longrightarrow \Gamma(A)\times\Gamma(B) \longrightarrow \Gamma(A\cup B)$$

and hence a long exact Mayer-Vietoris sequence.)

Proof. By hypothesis the vertical maps in

$$\begin{tikzcd}
\Gamma(A)\times\Gamma(B) \arrow[r, "(\alpha,\beta)\,\mapsto\,\alpha\beta"] & \Gamma(A\cup B) \\
\Gamma(A-B)\times\Gamma(A\cap B)\times\Gamma(A\cap B)\times\Gamma(B-A) \arrow[u, "{(xz',\,z''y)\,\leftarrow\!\shortmid\,(x,z',z'',y)}"'] \arrow[r, "{(x,z',z'',y)\,\mapsto\,(x,z'z'',y)}"'] & \Gamma(A-B)\times\Gamma(A\cap B)\times\Gamma(B-A) \arrow[u, "{xzy\,\leftarrow\!\shortmid\,(x,z,y)}"']
\end{tikzcd}$$

are heq's. The bottom arrow is a principal right $\Gamma(A\cap B)$-bundle with action

$$(x,z',z'',y)\cdot\gamma = (x, z'\gamma, \gamma^{-1}z'', y)$$

so does.

The relation with Anderson's chain functors. Suppose X, Y are pointed. Then we claim canonical map

$$\Gamma(X\vee Y) \longrightarrow \Gamma(X)\times_{\Gamma(pt)}\Gamma(Y)$$

is a heq. It suffices to show that

$$\Gamma(X) \times \Gamma(Y) / \Gamma(pt) \longrightarrow \Gamma(X) \times_{\Gamma(pt)} \Gamma(Y)$$

$$(\alpha, \beta) \longmapsto (\alpha\, \varepsilon(\beta), \varepsilon(\alpha)\beta)$$

is bijective, where $\varepsilon : \Gamma(X) \longrightarrow \Gamma(pt)$ is the augmentation. Clear.

Thus if we set

$$\bar{\Gamma}(X) = \mathrm{Ker}\{\Gamma(X) \xrightarrow{\varepsilon} \Gamma(pt)\}$$

for a pointed set, we have that

$$\bar{\Gamma}(X \vee Y) \longrightarrow \bar{\Gamma}(X) \times \bar{\Gamma}(Y)$$

is a heq., hence a chain functor à la Anderson.

Observation: Segal introduces category Γ consisting of finite sets, where a map $T_1 \longrightarrow T_2$ is a partition of a subset of T_2 indexed by T_1, i.e. ~~to each~~ a family of disjoint subsets of T_2 indexed by T_1. Such a thing is the same as a function

$$\overline{T}_1 \longleftarrow \overline{T}_2$$

where $\overline{T} = T \cup pt.$ Thus Γ is dual to the category of finite pointed sets. Hence Segal's special Γ-spaces and Anderson's chain functors are essentially the same thing.

October 18, 1971: More on Lang's theorem.

Recall that Lang's theorem: $G/G(\mathbb{F}_q) \xrightarrow{\sim} G$ for the general linear group amounts to the following. Let k be a separably closed field of char. p and let V be a fin. diml. vector space over k endowed with a semi-linear automorphism $F: V \to V$ satisfying $F(\lambda v) = \lambda^q F(v)$. Then

$$k \underset{\mathbb{F}_q}{\otimes} V^F \xrightarrow{\sim} V.$$

I want now to understand this result over a general ring A of characteristic p.

~~[illegible]~~

Proposition 1: Let M be a finitely presented A-module provided with an isomorphism

$$F: M^{(q)} \xrightarrow{\sim} M$$

where $M^{(q)} = A \underset{\sigma \nwarrow A}{\otimes} M \qquad \sigma(a) = a^q.$

Then M is a projective A-module.

Proof: Can suppose A - finitely generated over $\mathbb{Z}/p$, hence can suppose A noetherian. Then can suppose A local. Let

$$A^j \longrightarrow A^i \longrightarrow M \longrightarrow 0$$

be a minimal resolution. To prove $j = 0$, we can enlarge A by a faithfully flat extension A'. Thus as in appendix to ~~spectrum~~ paper, Part II, can suppose A with algebraically closed residue field k. Now one has that $M \otimes_A k = k \otimes_{\mathbb{F}_q} (M \otimes_A k)^F$. Should observe that A is canonically a k-algebra since Teichmüller section is an isomorphism. Suppose we find $m \in M$ such that

$$F(m) - m \in \mathfrak{m}^i M$$

where $\mathfrak{m}$ = max. ideal of A. Then

$$F(Fm) - Fm \in \mathfrak{m}^{qi} M$$

hence sequence $F^\nu m \in M$ converges to an element $m' \in M^F$. This shows that M^F generates M over A, because of the Lang theorem for a field. Also $M^F \cap \mathfrak{m}M = 0$; thus we have a minimal ~~[illegible]~~ surjection

$$A \otimes_{\mathbb{F}_q} M^F \xrightarrow{\pi} M \longrightarrow 0$$

Now apply same argument to the kernel K of π; it will be generated by elements of

$$K^F \subset M^F \cap \mathfrak{m}M = 0.$$

Thus π is an isomorphism and we are finished

General idea now is given M over A with an F as in the proposition, it is locally free. To obtain a generating subspace M^F, it is necessary to make an étale covering of A. Thus first make covering

$$A \otimes_{\mathbb{F}_p} \mathbb{F}_q \longleftarrow A$$

and assuming A over $\mathbb{F}_q$, we have a map

$$\mathrm{Sp}\, A \longrightarrow GL_n \xleftarrow{\ \sim\ } GL_n / GL_n(\mathbb{F}_q)$$

which gives us a principal covering of A with group $GL_n(\mathbb{F}_q)$.

Point from June 13, 1971 omitted above: Given $A \in GL_n$ to write $A = B(B^\sigma)^{-1}$, $\sigma(x) = x^q$, define F on $V = k^n$ by

$$F(e_i) = \sum a_{ji} e_i \qquad A = [a_{ij}]$$

Then if w_i is a basis for V^σ

$$e_i = \sum w_j b_{ji}$$

we have $\sum w_k b_{kj} a_{ji} = \sum e_j a_{ji} = F(e_i) = \sum w_j b_{ji}^\sigma$

so $BA = B^\sigma$ as desired.

October 21,1972: Summary of problems.

1. group-completion theorem for topological monoids; what is a torsor for a top. monoid?

2. good point of view simultaneously explaining the group-completion theorem, quasi-fibrations Segal's lemma on when $X \to Y$ is a h-fibration, and Sullivan's theory of rational h-type. and Friedlander's problem. One produces a cohomology theory over the space B, and the point is to check the homotopy axiom, and this can be done locally.

3. stable splitting of exact sequences theorem, nice model for $BGL(A)^+$, stability, the descent problem and the $I^q - 1$ exact sequence, products and ~~lambda~~ exterior power operations in exact sequence K-theory. Models for the gamma-filtration (Segal's suggestion). The Moore theorem-reduction of problem to representation of a cyclic group C; nice formula for the ~~gere~~ j-element of $\pi_{2i-1}Q(BC)$. Steinberg homology; any relation between this an gamma filtration K-theory of the dual numbers, and of curves

4. configurations and iterated loop spaces, braid groups, Barratt theorem, Tornehave problem, why $\underline{//}(\mathrm{loop}^n S^n)_{k^n}$ has the homotopy type of $BG(k^{-1})$ roughly.

Oct 21, 1971
Cornell
talk

$\boxed{\text{K-theory of discrete rings.}}$

A ring with 1

$$GL(A) = \bigcup_n GL_n(A)$$

$E_n A \subset GL_n A$ subgp. gen. by

$$\begin{bmatrix} 1 & & a & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad a \in A.$$

$$E(A) = \bigcup E_n(A).$$

One knows

$$E(A) = (E(A), E(A)) = (GL(A), GL(A)).$$

<u>Defn</u>:

$K_0 A =$ Groth group proj f.g. A-modules

(Bass) $K_1 A = GL(A)/E(A) = H_1(GL(A), \mathbb{Z})$

(Milnor) $K_2 A = H_2(E(A), \mathbb{Z})$.

Work of Bass + Tate + others shows these are interesting invariants

General definition of $K_n A$, $n \geq 0$:

Acyclic maps

Will work only with pointed conn. sp $\sim$ CW cxs.

<u>Defn</u>: X <u>acyclic</u> if

$$H_i(X,\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i>0. \end{cases}$$

Poincaré : $$\underset{\overset{\|}{0}}{H_1(X,\mathbb{Z})} = \pi_1(X)/(\pi_1(X),\pi_1 X)$$

so $\pi_1 X = (\pi_1 X, \pi_1 X)$. Such groups called <u>perfect</u>.

$\pi_1 X = 0$ means $X \sim 0$ Whitehead.

Given map $f : X \longrightarrow Y$,

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd}$$

$\bar{f}$ fibration

<u>Defn</u>: f <u>acyclic</u> if (homotopy-theoretic) fibres of $\bar{f}$ are acyclic spaces.

$$F \longrightarrow X \xrightarrow{f} Y$$

$$\pi_1 F \longrightarrow \pi_1 X \longrightarrow \pi_1 Y \longrightarrow \underset{\underset{*}{\parallel}}{\pi_0 F}$$

so $\pi_1(f)$ surjective &
$\operatorname{Ker} \pi_1(f)$ perfect.

equivalent:

<u>Prop 1:</u> (i) $f : X \to Y$ acyclic

(ii) for all L on Y, $f^* \xrightarrow{\sim}$

(iii) If

$$\begin{tikzcd} X \times_Y \tilde{Y} \arrow[r, "f'"] \arrow[d] & \tilde{Y} \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

universal covering

is cart., <u>then f' induces isos. on int. cohomology</u>

(i) $\Rightarrow$ (ii) $F \longrightarrow X \xrightarrow{f} Y$. Then

$$E_2^{pq} = H^p(Y, H^q(F, L)) \Rightarrow H^{p+q}(X, f^*L)$$

$$H^q(F, L) = \begin{cases} 0 & q > 0 \\ L & q = 0 . \end{cases}$$

(ii) $\Rightarrow$ (iii)

$$\begin{array}{ccc} H^*(\tilde{Y}, \mathbb{Z}) & \longrightarrow & H^*(X \times_Y \tilde{Y}, \mathbb{Z}) \\ \| & & \| \\ H^*(Y, p_*\mathbb{Z}) & \longrightarrow & H^*(X, f^*p_*\mathbb{Z}) \end{array}$$

(iii) $\Longrightarrow$ (i): $\pi_1 \tilde{Y} = 0$

fiber of f' must be acyclic.

$\quad\;\;$ '' fiber of f.

Cor 1: If

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

is cocartesian with f cofibration and acyclic, then f' is acyclic.

Proof: For any L on Y'

$$H^*(Y',X';L) \xrightarrow{\approx} H^*(Y,X;L) = 0$$

so done from.

$$H^*(X,L) \longrightarrow H^*(Y,L) \longrightarrow H^*(X,L)$$

Cor. 2: Given

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, dotted, "h"] & \end{tikzcd}$$

f acyclic + $\pi_1 g(\mathrm{Ker}\, \pi_1 f) = 0$

then $\exists$ h unique up to $\sim$

$\ni$ $hf \sim g$.

Proof: May assume f cofibration.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "f'"] \\ Y \arrow[ur, dotted] \arrow[r] & Z' \end{tikzcd}$$

Then f' acyclic. Van Kampen

~~$\pi_1(Z) \simeq \pi_1$~~

$$\pi_1 Z \xrightarrow{\simeq} \pi_1(Y) *_{\pi_1 X} \pi_1 Z \simeq \pi_1 Z'$$

f' heq by Whitehead

<u>Theorem</u>: Given X and $N \subset \pi_1 X$ perfect normal subgroup, there exists a $f: X \to Y$ acyclic $\ni$ $\pi_1(f) = N$. f unique up to $\sim$.

---

uniqueness clear from Cor 2.

existence: Let $p': X' \to X$ be covering with group $\pi_1 X' = N$. If can

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p'"'] & Y' \arrow[d, dashed] \\ X \arrow[r, dashed] & Y \end{tikzcd}$$

find $f': X' \to Y'$ ~~done~~ acyclic $\ni$ $\pi_1 Y' = 0$ done. So can assume $N = \pi_1 X$.

(Drop).

$$\begin{tikzcd} & X_3 \arrow[d] & \\ & X_2 \arrow[r] \arrow[d] & K(H_3 X_2, 3) \\ X = & X_1 \arrow[r] & K(H_2 X_1, 2) \end{tikzcd}$$

Assume have constructed

$$X_n \longrightarrow X \qquad n \geq 1$$

$$\begin{cases} \text{i)} & \pi_1 X_n \twoheadrightarrow \pi_1 X \\ \text{ii)} & H_i X_n = 0 \qquad 1 \leq i \leq n \end{cases}$$

$$H^{n+1}(X_n, A) \simeq \mathrm{Hom}(H_{n+1} X_n, A)$$

$$\chi \in H^{n+1}(X_n, H_{n+1} X_n) = [X_n, K(H_{n+1} X_n, n+1)]$$

__Defn:__

$$X_{n+1} \longrightarrow X_n \xrightarrow{\chi} K(H_{n+1} X_n, n+1)$$

$$H_{n+2} X_n \longrightarrow 0 \longrightarrow H_{n+1} X_{n+1} \longrightarrow H_{n+1} X_n \xrightarrow{\simeq} H_{n+1} X_n$$

$$H_{n+1} X_{n+1} = 0$$

$$X_\infty = \varinjlim X_n . \qquad X_\infty \text{ acyclic}$$

$$\pi_1 X_\infty \twoheadrightarrow \pi_1 X$$

set

$$X_\infty \longrightarrow X \xrightarrow{f} X/X_\infty = Y$$

$\therefore$ f acyclic $\qquad \pi_1 Y = 0$ van Kampen

$BGL(A) = K(GL(A), 1)$ classifying space

$\pi_1 BGL(A) = GL(A) \supset E(A)$ perfect normal.

$\exists!$ acyclic map

$$f : BGL(A) \longrightarrow BGL(A)^+$$

$\ni \operatorname{Ker} \pi_1(f) = E(A)$.

<u>Defn</u>: $K_i(A) = \pi_i\, BGL(A)^+ \quad i \geq 1.$

$K_1(A) = GL(A)/E(A)$

Recall

$$\begin{tikzcd} BE(A) \arrow[r, "f'", "\text{homology isom.}"'] \arrow[d, "p"] & BGL(A)^{+\sim} \arrow[d, "p"] \\ BGL(A) \arrow[r, "f"'] & BGL(A)^+ \end{tikzcd}$$

univ. cover.

$$K_2 A = \pi_2(BGL(A)^+) = \pi_2(BGL(A)^{+\sim})$$

$\|$ Hurewicz

$$H_2(BE(A)) \underset{f}{\cong} H_2(BGL(A)^{+\sim})$$

OKAY with Milnor.

$k = \overline{\mathbb{F}}_p$ $\qquad$ $k_d \subset k$ subfield of degree d

$$\bigcup_n BGL_n(k_{n!}) = BGL(k)$$

G finite group acting on f.d. k vector space V

$$\rho : G \longrightarrow \mathrm{Aut}_k(V).$$

Choose $\varphi : k^* \hookrightarrow \mathbb{C}^*$

$$\varphi_V : G \longrightarrow \mathbb{C}$$

$$\varphi_V(g) = \sum \varphi(\lambda_i)$$

where $\{\lambda_1, \dots, \lambda_{\dim V}\}$ eigenvalues of $\rho(g)$.

Thm: (Brauer) $\quad$ φ_V ~~generalized~~ $\mathbb{Z}$-combination of char. of $G/\mathbb{C}$.

$$\varphi_V \in R(G)$$

$$\varphi_V : BG \longrightarrow BU.$$

do for $GL_n(k_{n!}) \subset GL_n(k)$ acts on k^n

$$\bigcup_n BGL_n(k_{n!}) \xrightarrow{\varphi} BU$$

$$\|$$

$$\boxed{\varphi^{\#} : BGL(k)^{+} \longrightarrow BU}$$

<u>Thm</u>: ① Adams conjecture paper

$\varphi^{\#}$ induces isom. on $H_*(\mathbb{Z}/\ell\mathbb{Z})$

ℓ primes $\neq p$

② $H_*(BGL(k), \mathbb{Z}/p\mathbb{Z}) = 0 \quad * > 0.$

$$\begin{tikzcd}
 & & F \arrow[d] & \\
BGL(k)^{+} \arrow[urr, "\psi"] \arrow[rr] \arrow[drr, "\sim 0"'] & & BU \arrow[d] & \\
 & & BU \otimes \mathbb{Q} \arrow[r, Rightarrow, "ch"'] & \prod_{i \geq 0} K(\mathbb{Q}, 2i)
\end{tikzcd}$$

ψ induce isos over $\mathbb{Z}[p^{-1}]$. by ①.

Serre - Whitehead

$$\pi_i\, BGL(k)^+ \xrightarrow{\sim} \pi_i F$$ except for p torsion

② $\Rightarrow$ no p-torsion in $\pi_i\, BGL(k)^+$.

<u>Thm:</u>

$$K_i(k) \cong \begin{cases} \bigoplus_{\ell \neq p} \mathbb{Q}_\ell / \mathbb{Z}_\ell & i \text{ odd} \geq 1 \\ 0 & i \text{ even} \geq 1 \end{cases}$$

October 27, 1971

Let k be a field and suppose we consider the problem of proving the homotopy axiom: $k \to k[z]$ induces an isom on K-groups. Start with a representation of G on a f.t. proj. $k[z]$-module E. We want to show that E comes from k. Ideally we would like to produce a f.t. k-submodule L of E which is invariant under G and which generates. Replacing L by

$$L + zL + \cdots + z^n L$$

we can suppose that L is "involutive", i.e. that the conditions of the following hold.

Lemma: E finite type $A[z]$-module, A noetherian and L a f.t. A-submodule of E.

(i) $z^{-1}L \subset L$

(ii)

$$0 \longrightarrow A[z] \otimes_A (z^{-1}L \cap L) \xrightarrow{z\otimes in \,-\, 1\otimes z} A[z] \otimes_A L \longrightarrow E \longrightarrow 0$$

is exact.

(iii) L generates E over $A[z]$ and $L = \varphi^{-1}(M)$ where M is an f.t. $A[z^{-1}]$-submodule of $A[z, z^{-1}] \otimes_{A[z]} E$ and $\varphi(e) = 1 \otimes e$.

(iv) Set $L^{(n)} = L + zL + \cdots + z^n L$. Then

$$z^n : L/L^{(-1)} \xrightarrow{\sim} L^{(n)}/L^{(n-1)} \qquad n \geq 0$$

where $L^{(-1)} = z^{-1}L \cap L$.

Assume that $z^{-1}L \subset L$. The map
$$L/z^{-1}L \xrightarrow{z^n} L^{(n)}/L^{(n-1)} \qquad n \geq 1.$$
is clearly surjective always. Suppose
$$z^n x = \sum_{i<n} a_i z^i \qquad x, a_i \in L$$
Then
$$z\left(z^{n-1}x - \sum_{1\le i<n} a_i z^{i-1}\right) = a_0$$
so as $z^{-1}L \subset L$
$$z^{n-1}x \in \sum_{i<n} a_i z^{i-1} + L \subset L^{(n-2)}$$
so by induction $x \in z^{-1}L$. Thus (i) $\Rightarrow$ (iv).

Conversely, given $zx \in L$, let n be least $\ni$ $x \in L^{(n)}$, so that
$$x = \sum_{i\le n} a_i z^i.$$
If $n \geq 1$, then because $L^{(n)}/L^{(n-1)} \xhookrightarrow{z\cdot} L^{(n+1)}/L^{(n)}$ follows that $x \in L^{(n-1)}$. $\therefore$ $n = 0$. Thus (iv) $\Rightarrow$ (i).

Clearly (iii) $\Rightarrow$ (i). Conversely, let $M = A[z^{-1}]\cdot\varphi(L)$. If $\varphi(x) \in M$, then $z^N x \in L$ some N, so $x \in L$, hence $\varphi^{-1}(M) = L$.

(iv) $\Longrightarrow$ (ii) by filtering

(ii) $\Longrightarrow$ (iv) ?

If we can find an L f.t. over A, generating E and G-invariant, then we can suppose L involutive, and so by (ii)

$$0 \longrightarrow A[z]\otimes_A (z^{-1}L) \longrightarrow A[z]\otimes_A L \longrightarrow E \longrightarrow 0$$

which shows that E comes from k.

~~In general, we are going to have to choose an involutive L. Suppose $L' \subset L$ are both involutive. Then z^{-1} acts on L/L' and if z^{-1} kills L/L', we have an isomorphism~~

$$\cancel{z^{-1}L / z^{-1}L' \xrightarrow{\sim} L/L'}$$

Suppose L' and L are both involutive and that

$$L \subset L' \subset L + zL.$$

Then

$$\frac{L'}{L} \xrightarrow[\cong]{\cdot z} \frac{L'+zL'}{L+zL}$$

(It's onto because $(L+zL) + zL' \supset L' + zL'$. It's injective because $L' \cap z^{-1}(L+zL) = L' \cap L = L$.)

This is nice because it shows that the inclusion of pairs

$$(L+zL, L) \longrightarrow (L'+zL', L')$$

has contractible cokernel.

<u>Conjecture 1:</u> Consider the simplicial complex whose vertices are involutive L generating E and in which a q-simplex is a chain

$$L_0 \subset L_1 \subset \cdots \subset L_q$$

such that $L_q \subset L_0 + zL_0$. I conjecture this complex is <u>contractible</u>.

Evidence from Bruhat-Tits: They show that if one identifies L with $L^{(i)}$ for all $i > 0$, then ~~one~~ obtains a contractible complex.

<u>Conjecture 2:</u> Let I be an ideal in A with $I^N = 0$. If M is an A-module, consider [illegible] the simplicial complex whose vertices are submodules $M'_q \subset M$ and whose simplices are chains

$$M_0 \subset M_1 \subset \cdots \subset M_q$$

of submodules such that $IM_q \subset M_0$. Then this simplicial complex is contractible.

Reduction of conj. 1 to conj. 2: Let X_1 denote the complex of conj. 1. It may be identified with the subcomplex of f.g. $A[z^{-1}]$-modules M in $A[z, z^{-1}] \otimes_{A[z]} E$ such that $\varphi^{-1} M$ generates M. Thus X_1 is the subcomplex of the complex X_2 of conjecture 2 associated to $A[z^{-1}]$ and the ideal generated by z^{-1}. Enough to show X_2 contractible. Indeed given a finite subcomplex K of X_1 it contracts in X_2, so $z^n K$ contracts in X_1 for large enough n. Since $K \sim z^n K$??

<u>Theorem:</u> Let I be an ideal in a ring A and M an A-module such that $I^n M = 0$ for some n. Let $X(M)$ be the simplicial complex whose simplices are chains of A-submodules of M

$$M_0 \subset M_1 \subset M_2 \cdots \subset M_g$$

such that $IM_g \subset M_0$. Then $X(M)$ is collapsible to a point.

<u>Proof:</u> We collapse $X(M)$ to $X(IM)$. Given a submodule $L \subset M$ send it to $L \cap IM$. Claim it sends simplices to simplices:

$$L_0 \subset \cdots \subset L_g$$

$$L_0 \cap IM \subset \cdots \subset L_g \cap IM$$

and

$$L_g \cap IM / L_0 \cap IM \hookrightarrow L_g / L_0 \cap IM$$

and

$$L_g / L_0 \cap IM \hookrightarrow L_g / L_0 \times M/IM$$

Thus we have a simplicial map $X(M) \xrightarrow{f} X(IM)$ which is a retraction, i.e. $fi = id$ where $i : X(IM) \longrightarrow X(M)$ is the inclusion. But actually we have a homotopy

$$X(M) \times \Delta(1) \longrightarrow X(M)$$

$$\begin{array}{lll} L & \begin{cases} 0 \\ 1 \end{cases} & \begin{array}{l} L \cap IM \\ L \end{array} \\ & & \\ L & 0 < 1 & L \cap IM \subset L \end{array}$$

<u>Remark</u>: Proof shows more generally that $X_{A,I}(M)$ collapses to $X_{A,I}(M')$ where M/M' is killed by a power of I.