August 4, 1971.

Relation between $C^\wedge$ and $C^{\circ\wedge}$.

Let A denote the cofibred category over $C\times C^\circ$ defined by the functor

$$(X,Y) \longmapsto \mathrm{Hom}(Y,X).$$

Its objects are arrows $x \leftarrow y$ and a morphism from $(x\leftarrow y)$ to $(x'\leftarrow y')$ is given by a pair of arrows $x \to x'$ and $y' \to y$ s.t.

$$\begin{tikzcd} x \arrow[d] & y \arrow[l] \\ x' & y' \arrow[l] \arrow[u] \end{tikzcd}$$

commutes. The evident functors

$$\begin{array}{ccccc} C & \xleftarrow{\;t\;} & A & \xrightarrow{\;s\;} & C^\circ \\ x & \longleftarrow\!\!\shortmid & (x\leftarrow y) & \longmapsto & y \end{array}$$

are cofibrant, because A is cofibred over $C\times C^\circ$ and $pr_1 : C\times C^\circ \to C$ is cofibrant. Thus for G in $A^\wedge$

$$R^q t_*(G)_x = R^q \varprojlim_{C/x} G(x\leftarrow y)$$

(where $x\leftarrow y \longmapsto G(x\leftarrow y)$ is a covariant functor on C/x, hence the higher cohomology does not necessarily vanish identically, as C/x has a final, not initial, object.) But for $G = t^*F$, we have $G(x\leftarrow y) = F(x)$ is constant in y, hence $\mathcal{C}/x$ being contractible,* we have

$$R^q t_*(t^*F) = \begin{cases} F & q=0 \\ 0 & q>0, \end{cases}$$

hence t is <u>universally acyclic</u>.

~~[Remark: We use the cone construction at the point * to conclude that a category C with final object has~~ [redacted]

[At the point * we must prove, probably by explicit simplicial calculation, that for a category C with initial object $R^q \varprojlim F = 0$ $q>0$ for F [redacted] constant.]

Similarly $s: A \longrightarrow C^\circ$ is universally acyclic.

We can also consider the fibred category B over $C \times C^\circ$ defined by $(x,y) \longmapsto \mathrm{Hom}(x,y)$; [redacted] whose objects are arrows $x \longrightarrow y$ and whose morphisms [redacted] from $x \longrightarrow y$ to $x' \longrightarrow y'$ are diagrams

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \\ x' \arrow[r] & y' \arrow[u] \end{tikzcd}$$

Then

$$\begin{array}{ccccc} C & \xleftarrow{\bar{s}} & B & \xrightarrow{\bar{t}} & C^\circ \\ x & \longleftarrow\!\shortmid & (x \to y) & \longmapsto & y \end{array}$$

are fibrant with acyclic fibres, hence

$$\mathbb{L}\bar{s}_! \circ \bar{s}^* = id$$

and similarly for $\bar{t}$, showing again both $\bar{s}$ and $\bar{t}$ are **universally** acyclic.

~~Now define a functor from $D^+(C^\wedge)$ to $D^+(C^{o\wedge})$ by taking the composition~~

$$\cancel{D^+(C^\wedge) \xrightarrow{t^*} D^+(A^\wedge) \xrightarrow{\mathbb{R}s_*} D^+(C^{o\wedge})}$$

<u>Proposition 1</u>: $t^* : D^+(C^\wedge) \longrightarrow D^+(A^\wedge)$ is fully faithful and its essential image consists of complexes K whose homology sheaves $\mathcal{H}^q(K)$ are of the $t^*(F)$ for $F \in$ ~~[illegible]~~ $C^\wedge$.

Proof: Fully faithful results from univ. acyclicity:

$$\mathrm{Hom}_{D^+(A^\wedge)}(t^*K, t^*L) = \mathrm{Hom}_{D^+(C^\wedge)}(K, \underset{\underset{L}{\parallel}}{\mathbb{R}t_* t^* L})$$

The image of t^* is a triangulated subcategory, hence, by induction on amplitude, contains all bounded cxs. with homology sheaves in the image of t^*. Now given K in $D^+(A^\wedge)$ write it as inductive limit of its Postnikov system $K^{(n)}$ and if $K^{(n)} = t^* L^{(n)}$ for each n, let L be a weak limit of the $L^{(n)}$. Then both K and t^*L are weak limits of $t^*L^{(n)} = K^{(n)}$, hence ~~[illegible]~~ t^*L and K are isomorphic. q.e.d.

<u>Remarks</u>: It seems reasonable to expect the preceding to hold for the full derived categories, and that moreover t^* has the left adjoint $Lt_!$ and the right adjoint Rt_*. We will assume this without checking if necessary. In any case where C has finite homological + coh. dim. there is no problem with existence of the adjoint functors.

So we have the following full embeddings

$$\begin{array}{ccc} D_{lc}(C^\wedge) & \longrightarrow & D(C^\wedge) \\ \downarrow & & \downarrow{\scriptstyle s^*} \\ D(C^{o\wedge}) & \xrightarrow{t^*} & D(A^\wedge) \end{array}$$

which is cartesian, because a functor $F \in A^\wedge$ is both in the image of s^* and t^* iff it comes from a locally constant functor on C.

Now consider the functor

$$D^+(C^\wedge) \xrightarrow{s^*} D^+(A^\wedge) \xrightarrow{Rt_*} D^+(C^{o\wedge}).$$

This converts a complex of sheaves K into a complex of cosimplicial sheaves with the same cohomology:

$$\begin{aligned} H^q(C^\wedge, K) &\xrightarrow{\ \sim\ } H^q(A^\wedge, t^*K) \\ &= H^q(C^{o\wedge}, \mathbb{R}s_* \cdot t^*K). \end{aligned}$$

As

$$R^q s_*(G)_y = R^q \varprojlim_{x \leftarrow y} G(x \leftarrow y)$$

$$\boxed{R^q s_*(t^*F)_y = R^q \varprojlim_{y \to x} F(x)}$$

(Observe that F is contravariant so that the limit as x runs away from y. On the other hand given $y' \to y$ we have $y\backslash \mathcal{C} \longrightarrow y'\backslash \mathcal{C}$ hence the above is covariant in y.)

The functor $\mathbb{R}s_* \circ t^*$ has the left adjoint $\mathbb{L}t_! \circ s^*$ defined at least on D^-, and hence it seems unlikely that it is an equivalence of categories.

(It seems that the other functor $\mathbb{R}\bar{t}_* \circ \bar{s}$ might have the same effect. Indeed $R^q\bar{t}_*(\bar{s}^*F)$ is computed as the limit over the category of $(x \longrightarrow y \longleftarrow z)$ with arrows

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y & z \arrow[l] \arrow[dl] \\ x' \arrow[r] & y' \arrow[u] & \end{tikzcd}$$

of the functor $(x \to y \leftarrow z) \longmapsto F(x)$. Now by virtue of the diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y & z \arrow[l] \arrow[dl] \\ y \arrow[r, "id"] & y \arrow[u, "id"'] \end{tikzcd}$$

it seems reasonable that the derived limits over the category of $(x \longrightarrow y \longleftarrow z)$ might be the same as over the category of $y \longleftarrow z$.)

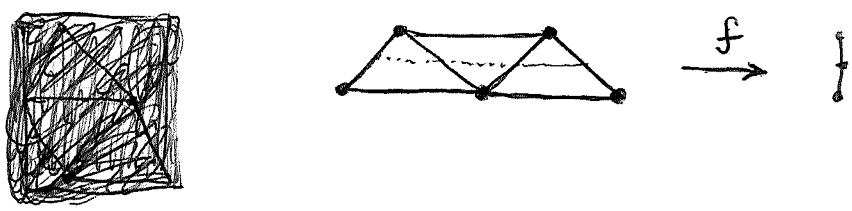
August 5, 1971

Sheaves associated to a simplicial complex:

Let $f : X \longrightarrow Y$ be a map of simplicial complexes. A basic geometric fact is that if y is in the interior of a simplex σ, then there is a homeo.

$$f^{-1}(\mathrm{Int}\,\sigma) = (\mathrm{Int}\,\sigma) \times f^{-1}\{y\}.$$

Standard picture:



Consequently for any abelian group A, $R^q f_*(A)$ is a sheaf on Y which is constant when restricted to any open simplex.

Let F denote a sheaf over a simplicial complex X which is constant over each open simplex. Given a simplex σ let U_σ denote its star; it is the union of the open simplices τ containing σ as a face, or equivalently

$$U_\sigma = \bigcap_{v \in \sigma} U_v$$

where $U_v = \{x \mid v\text{-th coordinate of } x > 0\}$. Then

$$(*) \qquad \Gamma(U_\sigma, F) \xrightarrow{\sim} F_\sigma$$

where the latter denotes the stalk at any interior point of σ. Indeed a germ of section s at an interior point b_σ of σ is defined in some nbd. of b_σ, hence gives rise to an element of F_τ for any $\tau \supset \sigma$. Thus any such sheaf F gives rise to a covariant functor on the category of simplices of X, notation $\mathrm{Simp}(X)$.

Conversely let $\bar{X}$ denote ~~the quotient space of X obtained by identifying points in the same open simplex~~ the topological space whose points are ~~the~~ simplices and having for a basis* of its open sets the subsets

$$\bar{X}_\sigma = \{\tau \mid \tau \supset \sigma\}.$$

A sheaf on $\bar{X}$ is the same thing as a covariant functor on $\mathrm{Simp}(X)$. ($\bar{X}_\sigma$ is the smallest open set of $\bar{X}$ containing σ. If $\sigma < \sigma'$, then $\bar{X}_\sigma \supset \bar{X}_{\sigma'}$, hence $\Gamma(\bar{X}_\sigma, F) \longrightarrow \Gamma(\bar{X}_{\sigma'}, F)$.)

~~The ... map~~

~~$f: X \longrightarrow \bar{X}$~~

~~is continuous~~

* If $\bar{X}_\sigma \cap \bar{X}_{\sigma'} \neq \phi$, then $\exists\, \tau$ containing σ and σ', hence $\sigma \cup \sigma'$ is a simplex, and we have

$$\bar{X}_\sigma \cap \bar{X}_{\sigma'} = \bar{X}_{\sigma \cup \sigma'}$$

Define the map

$$g : X \longrightarrow \overline{X}$$

by $g(x) = \sigma$ if x lies in the interior of σ.
Then

$$g^{-1}(\overline{X}_\sigma) = \bigcup_{\tau \supset \sigma} \operatorname{Int} \tau = U_\sigma$$

hence g is continuous and defines a morphism of topoi

$$g : \operatorname{Top}(X) \longrightarrow \operatorname{Top}(\overline{X}) = \operatorname{Simp}(X)^{\vee} \qquad \text{(cov. functors).}$$

g_* sends a sheaf F into the functor

$$\sigma \longmapsto \Gamma(U_\sigma, F).$$

while g_* takes a functor G into a sheaf constant on each simplex with stalk $G(\sigma)$ at each interior point of σ. In virtue of (*) p. 7 on has

$$G \xrightarrow{\sim} g_* g^* G \qquad \text{for all } G$$

while

$$g^* g_* F \xrightarrow{\sim} F$$

iff F is constant on each open simplex.

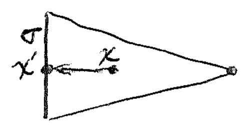
I want to prove that g is universally acyclic, which amounts to the formula

$$H^q(U_\sigma, F) = 0 \qquad q > 0$$

if F is constant on each open simplex. We use the homotopy axiom: Start with the canonical deformation of U_σ into $\mathrm{Int}(\sigma)$. If $x \in U_\sigma$ then $x = \lambda x' + (1-\lambda) x''$, where $x' \in \mathrm{Int}(\sigma)$ and x'' is a linear combination of vertices not in σ, and $\lambda > 0$ (if $\lambda = 1$, x'' is empty, $x = x'$). Let

$$h(x,t) = t x' + (1-t) x.$$

(Picture:



The point is that if $\tau > \sigma$, then τ is the join of σ and the complementary simplex $\tau - \sigma$.)
Then

$$h : U_\sigma \times I \longrightarrow U_\sigma$$

$$h_0 = id \qquad h_1 = r$$

where $rx = x'$ is the canonical retraction of U_σ onto $\mathrm{Int}\,\sigma$. Define

$$u : h^*(F) \longrightarrow pr_1^* F$$

as follows:

$$h^*(F)_{(x,t)} = F_{h(x,t)} \cong \begin{cases} F_x & 0 \le t < 1 \\ F_{x'} & t = 1 \end{cases}$$

$$pr_1^*(F)_{(x,t)} = F_x \qquad \text{all } t$$

one can take $u_{(x,t)}$ ~~to~~ be the identity for $0 \le t < 1$ and the specialization map

$$F_{x'} \longrightarrow F_x$$

for $t = 1$. Using this and the homotopy formula

$$H^*(U_\sigma, F) \xrightarrow{\sim} H^*(U_\sigma \times I, pr_1^* I)$$

we find that the composition of the map

$$(U_\sigma, F) \longrightarrow (\mathrm{Int}(\sigma), F|\mathrm{Int}(\sigma))$$

given by $r : U_\sigma \to \mathrm{Int}(\sigma)$ and the specialization maps $F_{x'} \to F_x$, together with the inclusion map

$$(\mathrm{Int}(\sigma), F|\mathrm{Int}(\sigma)) \hookrightarrow (U_\sigma, F)$$

induces the identity on $H^*(U_\sigma, F)$. Thus $H^*(U_\sigma, F)$ is a retract of

$$H^*(\mathrm{Int}(\sigma), F|\mathrm{Int}(\sigma))$$

which is zero in positive degrees as $\mathbf{F}$ is constant on $\mathrm{Int}(\sigma)$, and $\mathrm{Int}(\sigma)$ is contractible. So we have established

Proposition: If F is a sheaf on X constant on each open simplex, then

$$H^q(U_\sigma, F) \simeq \begin{cases} F_\sigma & q = 0 \\ 0 & q > 0 \end{cases}$$

~~Corollary: $g: X \to \bar{X}$ is universally acyclic, hence for any F constant on each open simplex we have~~

Cor: $g: \mathrm{Top}(X) \longrightarrow \mathrm{Top}(\bar{X})$ is universally acyclic.

In particular we have

$$H^n(X, g^*G) = H^n(\bar{X}, G)$$

$$= R^n \varprojlim_{\mathrm{Simp}(X)} G$$

Now the cohomology of $\bar{X}$ can be computed using the spectral sequence for the covering $\bar{X}_v$ as v runs over the vertices of X. Since any finite intersection

$$\bigcap_{v \in \sigma} \bar{X}_v = \bar{X}_\sigma$$

has trivial cohomology, we conclude that the s.s. deg., so

$$H^n(\bar{X}, G) = H^n(C^*(X, G))$$

where $C^*(X,G)$ is the complex with

$$C^p(X,G) = \prod_{|\sigma|=p} G(\sigma)$$

and with δ = alternating sum of face operators so we've proved.

<u>Prop</u>: Let X be a simplicial complex, and $Simp(X)$ the category of its simplices. Then for any $G: Simp(X) \longrightarrow \mathcal{Ab}$, we have

$$R^n \varprojlim_{Simp(X)} G = H^n(C^*(X,G))$$

where

$$C^p(X,G) = \prod_{|\sigma|=p} G(\sigma).$$

Skeletal filtration: Let $X_{(p)}$ denote the p skeleton of X. This gives an increasing filtration of X by closed subsets

$$X_{(0)} \subset X_{(1)} \subset X_{(2)} \cdots\cdots$$

hence in the case of a map $f: E \longrightarrow X$ to an exact couple

$$E_1^{pq} = H^{p+q}(f^{-1}X_{(p)}, f^{-1}X_{(p-1)}; A) \Longrightarrow H^{p+q}(E; A).$$

The sheaf-theoretic significance of this filtration is not immediately clear. However by excision, one has

$$E_1^{pq} = \prod_{|\sigma|=p} H^{p+q}(f^{-1}(\sigma), f^{-1}(\sigma^\circ); A)$$

$$= \prod_{|\sigma|=p} H^q(f^{-1}(\sigma); A)$$

where the last isomorphism comes from the fact there is a homeomorphism

$$\cancel{f^{-1}(\sigma) - f^{-1}(\sigma^\circ) = f^{-1}(b_\sigma) \times (\sigma - \sigma^\circ)}$$

$$f^{-1}(\sigma) - f^{-1}(\sigma^\circ) = f^{-1}(\mathrm{Int}\,\sigma) = f^{-1}(b_\sigma) \times \mathrm{Int}\,\sigma.$$

It is very likely that the resulting spectral sequence is the Leray spectral sequence of f. Moreover, it should be so that for any simplex-constant F on X, there should be a similar exact couple, leading to a spectral sequence degenerate from E_2 on:

$$H^*(C^\cdot(X, F)) = H^*(X, F).$$

Unfortunately I don't see how to get this spectral sequence in terms of some filtration by families of supports on $\bar{X}$.

<u>Grothendieck's filtration:</u>

Now working on the space $\bar{X}$ set

$$\bar{Z}_p = \{\sigma \mid \operatorname{codim} \sigma \geq p\}.$$

This is closed under specialization, hence closed; so we have a decreasing filtration by closed sets

$$\bar{X} = \bar{Z}_0 \supset \bar{Z}_1 \supset \cdots\cdots\cdots$$

and hence have an associated spectral sequence

$$E_1^{pq} = H^{p+q}_{Z_p/Z_{p+1}}(\bar{X}, \bar{F}) \Longrightarrow H^{p+q}(\bar{X}, \bar{F})$$

$$\| $$

$$\bigoplus_{\operatorname{cod}(\sigma)=p} H^{p+q}_{\sigma}(\bar{F}).$$

Here $H^*_\sigma(\bar{F})$ denotes the cohomology with supports in σ modulo that with supports in $\dot{\sigma}$. Recall that in general if $L \subset K$ are closed subsets of a space X, then we have

$$H^i_{K/L}(F) = \operatorname{Ext}^i(\mathbb{Z}_{K-L}, F)$$

where there are exact sequences

$$0 \longrightarrow \mathbb{Z}_{K-L} \longrightarrow \mathbb{Z}_K \longrightarrow \mathbb{Z}_L \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}_{X-K} \longrightarrow \mathbb{Z}_{X-L} \longrightarrow \mathbb{Z}_{K-L} \longrightarrow 0.$$

This shows that

$$H^i_{K/L}(F) = H^i(X-L, X-K; F)$$

where the left is the topologists notation. Now take $K = \bar{\sigma}$ and $L = \dot{\sigma}$ which are subcomplexes of X. Recall that the map $g: X \to \bar{X}$ is universally acyclic, hence for any open set $\bar{U}$ of $\bar{X}$ we have

$$H^*(\bar{U}, \bar{F}) \xrightarrow{\sim} H^*(U, F). \qquad (F = g^*\bar{F})$$

Then

$$H^i_\sigma(\bar{F}) = H^i(X-\dot{\sigma}, X-\bar{\sigma}; F).$$

~~If σ is a vertex, this is the local cohomology of F at σ. In general, this is the cohomology of U_σ [illegible] with support in σ, and there is a homeomorphism~~

In general if $K-L$ is closed in the open set U one has

$$H^i_{K/L}(F) = H^i_{K-L}(U; F)$$

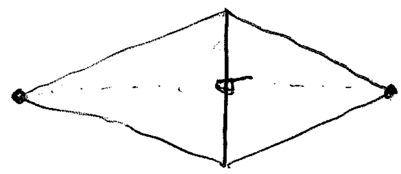
as there is an exact sequence

$$0 \to \mathbb{Z}_{U-(K-L)} \longrightarrow \mathbb{Z}_U \longrightarrow \mathbb{Z}_{K-L} \longrightarrow 0.$$

Note $\sigma = \bar{\sigma} - \dot{\sigma}$ is closed in U_σ, hence we have the better formula

$H^i_\sigma(\bar{F}) = H^i(U_\sigma, U_\sigma -$ [redacted] $\sigma; F)$.

Now a nbd of [redacted] σ is the product of $\mathrm{Int}\,\sigma$ and the cone on the link $L(\sigma)$



$\leftarrow L(\sigma)$

Cone $L(\sigma)$.

where $L(\sigma)$ is the simplicial complex whose vertices are those that can be added to σ in X. F induces a sheaf $\tilde{F}$ on Cone $L(\sigma)$, so we have by homotopy axiom

$H^i(U_\sigma, U_\sigma -$ [redacted] $\sigma; F) = H^i_{\{*\}}(\text{Cone } L(\sigma); \tilde{F})$.

We also have the formulas

$$H^i(U_\sigma, U_\sigma - \sigma; F) = H^{i+d}(X, X - \{x\}; F)$$

[redacted] $= H^{i+d}_c(U_\sigma; F)$

if $d = \dim \sigma$ and x is an interior point of σ, e.g. the barycenter.

Now let suppose that $n = \dim X$ and that every simplex is the face of an n-simplex, whence

$$\mathrm{cod}(\sigma) = n - \dim(\sigma)$$

[redacted] Claim that $\sigma \mapsto H^{0}(X, X - \{b_\sigma\}; F)$ is a contravariant functor of simplices. In effect given $\sigma \subset \tau$ have $U_\sigma \supset U_\tau$ hence have a map

$$H^q_c(U_\tau; F) \longrightarrow H^q_c(U_\sigma; F).$$

Summary: Given a simplicial complex X, let Z_p be the subcomplex consisting of simplices of codimension $\geq p$. Then there is a decreasing filtration by closed sets

$$X = Z_0 \supset Z_1 \supset \cdots$$

leading to a spectral sequence

$$E_1^{pq} = H^{p+q}_{Z_p/Z_{p+1}}(F) \Longrightarrow H^{p+q}(X, F)$$

in local cohomology. If F is simplex-constant, then

$$H^{p+q}_{Z_p/Z_{p+1}}(F) = \bigoplus_{\operatorname{cod}(\sigma)=p} H^{p+q}_\sigma(F)$$

where

$$H^m_\sigma(F) = H^m(U_\sigma, U_\sigma - \sigma; F)$$
$$= H^{m+d(\sigma)}_c(U_\sigma; F)$$

depends contravariantly on the simplex σ.

(see p. 21) If now $\operatorname{cod}(\sigma) = N - d(\sigma)$, then

$$E_1^{pq} = \bigoplus_{d(\sigma)=N-p} H_c^{N+q}(U_\sigma, F) \Longrightarrow H^{p+q}(X, F)$$

which probably can be rewritten after shifting

$$E_2^{pq} = H_{N-p}\left(X, \sigma \mapsto H_c^{N+q}(U_\sigma, F)\right) \Longrightarrow H^{p+q}(X, F),$$

and this probably degenerates in the case of a (constant coeffs + a) manifold to P.D.

$$H_{N-p}(X, \omega) = H^{p}(X).$$

We have already seen that the local system

$$x \longmapsto \boxed{H^q(X, X-\{x\}; F) = H^q_c(U_\sigma; F) \quad \text{if } x \in \sigma}$$

depends contravariantly on σ.
Similarly

$$x \longmapsto H_q(X, X-\{x\}; A)$$

is a sheaf.

<u>Basic geometry</u>: If X is a simplicial complex and A is an abelian group, then

$$x \longmapsto H_q(X, X - \{x\}; A)$$

is a sheaf on X locally constant on each simplex. Moreover, if $f : X \longrightarrow pt$, then the dualizing complex $f^!(A)$ has these for homology sheaves

$$\mathcal{H}^{q}(f^!(A))_x = H_{-q}(X, X - \{x\}; A)$$

and there is a spectral sequence (due to Zeeman)

$$E_2^{pq} = H^p(X, x \mapsto H_{-q}(X, X - \{x\}; A)) \Rightarrow H_{-p-q}(X; A)$$

which is the hypercohomology spectral sequence associated to the complex $f^! A$, together with the isom.

$$H^n(X, f^! A) = H_{-n}(X, A).$$

All of this follows from the adjointness formula

$$\mathrm{Hom}^n_{D(pt)}(\mathbb{R}f_*(M), N) = \mathrm{Hom}^n_{D(X)}(M, f^! N)$$

which also furnishes the formulas:

$$H_{-n}(U, A) = H^n_c(U, f^! A)$$

$$\mathcal{H}^q_\sigma(f^! A) = \begin{cases} 0 & q \neq -d(\sigma) \\ A & q = -d(\sigma). \end{cases}$$

Unfortunately there doesn't seem to be any relation between the above-mentioned ways of converting covariant functors on $\mathrm{Simp}(X)$ to contravariant ones, namely

$$F \longmapsto (\sigma \mapsto H^*_c(U_\sigma, F))$$

and the one introduced on Aug 4:

$$F \longmapsto (\sigma \mapsto H^*(\bar{\sigma}; F))$$
$$\parallel$$
$$R^* \varprojlim_{\tau\subset\sigma} F(\tau)$$

Let X_p be the p-skeleton of X, so that we have a decreasing filtration by closed sets

$X = X_{-0} \cdots \supset X_{-p} \supset X_{-p-1}$, i.e. $Z_p = X_{-p}$. Again have a spectral sequence

$$E_1^{pq} = H^{p+q}_{X_{-p}/X_{-p-1}}(F) \Longrightarrow H^{p+q}(X, F)$$
$$\parallel$$
$$\bigoplus_{|\sigma|=-p} H^{p+q}_\sigma(F) = \bigoplus_{|\sigma|=-p} H^q_c(U_\sigma, F)$$

which is undoubtedly the Mayer-Vietoris (spectral) sequence associated to the covering $\{U_\sigma\}$, i.e. the spec. seq. assoc. to the complex which in degree $-k$ is

$$\bigoplus_{|\sigma|=k} F_{U_\sigma} .$$

Assuming what should happen with d_1 does happen, we obtain a spectral sequence

$$\boxed{E_2^{pq} = H_{-p}(X, \sigma \mapsto H_c^q(U_\sigma, F)) \Longrightarrow H^{p+q}(X, F)}$$

which is a better form of the spectral sequence on page 19.

August 6, 1971

Simplicial sheaves over simplicial sets.

Let $f: X \longrightarrow Y$ be a map of simplicial sets. Associating to a simplicial set X the topos $\Delta^\wedge/X$ of simplicial sheaves on X, there is a Leray spectral sequence

(1) $$E_2^{pq} = H^p(Y, R^q f_*(F)) \Longrightarrow H^{p+q}(X, F)$$

where

$$R^q f_*(F)_y = H^q(f^{-1}(y); F)$$

$f^{-1}(y)$ = pull back by f of map $\Delta(d(y)) \longrightarrow Y$ belonging to y.

and where cohomology is defined by

$$H^q(X, F) = R^q \varprojlim_{\Delta/X} F$$

i.e in terms of the topos.

On the other hand, Moore obtains a spectral sequence by filtering X by $f^{-1}Y_{(p)}$:

$$E_1^{pq} = H^{p+q}(f^{-1}Y_{(p)}, f^{-1}Y_{(p-1)}; F) \Longrightarrow H^{p+q}(X; F)$$

(2) $\qquad\qquad \|$ excision

$$\prod_{y \in Y_p^+} H^{p+q}(f^{-1}(y), f^{-1}(\dot{y}); F) \qquad Y_p^+ = \text{non-degenerate part of } Y.$$

where $f^{-1}(y)$ and $f^{-1}(\dot{y})$ denote resp. the pull-backs by f of the map $\Delta(d) \longrightarrow Y$ associated to y

and its restriction to $\Delta(d)^\circ$. This spectral sequence is more suitable than (1) for Friedlander's thesis as it is a homotopy invariant of f from E_2 on, when F is locally constant. It would be nice to know whether (1) and (2) are the same, or not.

Computation of $H^*(\Delta(d), \Delta(d)^\circ; F)$, where F is a simplicial sheaf over $\Delta(d)$. Observe that for each $\sigma \subset \{0, \ldots, d\}$ there is an injection $\hat{\sigma}: \Delta(d') \to \Delta(d)$, hence F gives rise to a contravariant functor $\sigma \mapsto \Gamma(\hat{\sigma}, F) = F(\sigma)$ on the category of simplices $\sigma \subset \{0, \ldots, d\}$, hence we can form the complex of chains

$$C_p(\Delta(d), F) = \prod_{|\sigma| = p} F(\sigma).$$

We claim that

$$\boxed{H^q(\Delta(d), \Delta(d)^\circ; F) = H_{d-q}\Big(\nu \mapsto \prod_{\substack{\sigma \subset \{0, \ldots, p\} \\ |\sigma| = \nu}} F(\sigma)\Big).}$$

For example,

$d = 0$ $\qquad H^q(\Delta(0), \phi; F) = \begin{cases} 0 & q > 0 \\ F_0 & q = 0 \end{cases}$

$d = 1$ $\qquad H^*(\Delta(1), \Delta(1)^\circ; F) =$ homology of

$$F_{01} \longrightarrow F_0 \times F_1$$

degree (0) $\qquad$ (1)

To prove this we first have

$$(*) \qquad \to H^q(\Delta(d), \Delta(d)^\circ; F) \longrightarrow \underset{\substack{\| \\ \begin{cases} F_{0\cdots d} & q=0 \\ 0 & q\neq 0. \end{cases}}}{H^q(\Delta(d); F)} \longrightarrow H^q(\Delta(d)^\circ; F) \xrightarrow{\delta}$$

and $\Delta(d)^\circ$ as an object of the simplicial sheaves over $\Delta(d)$ is covered by the family of faces

$$\partial_i = \Delta(d-1) \longrightarrow \Delta(d)^\circ$$

leading to a Čech spectral sequence

$$E_1^{pq} = \prod_{0 \le i_0 < \cdots < i_p \le d} H^q\Big(\bigcap_{j=0}^{p} \operatorname{Im} \partial_{i_j}, F\Big) \Rightarrow H^{p+q}(\Delta(d)^\circ, F).$$

As the intersection is a simplex σ of $\Delta(d)$, this spectral sequence degenerates yielding isos.

$$H^n(\Delta(d)^\circ, F) = H^n\Big(\nu \mapsto \prod_{|\sigma| = d-\nu-1} F(\sigma)\Big)$$

so that taking into account the exact sequence $(*)$ above and the dimension shift, we obtain the desired formula.

<u>Corollary</u>: Given $\theta : \Delta(d') \longrightarrow \Delta(d)$ and a simplicial sheaf F over $\Delta(d)$, there is an induced map

$$H^q(\Delta(d'), \Delta(d')^\circ; \theta^* F) \longrightarrow H^{q+d-d'}(\Delta(d), \Delta(d)^\circ; F)$$

Proof: The former is isomorphic to the homology of dimension $d'-q$ of the d'-simplex with coefficients in the contravariant system $\sigma' \mapsto (\theta^*F)(\sigma') = F(\theta\sigma')$; the latter is isom. to homology of dimension $d-(q+d-d') = d'-q$. So the desired result follows from the covariant nature of homology.

Given $x \in X_p$ and a cosimplicial sheaf F over X denote by x^*F the contravariant system on the q-simplex given by

$$\sigma \subset \{0, \dots, p\} \longmapsto F(\sigma^* x)$$

where $\sigma^* x$ denotes the face of x corresponding to σ. Then we have established above the formula

$$\boxed{H^{p+q}(x, \dot{x}; F) = L_{-q} \varinjlim_{\sigma \subset \{0,\dots,p\}} F(\sigma^* x)}$$

and so we obtain a spectral sequence

$$E_2^{pq} = \check{H}^p\Big((\Delta^\circ/X)^\wedge ; x \mapsto L_{-q} \varinjlim_{\sigma \subset \{0,\dots,dx\}} F(\sigma^* x)\Big) \Rightarrow H^{p+q}(X;F).$$

The coefficient term is probably the same as the limit over the category of arrows $y \to x$ in Δ/X

$$L_{-q} \varinjlim_{y \to x} F(y)$$

suggesting that the map from $D(C^\wedge)$ to $D(C^{o\wedge})$, ~~using~~ $\mathbb{L}^q t_! \circ s^*$ preserves cohomology, at least for $C = \Delta/X$. Somehow this must be special to the case of simplicial sets.

The exact relation between spectral sequences (1) + (2) remains in doubt.

(simp.) sheaves over simplicial sets.

Let X be a simplicial set and Δ/X the associated fibred category over Δ. We consider the topos $(\Delta/X)^\wedge$, whose objects we call sheaves on X. As $(\Delta/X)^\wedge = \Delta^\wedge/X$ such a sheaf may be identified with a simplicial set over X.

If x is a simplex of ~~X~~, say of degree d, it defines an object of Δ/X, and hence gives rise to a representable sheaf h_x satisfying

$$\mathrm{Hom}_{(\Delta/X)^\wedge}(h_x, F) = F(x).$$

We can also think of h_x as the simplicial set over X furnished by the canonical map

$$\Delta(d) \longrightarrow X$$

carrying the distinguished d-simplex of $\Delta(d)$ to x. When no confusion is possible we shall regard Δ/X as embedded in $(\Delta/X)^\wedge$ by the functor $x \longmapsto h_x$, and we identify h_x and x.

Let $f: X \longrightarrow Y$ be a morphism of simplicial sets. One then has adjoint functors

$$\begin{tikzcd} \Delta^\wedge/X \arrow[r, shift left=3, "f_!"] \arrow[r, shift right=3, "f_*"'] & \Delta^\wedge/Y \arrow[l, "f^*" description] \end{tikzcd}$$

given by the formula

$$f_!(F \xrightarrow{u} X) = (F \xrightarrow{fu} Y)$$

$$f^*(G \longrightarrow Y) = \left(X \times_G Y \xrightarrow{pr_1} X\right)$$

$$(f_* F)(y) = \mathrm{Hom}_{\Delta^\wedge/X}\left(f^*(y), F\right).$$

Here $f^*(y)$ denotes the simplicial set over X which is the inverse image of h_y, so there is a cartesian square

$$\begin{tikzcd} f^*(y) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta(d) \arrow[r, "h_y"'] & Y \end{tikzcd} \qquad d = \deg(y).$$

The pair (f^*, f_*) constitutes a morphism of topoi which we denote by

$$f : \Delta^\wedge/X \longrightarrow \Delta^\wedge/Y$$

when no confusion results.

~~If U and F are sheaves over X and F is abelian, we denote by~~

$$\cancel{H^q(U, F) = \mathrm{Ext}^q_{(\Delta^\wedge/X)_{ab}}(\mathbb{Z}[U], F)}$$

~~the derived functors of $F \mapsto \mathrm{Hom}_{(\Delta^\wedge/X)}(U, F)$ on $(\Delta^\wedge/X)_{ab}$.~~

~~With f as above, we have the formula~~

$$\cancel{R^q f_*(F)(y) \simeq H^q(f^*(y), F)}$$

Denote by $F \longmapsto H^q(X, F)$ the derived functors of

$$H^0(X, F) = \mathrm{Hom}_{\Delta^\wedge/X}(e, F)$$

on $(\Delta^\wedge/X)_{ab}$. As $f : \Delta^\wedge/X \longrightarrow \Delta^\wedge/Y$ is ~~the~~ an induced topos morphism, one knows that f^* carries injectives into injectives, hence one knows the functors on $(\Delta^\wedge/Y)_{ab}$

$$G \longmapsto H^q(X, f^*G)$$

are the derived functors of $H^0(X, f^*?)$. For this reason when U and F are both sheaves over X with F abelian, we write simply

$$H^q(U, F)$$

instead of $H^q(U, j^*F)$, $j : U \longrightarrow X$ being the structural map of U.

We have the formula

$$R^q f_*(F)(y) = H^q(f^*(y); F)$$

as both sides are derived functors on $(\Delta^\wedge/X)_{ab}$ coinciding in degree zero.

This in turn implies ~~that~~ the base change formula

$$g^* R^q f_*(F) \simeq R^q f'_*(g'^*F)$$

for any cartesian square [redacted] of simplicial sets

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

[redacted] and abelian sheaf F on X.

As a particular case consider the projection

$$\begin{tikzcd} X \times S \arrow[r, "pr_1"] \arrow[d, "f\times id"] & X^{F} \arrow[d, "f"] \\ Y \times S \arrow[r, "pr_1"] & Y \end{tikzcd}$$

and we have that

$$pr_1^* R^q f_*(F) \xrightarrow{\sim} R^q (f\times id)_*(pr_1^* F).$$

[redacted] ~~would be true a~~

~~Homotopy: We wish to prove that~~

~~$pr_1 : X \times \Delta(d) \longrightarrow X$~~

~~is acyclic. It suffices by the above-mentioned base change formula to prove this for the map.~~

~~$\Delta(d) \longrightarrow e$~~

~~We must therefore show that for any abelian sheaf F over $\Delta(d')$~~

In the case of topological spaces, one knows that the projection $p : X \times I \longrightarrow X$ is acyclic, i.e.

$$F \xrightarrow{\sim} p_* p^* F \qquad R^q p_* (p^* F) = 0 \qquad q > 0$$

for all sheaves F on X. Unfortunately this is not true in the simplicial setup. For example, the first isomorphism implies that liftings

$$\begin{tikzcd} & F \arrow[d, "u"] \\ X \times I \arrow[r] \arrow[ur, dotted] & X \end{tikzcd} \qquad I = \Delta(1)$$

are all obtained from sections of u, which is nonsense already for $X = pt$.

<u>Homotopy property</u>: If F is locally constant on X, then

$$F \xrightarrow{\sim} p_* p^* F \quad \text{and} \quad R^q p_* (p^* F) = 0 \qquad q > 0.$$

where $p : X \times I \longrightarrow X$ is the projection.

This may be proved by using the isomorphism

$$H^*(X, F) = H^*(\nu \longmapsto \Gamma(X_\nu, F^{-1}))$$

for any locally constant sheaf F. This formula reduces one to the case of cosimplicial ab. groups where the homotopy axiom is verified by computation.

<u>August 7, 1971</u> Homotopy theory using $D_{lc}(X)$.

Let X be a topos and let $D_{lc}(X)$ denote the full subcategory of $D(X)$ consisting of complexes K whose homology sheaves $\mathcal{H}^q(K)$ are locally constant. Denote by

$$(1) \qquad i^* : D_{lc}(X) \longrightarrow D(X)$$

the inclusion functor. Intuitively, i^* is the inverse image for the map from X to its "homotopy type." Let us pretend such a thing exists, and denote the "homotopy type" of X by $\bar{X}$ and by

$$i : X \longrightarrow \bar{X}$$

the canonical map. Thus i^* should be an equivalence of $D(\bar{X})$ with $D_{lc}(X)$.

Implication A: $D_{lc}(X)$ is a triangulated subcategory of $D(X)$, that is, the subcategory of locally constant abelian sheaves is closed under kernels, cokernels, and extensions.

NO (p.3) If X is locally connected, this is the case,* because then one knows (existence of a fundamental pro-group) that the locally constant sheaves on X form a topos.

Implication B: i^* admits a right adjoint Ri_*, (defined at least on $D^+(X)$). Observe that this requires that i^* preserves direct sums, hence that locally constant sheaves be closed under direct sums, which is the case when X is locally connected.

~~Suppose $f: X \to Y$ is a morphism of topoi.~~

Furthermore, since i^* is fully faithful, $\mathbb{R}i_* \circ i^* = id$, [redacted] i.e. i is univ. acyclic.

Suppose we have a morphism of topoi $f: X \to Y$:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & \bar{X} \arrow[d, "\bar{f}"] \\ Y \arrow[r, "i'"'] & \bar{Y} \end{tikzcd}.$$

Then [redacted]

$$\mathbb{R}\bar{f}_* = \mathbb{R}\bar{f}_* \cdot \mathbb{R}i_* \cdot i^* = \text{[redacted]} \ \mathbb{R}i'_* \cdot \mathbb{R}f_* \cdot i^* .$$

Assume now that $F = i^* \bar{F}$ is in $D_{lc}(X)$, [redacted] and that $\mathbb{R}f_*(F)$ belongs to $D_{lc}(Y)$. Then one has that

$$i'^* \, \mathbb{R}\bar{f}_*(\bar{F}) = \mathbb{R}f_*(F)$$

hence as i and i' are universally acyclic, the Leray spec. sequences of $(\bar{f}, \bar{F})$ and (f, F) are the same.

* Locally connected implies that locally constant sheaves form an abelian ~~category~~ subcategory, but it doesn't imply ~~that all not non-trivial extensions~~ ~~for if A', A'' are locally constant~~ ~~$H^1(X, \mathcal{H}om(A', A'')) \to Ext^1(A', A'') \to H^0(X, \mathcal{E}xt^1(A', A''))$~~ ~~[illegible] 0~~ that extensions of loc. constant sheaves are locally constant. To know this we assume locally-simply connected.

Construction of the adjoint Ri_*. Let K be a complex in X and assume we have found a map $u_n : K_{(n)} \longrightarrow K$ such that $K_{(n)} \in D_{lc}(X)$ and

$$\mathrm{Hom}^q(L, K_{(n)}) \longrightarrow \mathrm{Hom}^q(L, K) \qquad \begin{array}{ll} \text{iso.} & q < n \\ \text{inj.} & q = n \end{array}$$

for all locally constant L. (Thus if $K \in D^+(X)$, then we can take $K_{(n)} = 0$ for $n \ll 0$.) Equivalently if $K^{(n)}$ is the cofibre of u_n, we have

$$\mathrm{Hom}^q(L, K^{(n)}) = 0 \qquad q < n.$$

The functor

$$L \longmapsto \mathrm{Hom}^n(L, K^{(n)})$$

is therefore left exact; as it transforms sums into products it is representable

$$\mathrm{Hom}^n(L, K^{(n)}) = \mathrm{Hom}(L, L_n)$$

where

$$L_n(x) = \mathrm{Hom}^n(\mathbb{Z}[\pi_1(X,x)], K^{(n)}).$$

Let v_n denote the canonical element of

$$\mathrm{Hom}(L_n[-n], K^{(n)}) = \mathrm{Hom}^n(L_n, K^{(n)})$$

and define $K_{(n+1)} \to K$ by the pull-back diagram

$$\begin{tikzcd}
K_{(n)} \arrow[r] \arrow[d, equal] & K_{(n+1)} \arrow[r] \arrow[d] & L_n[-n] \arrow[d] \\
K_{(n)} \arrow[r] & K \arrow[r] & K^{(n)}.
\end{tikzcd}$$

so that $K_{n+1} \in D_{lc}(X)$.

~~Then it is clear from the 5 lemma that~~

$$\cancel{\mathrm{Hom}^q(L, X_{n+1}) \to \mathrm{Hom}}$$

~~By construction v_n induces~~

$$\cancel{\mathrm{Hom}^q(L, L_n) \to}$$

$$\cancel{\mathrm{Hom}^q(L, X_n) \leftarrow \mathrm{Hom}(}$$

$$\cancel{\uparrow}$$

$$\cancel{\mathrm{Hom}^q(L, X_n)}$$

$$\begin{tikzcd}
\mathrm{Hom}^{q-1}(L, L_n[-n]) \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, equal] & \mathrm{Hom}^q(L, K_{n+1}) \arrow[r] \arrow[d] & \mathrm{Hom}^q(L, L_n[-n]) \arrow[r] \arrow[d] & \cdot \arrow[d, equal] \\
\mathrm{Hom}^{q-1}(L, K^{(n)}) \arrow[r] & \cdot \arrow[r] & \mathrm{Hom}^q(L, K) \arrow[r] & \mathrm{Hom}^q(L, K^{(n)}) \arrow[r] & \cdot
\end{tikzcd}$$

The five lemma shows us that

$$\mathrm{Hom}^q(L, K_{n+1}) \xrightarrow{\sim} \mathrm{Hom}^q(L, K)$$

for $q \leq n$, and that it will be injective for $q = n+1$ provided

$$\begin{array}{c} \mathrm{Hom}^{n+1}(L, L_n[-n]) \hookrightarrow \mathrm{Hom}^{n+1}(L, K^{(n)}) \\ \| \\ \mathrm{Hom}^1(L, L_n) \end{array}$$

Now one knows by general derived functor theory that this will be the case provided $\mathrm{Hom}^1(L, L_n)$ is effaceable as a functor of L. But cohomology of degree 1 is that of the fundamental group, so this is clear.

Thus by induction we construct a sequences of complexes K_n. Let K_∞ be a weak limit of this sequences, i.e. realize the maps $K_n \to K_{n+1}$ by inclusions and take the inductive limit. Then we have a short exact sequence

$$0 \longrightarrow R^1 \varprojlim_n \mathrm{Hom}^{-1}(K_n, K) \to \mathrm{Hom}(K_\infty, K) \longrightarrow \varprojlim \mathrm{Hom}(K_n, K) \to 0$$

so that there is at least one maps $K_\infty \to K$ compatible with the maps $K_n \to K$. By "way-out" ness, we have

$$\mathrm{Hom}^q(L, K_\infty) \xrightarrow{\sim} \mathrm{Hom}^q(L, K)$$

for all L and q. It follows then in standard fashion that

$$\mathrm{Hom}(M, K_\infty) \xrightarrow{\sim} \mathrm{Hom}(M, K)$$

for all M in $D_{lc}(X)$, whence $K_\infty = \mathbb{R}i_*(K)$.

The above argument proves the following:

Proposition: Let X be a topos such that the category $\mathcal{L}$ of locally constant abelian sheaves is closed under kernels, cokernels, direct sums and extensions in the category of all abelian sheaves on X, whence $D_{lc}(X)$ is a full triangulated subcategory of $D(X)$ closed under sums. (Assume $\mathcal{L}$ has generators) Then the inclusion i^* of $D_{lc}(X)$ in $D(X)$ has a right adjoint ~~[illegible]~~

$$\mathbb{R}i_* : D^+(X) \longrightarrow D^+_{lc}(X)$$

The point is that the proof uses the following facts about the category $\mathcal{L}$ constant sheaves

~~[illegible]~~ i) representability of $F : \mathcal{L}^\circ \to \mathcal{A}b$ if it transforms $\varinjlim$'s to $\varprojlim$'s

ii) $$\mathrm{Hom}_{D(X)}(L, L') = \mathrm{Hom}_{\mathcal{L}}(L, L') \qquad \mathcal{L} \text{ full subcat}$$

$$\mathrm{Hom}^1_{D(X)}(L, L') = \mathrm{Ext}^1_{\mathcal{L}}(L, L') \qquad \mathcal{L} \text{ closed under extensions}$$

Conjecture: In the good theory $D(\bar{X}) \neq D_{lc}(X)$ except when X locally n-contractible for all n.

Application: Suppose we have morphisms

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

and $F \in D^+_{lc}(Y)$. Assume

(i) u homotopy equivalence, i.e. $u^* : D^+_{lc}(Y) \to D^+_{lc}(X)$ is an equivalence

(ii) $\mathbb{R}g_*(F)$ and $\mathbb{R}f_*(u^*F) \in D^+_{lc}(S)$.

Then

$$\mathbb{R}g_*(F) \xrightarrow{\sim} \mathbb{R}f_*(u^*F)$$

Proof: Immediate application of the yoga of pages 1+2, the point being that $\bar{u} : \bar{X} \to \bar{Y}$ is is univ. acyclic, hence

$$\mathbb{R}\bar{g}_*(\bar{F}) \xrightarrow{\sim} \mathbb{R}\bar{f}_*(\bar{u}^*F)$$

On the other hand the hypothesis (ii) implies that these may be identified respectively with $\mathbb{R}g_*(F)$ and $\mathbb{R}f_*(\bar{u}^*F)$.

Actually the argument is a bit more elementary and amounts to the following

$$\begin{aligned}\mathrm{Hom}_S(M, \mathbb{R}g_*(F)) &= \mathrm{Hom}_Y(g^*M, F)\\ &= \mathrm{Hom}_X(f^*M, u^*F)\\ &= \mathrm{Hom}_S(M, \mathbb{R}f_*(u^*F))\end{aligned}$$

where the important step is that u^* is fully faithful as far as locally constant things are concerned.

We see that the argument is more elementary than the comparison theorem for spectral sequences. In effect to show a map of complexes induces an isom enough to show same effect on $\mathrm{Hom}^\bullet(L, ?)$ where L runs over any class big enough to contain the homology ~~groups~~ sheaves of the two complexes.

Application to a problem related to the Friedlander thesis. Suppose we have a diagram of categories

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "f"'] & P \arrow[d, "g"] \\ & Y \end{tikzcd}$$

and a locally constant L in $P^\wedge$. Assume

(i) u heq

(ii) $\mathbb{R}g_*(L) \in D^+_{lc}(Y^\wedge)$

(iii) $\mathbb{R}f^\circ_*(L^{-1}) \in D^+_{lc}(Y^{\circ\wedge})$

Then we want to show that

$$R^\delta g_*(L)^{-1} \simeq R^\delta f_*(L^{-1}).$$

To prove this we consider

$$\begin{tikzcd} X^\circ \arrow[d, "f^\circ"] & A(X) \arrow[l] \arrow[r] \arrow[d] & X \arrow[r, "u"] \arrow[d, "f"] & P \arrow[d, "g"] \\ Y^\circ & A(Y) \arrow[l] \arrow[r] & Y \arrow[r, equal] & Y \end{tikzcd}$$

where all horizontal arrows are weq's, and we use that the equivalences

$$D_{lc}(Y^{\circ\wedge}) \longrightarrow D_{lc}(A(Y)^\wedge) \longleftarrow D_{lc}(Y^\wedge)$$

transform L to L^{-1}. The rest is clear from the yoga, since when $-$ is applied, all the horizontal maps become isos.

Acyclic maps in homotopy theory.

A map of 'spaces' (as in homotopy theory) $f: X \longrightarrow Y$ will be called <u>acyclic</u> if the following equivalent conditions are satisfied:

(i) for all locally constant sheaves L on Y

$$H^*(Y, L) \xrightarrow{\sim} H^*(X, f^*L)$$

(ii) $f^*: D^+_{lc}(Y) \longrightarrow D^+_{lc}(X)$ is fully faithful

(iii) as a map of topoi, f is acyclic relative to $\underline{\Pi}Y$, the topos of locally constant sheaves on Y.

(iv) if $\tilde{Y}$ is a universal covering of Y and $\tilde{X} = f^{-1}\tilde{Y}$, then $\tilde{X} \longrightarrow \tilde{Y}$ induces isos. on homology.

(v) all the homotopy-theoretic fibres of f are acyclic, i.e. have trivial homology.

Equivalences: (i) $\equiv$ (iii) by definition. (ii) $\Rightarrow$ (i) trivial. (i) $\Rightarrow$ (ii) use Postnikov ~~spectral~~ sequence to reduce to proving f^* induces isos. on Ext^* of two locally constant sheaves. Then one uses spec. seq.

$$E_2^{pq} = H^p(Y, \underline{Ext}^q(L, L')) \Longrightarrow Ext_Y^{p+q}(L, L')$$

+ isom.

$$\underline{Ext}^q(L, L')_y = \underline{Ext}^q_{\mathbb{Z}}(L_y, L'_y)$$

which results from local contractibility. (i)$\Leftrightarrow$(iv). May assume Y connected. Use formulas

$$H^*(Y, L) \Leftarrow \mathrm{Ext}^*_{\pi_1 Y}(H_* Y, L) \qquad \pi_1 Y = \mathrm{Aut}(\tilde{Y}/Y)$$

$$H^*(\tilde{Y}, A) = H^*(Y, p_* A). \qquad p : \tilde{Y} \to Y$$

and similar ones for the covering $\tilde{X} \to X$. (iv)$\Leftrightarrow$(v). The fibres of f and $\tilde{X} \to \tilde{Y}$ are the same, hence can assume Y simply-connected. Then use the Leray spectral sequence for the equivalent fibration.

Acyclic maps are stable under composition and homotopy-theoretic base change. (Problem: Find manageable criteria guaranteeing that a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

be homotopy-cartesian, in the sense that we have base change for the locally constant derived categories

$$g^* Rf_* = Rf'_* \, g'^* .$$

sufficient for f or g to be a fibration, or better that it be locally fibre homotopic to a product.)

Classification of acyclic maps

Suppose X fixed and $f: X \to Y$ is an acyclic maps. (Work in homotopy category of pointed connected CW complexes.) Then $\pi_1(X)$ maps onto $\pi_1(Y)$ and the kernel N is perfect. (In effect $\pi_0 \tilde{X} = pt \Rightarrow \pi_1(f)$ surj. Since $H_1 \tilde{X} = N^{ab} = 0$, N is perfect.) Conversely given a perfect normal subgroup N of $\pi_1 X$, there is a unique map $f: X \to Y$ in the pointed homotopy category which is acyclic with $N = \operatorname{Ker} \pi_1(f)$. f is universal map killing N.

If E_1 and E_2 are perfect subgroups of a group G, so is $\langle E_1, E_2 \rangle$ because

$$E_i = (E_i, E_i) \subset (\langle E_1, E_2 \rangle, \langle E_1, E_2 \rangle) \qquad \text{for } i = 1, 2.$$

Better this holds for a whole family.
Similarly if N is the smallest normal subgroup of G contain a perfect group E, then

$$E = (E, E) \subset (N, N)$$

so N is perfect. Thus G contains a largest perfect subgroup N which is normal, and G/N has no perfect subgroup, since any extension of perfect groups is perfect.

From the point of the locally const. derived cats. $D_{lc}(Y)$ is the full subcategory of $D_{lc}(X)$ consisting of complexes whose homology sheaves are $\pi_1 Y = \pi_1 X/N$-modules. The condition that N be perfect implies that the category of $\pi_1 Y$-modules is closed under extensions within the category of $\pi_1 X$-modules:

$$0 \to H^1(G/N, M) \xrightarrow{\sim} H^1(G, M) \to H^1(N, M)^{G/N}$$
$$\to H^2(G/N, M) \hookrightarrow H^2(G, M) \qquad H^1(N,M) = 0$$

$$0 \to H^1(G/N, \operatorname{Hom}_{\mathbb{Z}}(M'', M')) \to \operatorname{Ext}^1_{G/N}(M'', M') \to \operatorname{Ext}^1_{\mathbb{Z}}(M'', M')^{G/N} \to H^2(G/N, --)$$
$$\downarrow\cong \qquad\qquad\qquad\qquad \downarrow\cong \qquad\qquad \downarrow$$

Now we want to understand acyclic maps with Y fixed. Break this up into understanding "acyclic" spaces, and then how they might be "extended" by Y.

~~Basic system of an acyclic space X~~.

Let X be a space, connected with basepoint, such that $\pi_1 X$ is perfect. Assume X is $(n-1)$-acyclic, i.e. for all abelian A

$$H^q(X, A) = 0 \qquad q < n. \qquad (n \geq 2)$$

Then

$$H^n(X, A) = \operatorname{Hom}(H_n X, A)$$

and there is a canonical class in

$$H^n(X, H_n X) = [X, K(H_n X, n)]$$

whence, if X' is the fibre of $X \to K(\pi_n X, n)$, then X' is n[redacted]-acyclic. This leads to the Dror system of a space with perfect fundamental group

$$\begin{tikzcd}
K(H_3X_1,2) \arrow[r] & X_2 \arrow[r] \arrow[d] & K(H_4X_2,4) \\
K(H_2X_0,1) \arrow[r] & X_1 \arrow[r] \arrow[d] & K(H_3X_1,3) \\
& X = X_0 \arrow[r] & K(H_2X_0,2)
\end{tikzcd}$$

The limit X_∞ of the tower is acyclic. [redacted] Observe that any map $A \to X$, where A is acyclic, lifts uniquely to X_∞, hence $X_\infty \to X$ is a universal map from an acyclic space to X. [redacted] However the same is true of the fibre of the map $X \to X^+$, hence X_∞ = this fibre.

The above system is the [redacted] relative Postnikov system for the map $X_\infty \to X$. In particular as the fibre of the latter is ΩX^+, we have

$$\pi_{i+1}(X^+) = \pi_i(\Omega X^+) = H_{i+1}(X_{i-1})$$

for $i \geq 1$.

Given an acyclic space Z, one can form its Postnikov system $Z_{(n)}$. $\pi_1 Z$ is perfect, hence the subgroup

$$(\pi_1 Z, \pi_n Z) \subset \pi_n Z$$

is a $\pi_1(Z)$-perfect group, i.e. $\pi_1 Z$ acts on it and the only trivial-action quotient is zero. If it should happen that $\pi_1 Z$ acts trivially on $\pi_n Z$ for all n, then $Z = X_\infty$, where $X = K(\pi_1 Z, 1)$. In effect, the Postnikov system of Z, then takes the form

$$\begin{tikzcd}
K(\pi_n Z, n) \arrow[r] & Z_{(n)} \arrow[d] & \\
& Z_{(n-1)} \arrow[r] \arrow[d] & K(\pi_n Z, n+1) \\
& \vdots & \\
K(\pi_1 Z, 1) = & Z_{(1)} \arrow[r] & K(\pi_2 Z, 3)
\end{tikzcd}$$

and one has that

$$H^i(Z_{(n-1)}; A) \xrightarrow{\sim} H^i(Z_{(n)}; A) \qquad i<n$$

$$0 \to H^n(Z_{(n-1)}; A) \to H^n(Z_{(n)}; A) \to \mathrm{Hom}(\pi_n Z; A)$$

$$\xrightarrow{d_{n+1}} H^{n+1}(Z_{(n-1)}, A) \to H^{n+1}(Z_{(n)}, A)$$

Since Z is acyclic, it follows that

$$H^i(Z_{(n-1)}; A) = 0 \qquad 0 < i \leq n$$

$$H^{n+1}(Z_{(n-1)}; A) = \mathrm{Hom}(\pi_n Z; A)$$

which means that $Z_{(n)}$ is the Dror system of the space $Z_{(1)} = K(\pi_1 Z, 1)$.

In general, Dror somehow associates a basic acyclic space [redacted] to a [redacted] super-perfect group π_1 acting perfectly on an abelian group π_n', and he [redacted] then shows how an acyclic space then admits a Postnikov-like decomposition into such building blocks.

Cohomology of the loop space.

Let X be a space, connected with basepoint, and let $j : pt \longrightarrow X$ be the inclusion of the basepoint. Then we have constructed a functor

$$\mathbb{R}j_* : D^+_{lc}(pt) \longrightarrow D^+_{lc}(X)$$

right adjoint to j^*. Intuitively this is the map

$$D^+_{lc}(pt) \xrightarrow{\sim} D^+_{lc}(P) \xrightarrow{\mathbb{R}f_*} D^+_{lc}(X)$$

where P is the space of paths starting at the basepoint, and $f : P \longrightarrow X$ is the endpoint map. Consequently $\mathbb{R}j_*(\mathbb{Z})$ is the complex [redacted] whose stalk [redacted] over x gives the cochain complex for the space of paths joining the basepoint to x. In particular

$$j^*\mathbb{R}j_* = C^\cdot(\Omega X)$$

Proposition: Let X be a 1-connected topos. ~~satisfying the conditions of the proposition on p. 6 (category of locally constant sheaves nice, e.g. if X locally 1-connected).~~ Then there is a complex K in $D^+_{lc}(X)$ such that

$$H^q(X; K) = \begin{cases} 0 & q \neq 0 \\ \mathbb{Z} & q = 0 \end{cases}$$

and moreover it is unique up to canonical isom.

Existence. Construct the Postnikov system starting with $K_{(1)} = \mathbb{Z}$:

$$\begin{tikzcd}
H^2(K_{(1)})[-2] \arrow[r] & K_{(1)} = \mathbb{Z} \arrow[d] \\
H^3(K_{(2)})[-3] \arrow[r] & K_{(2)} \arrow[d] \\
 & K_{(3)}
\end{tikzcd}$$

The point is that if we assume by induction that

$$\mathrm{Hom}^q(A, K_{(n)}) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & 0 \neq q \leq n \end{cases}$$

then $A \mapsto \mathrm{Hom}^{n+1}(A, K_{(n)})$ is left exact, hence represented by

$$\mathrm{Hom}^{n+1}(\mathbb{Z}, K_{(n)}) = H^{n+1}(K_{(n)})$$

and there is a canonical map

$$H^{n+1}(K_{(n)}) \xrightarrow{[-n-1]} K_{(n)}$$

whose cofibre is $K_{(n+1)}$. Then the limit $K_{(\infty)}$ is the desired complex K.

Uniqueness. Given another acyclic complex K, it is clear that ~~the~~ map $\mathbb{Z} \longrightarrow K$ extends inductively to a map $K_{(n)} \longrightarrow K$, hence to a map $K_{(\infty)} \longrightarrow K$. It is also clear that this map inductively has to be an isomorphism on the terms of the Postnikov system.

In the non-simply-connected case one proves the existence of a complex K such that

$$\mathcal{H}om^q(L, K) = \begin{cases} 0 & q \neq 0 \\ \mathrm{Hom}_{\mathcal{L}}(L, L_0) & q = 0. \end{cases}$$

where L_0 is a given locally constant sheaf.

Given a basepoint x_0 of X one can consider the spaces of paths $x_0 \times_X X^I$ over X, or one can consider the space of loops $X \times_{X^2} X^I$. The associated direct images complexes are different, e.g. look at the $\mathcal{H}^0$. In the former it is the functions on $\pi_1(X, x_0)$ where π_1 acts by translation, and in the latter it is the functions where π_1 acts by conjugation.

August 8, 1970: The stability problem.

One wants to show that $GL_{n-1}A \longrightarrow GL_n A$ induces isos. on homology in a range depending on $d = \dim(\operatorname{Max} A)$. Classically $A = \mathbb{R}$ one sees that $GL_n A / GL_{n-1} A$ is a space with homotopy type of S^{n-1} and the idea to be used is to make $GL_n A$ act on a simplicial set which is highly-connected and whose isotropy groups are related to $GL_{n-1} A$.

I understand some ingredients which ought to appear in the proof because of the following geometric example. I recall the first stability result is Serre's thm. which says that if E is projective over A of rank r, then $E \simeq A^{r-d} \oplus F$. Geometrically one considers E as a vector bundle over a variety X over k which is spanned by a k-vector space $V \subset \Gamma(X, E)$. Then one proves E contains a trivial sub-line-bundle by noting that the $\{v \in V \mid v(x) = 0 \text{ some } x\}$ is the image of the map of varieties

$$\begin{array}{ccc} F & \longrightarrow & V \\ \| & & \\ \{(v,x) \mid v(x)=0\} & & \end{array}$$

and F is of dimension $N - r + d$ while V is of dimension N. By dimension theory the set of non-vanishing v contains a Zariski-dense open set.

I can form a simplicial complex Q as follows. An i-simplex of Q is defined to be a family $\{v_0, \dots, v_i\} \in V$

such that these sections are everywhere independent. Claim that Q is highly-connected. In fact what we show is that if $i+1 < r-d$, then for all v in a Zariski open dense subset of V, $\{v_0, \dots, v_i, v\}$ is an $(i+1)$-simplex of Q. This implies that if K is any finite subcomplex of Q of dimension $< r-d$, then Q contains the cone on Q, hence $\pi_i Q = 0$ $\quad i < r-d-1$.

I wish to compute the dimension of $\{v \mid v_0, \dots, v_i, v$ not an $(i+1)$-simplex$\} = B$. Let $W = \langle v_0, \dots, v_i \rangle$; its of dimension $i+1$

Form cartesian square

$$\begin{tikzcd} S \arrow[r] \arrow[d] & F \arrow[d] \\ V \arrow[r] & V/W \end{tikzcd} \qquad \begin{tikzcd} (v, x) \arrow[d, mapsto] \\ v+W \end{tikzcd} \qquad v(x) = 0.$$

Then B is the image of $S \to V$. $\big(v \in B \Leftrightarrow v + W \supset$ a vanishing section $\Leftrightarrow \exists\ (v, x, w) \ni (v+w)(x) = 0\big)$. Now S is an affine W bundle over F so

$$\dim S = N - r + d + i + 1 < N$$

because we have assumed $i+1 < r-d$. Again done by dimension theory.

Note that nowhere in this argument did N appear and hence we could take $V = \Gamma(X, E)$ and we still have a simplicial complex beginning in dim $r-d-1$. (If $r \geq d+1$ then Q non-empty and $r \geq d+2$ then Q connected.)

In the geometric case where A = coordinate ring of an affine variety of dimension d over an alg. closed field k, I want to see what range of dimensions I get for $GL_{n-1}A \longrightarrow GL_n A$ to be a homology isomorphism. So take $E = A^n$ and let Q be the simplicial complex whose vertices are the unimodular vectors. In general an i-simplex of Q is defined as a family $\{v_0, .., v_i\}$, $v_i \in E$ $\ni$ $A^{i+1} \longrightarrow E$ is a direct injection.

I want to show that $GL_n A$ acts transitively on the set of i-simplices for $i+1 < n-d$ and that if K is any finite subcomplex of dimension $< n-d-1$ of Q then a cone on it can be found in Q.

Start with K and choose V to be a k-subspace of E containing all of the vertices of K and also generators for E over A. Suppose $\dim V = N$ and let F be the variety of pairs $\{(v, x) \mid v \in V, x \in X \ni v(x) = 0\}$. Then $\dim F = N - n + d$. Let $\{v_0, .., v_i\}$ be an i-simplex of K and set $W = \langle v_0, .., v_i \rangle$. Now let $B = \{v \mid \{v_0, .., v_i, v\}$ not an $(i+1)$-simplex$\}$, i.e. $v + W = v_0 + W$ where $v_0(x) = 0$ for some x. (If $v + W > W$ and $v + W$ not independent, then $v + W \supset v_0$ and v_0 necess. ind. of W).

$$\begin{tikzcd} S \arrow[r] \arrow[d] & F \arrow[d] & (v_0, x) \arrow[d, mapsto] \\ V \arrow[r] & V/W & v_0 + W \end{tikzcd}$$

$$v \longmapsto v + W$$

Thus $B = \mathrm{Im}\{S \to V\}$ is constructible of dimension at most $\dim S = \dim W + \dim F = i+1 + N-n+d < N$ as $i+1 < n-d$.

Next we want to prove transitivity. Given $\{v_0, \dots, v_i\}$ an i-simplex we want to show it is conjugate to the standard one $\{e_1, \dots, e_{i+1}\}$ $e_i = i$th basis element of A^n. ~~[illegible]~~ Given another i-simplex $\{v_0', \dots, v_i'\}$. As $i+1 < n-d$ there is a v such that $\{v_0, \dots, v_i, v\}$ and $\{v_0', \dots, v_i', v\}$ are $(i+1)$-simplices. Now all you have to do is to show ~~[illegible]~~ that any two faces ~~are conjugate~~ of an $(i+1)$-simplex are conjugate. One supposes that $v_0 = v_0', \dots, v_j = v_j'$ and adds a new common vertex and deletes different ones.

~~Now we deduce the cohomological implications of this~~

? Q. The chains on Q give us a $GL_n A$ resolution

$$\to \mathbb{Z}Q_{n-d-1} \to \mathbb{Z}Q_{n-d-2} \to \cdots \to \mathbb{Z}Q_2 \to \mathbb{Z}Q_1 \to \mathbb{Z}Q_0 \to \mathbb{Z} \to 0$$

not acyclic here

$$Q_i = GL_n A / \text{stabilizer of } \{e_1, \dots, e_{i+1}\}.$$

Actually what we know is that there are spectral sequences

$$E_2^{pq} = H^p(GL_n A, H^q(Q)) \Longrightarrow H^{p+q}_{GL_n A}(Q)$$

$$H^p(Q/GL_n A, \theta \mapsto H^q\ ) \Longrightarrow$$

August 9, 1970 stability

Existence of a stable range for the symmetric groups

Proposition: The inductive system $H_i(\Sigma_n) \longrightarrow H_i(\Sigma_{n+1}) \longrightarrow$ is eventually constant, i.e. for each i, $\exists N \ni H_i(\Sigma_n) \xrightarrow{\sim} H_i(\Sigma_{n+1})$ for $n \geq N$.

Proof: Use induction on i; true for $i=0$. Make Σ_n act on the $(n-1)$-simplex permuting the vertices, giving a contractible simplicial set without degeneracies

$$\Sigma_n/\Sigma_{n-3} \Rrightarrow \Sigma_n/\Sigma_{n-2} \rightrightarrows \Sigma_n/\Sigma_{n-1}$$

(non-degenerate part of standard Čech cx. for $\Sigma_n/\Sigma_{n-1} \to pt$)

and giving rise to a spectral sequence

$$E^1_{pq} = H_q(\Sigma_{n-p-1}) \Longrightarrow H_{p+q}(\Sigma_n)$$

where the d_1 is the map

$$\begin{aligned} \text{res} : \quad & H_q(\Sigma_{n-p-1}) \longrightarrow H_q(\Sigma_{n-p}) && \text{if } p \text{ odd} \\ d_1 = 0 & && \text{if } p \text{ even} \end{aligned}$$

By induction we know that for $q<i$ the restriction homomorphism is an isomorphism for n sufficiently large, hence for $E^2_{pq} = 0$ for $0<p\leq m$, $q<i$ for any m, if n is sufficiently large, so the edge homomorphism

$$H_i(\Sigma_{n-1}) \longrightarrow H_i(\Sigma_n)$$

is an isomorphism for all large n, completing the induction.

How good an estimate can we obtain in this way. Thus to get $H_i(\Sigma_{n-1}) \longrightarrow H_i(\Sigma_n)$ onto we need to know that $E^2_{p,i-p} = 0 \quad p = 1, \ldots, i$, hence I must know that

$$H_{i-p}(\Sigma_{n-p-2}) \twoheadrightarrow H_{i-p}(\Sigma_{n-p-1}) \qquad p \text{ odd}$$

$$H_{i-p}(\Sigma_{n-p-1}) \hookrightarrow H_{i-p}(\Sigma_{n-p}) \qquad p \text{ even } > 0$$

and for injectivity I want $E^2_{p,i-p+1} = 0$ so I must have

$$H_{i-p+1}(\Sigma_{n-p-2}) \twoheadrightarrow H_{i-p+1}(\Sigma_{n-p-1}). \qquad p \text{ odd}$$

$$H_{i-p+1}(\Sigma_{n-p-1}) \hookrightarrow H_{i-p+1}(\Sigma'_{n-p}) \qquad p \text{ even } > 0$$

August 10, 1970: stability.

Let A be a finitely-generated algebra over an infinite field k. (This hypothesis on k insures that a non-empty Zariski open subset of affine space over k contains lots of rational points.)

Fix an integer r and let Q be the following simplicial complex: The vertices of Q are vectors $v \in A^{r+n}$ such that $\{e_1, .., e_r, v\}$ is a $(r+1)$-frame, i.e. the associated map $A^{r+1} \longrightarrow A^{r+n}$ is a direct injection. Here $e_i = (0, .., \underset{i}{1}, 0, .., 0)$. A family of vertices $v_0, .., v_i$ forms an i-simplex of Q iff $\{e_1, .., e_r, v_0, .., v_i\}$ is an $(r+i+1)$-frame.

Note that if n is sufficiently large then for any i-simplex $\{e_1, .., e_r, v_0, .., v_i\}$ we have that $\{e_1, .., e_r, v_0, .., v_i, v\}$ is an $(i+1)$-simplex for almost all v. In effect choose a k-subspace V of A^{r+n} containing $e_1, .., e_r, v_0, .., v_i$ and the basis vectors. Then the set of good $v \in V$ contains the set of rational points in a Zariski open-dense subset of $V \otimes_k \bar{k}$. This shows that the connectivity of Q increases with n.

Now let $\Gamma_{r,n}$ be the group of autos. of A^{r+n} which preserve $\{e_1, .., e_r\}$. ~~The Γ_n acts transitively on Q_n iff n is sufficiently large~~

Let $Q_i =$ the set of $(i-1)$-simplices. Then

$$\Gamma_{r,n} / \Gamma_{r+i,n-i} \longrightarrow Q_i$$

X forgot symm. gp.

and we have that

$$\cdots \xrightarrow{d} \mathbb{Z}Q_2 \xrightarrow{d} \mathbb{Z}Q_1 \xrightarrow{\varepsilon} \mathbb{Z}Q_0 \to 0$$

is exact in a range increasing with n. I want to prove that $H_q(\Gamma_{r+1,n-1}) \xrightarrow{\sim} H_q(\Gamma_{r,n})$ is an isomorphism for n sufficiently large. Use induction on q. ~~Let $I_i = \mathrm{Im}\{\mathbb{Z}Q_{i+1} \to \mathbb{Z}Q_i\}$. Then there are exact sequences~~

$$\cancel{0 \to I_i \to \mathbb{Z}Q_i \to I_{i-1} \to 0}$$

~~and long exact sequence~~

$$\cancel{\to H_j^{\Gamma_{r,n}}(\mathbb{Z}Q_i) \to H_j^{\Gamma_{r,n}}(I_{i-1}) \xrightarrow{\partial} H_{j-1}^{\Gamma_{r,n}}(I_i)}.$$

Now the important thing to notice is that the square

$$\begin{array}{ccc}
H_*^{\Gamma_{r,n}}(\mathbb{Z}Q_i) & \xrightarrow{\quad d \quad} & H_*^{\Gamma_{r,n}}(\mathbb{Z}Q_{i-1}) \\
\wr\| & & \wr\| \\
H_*^{\Gamma_{r+i,n-i}}(\mathbb{Z}) & \xrightarrow{\left\{\begin{smallmatrix} \text{res} & i \text{ odd} \\ 0 & i \text{ even}\end{smallmatrix}\right.} & H_*^{\Gamma_{r+i-1,n-i+1}}(\mathbb{Z})
\end{array}$$

commutes. The map $d = \sum_{j=1}^{i} (-1)^j d_j$ where $d_j : Q_i \to Q_{i-1}$ is the map induced by the inclusion $\Gamma_{r+i, n-i} \longrightarrow \Gamma_{r+i-1, n-i+1}$ coming from the fact that a matrix stabilizing $e_1, \dots, e_{r+i}$ also stabilizes $e_1, \dots, \hat{e}_{r+j}, \dots, e_{r+i}$. But these inclusions are conjugate in $\Gamma_{r+i-1, n-i+1}$, hence all d_j induces the same map on homology, so the boundary is zero or restriction depending whether i is even or odd.

By induction hypothesis this restriction homomorphism is an isomorphism in dimensions $< q_0$. Hence in the spectral sequence

0 for n large
$\|$ $(p+q)$-bounded

$$E^1_{pq} = H_q^{\Gamma_{r,n}}(\mathbb{Z}Q_p) \Longrightarrow H_{p+q}^{\Gamma_{r,n}}(\mathbb{Z}Q_\bullet)$$

$\| \longleftarrow$ n large, p bdd.

$$H_q^{\Gamma_{r+p, n-p}}(\mathbb{Z})$$

one has $E^2_{pq} = 0$ for $p \geq 2$, $q < q_0$. Hence it must also be true for $q = q_0$, i.e.

$$H_{q_0}(\Gamma_{r+1, n-1}) \xrightarrow{\sim} H_{q_0}(\Gamma_{r,n})$$

for n large.

Next stage: to show that $\Gamma_{r,n} \hookrightarrow GL_n A$ induces isomorphisms on H_i for n large. Enough to consider $r=1$ because

$$\begin{tikzcd}
\Gamma_{r,n} \arrow[r, hook] & \Gamma_{1,n+r-1} \\
& GL_{n+r-1} = \Gamma_{0,n+r-1} \arrow[u, hook] \\
& \Gamma_{1,n+r-2} \arrow[u, hook] \\
& GL_{n+r-2} = \Gamma_{0,n+r-2} \arrow[u, hook] \\
& \vdots \arrow[u, hook] \\
& GL_n A \arrow[uuuuul, bend left]
\end{tikzcd}$$

~~So make $\Gamma_{1,n}$ act on something highly-connected. So make $\Gamma_{1,n}$ act on the simplicial complex whose i-simplices are vectors in A^n plus an i-frame i.e. a family $\{v_0, v_1, \dots, v_i\}$ such that the $\{v_j - v_0\}_{j=1}^{i}$ forms a frame. Connectivity still the same because for almost all v $v - v_0$ can be added to $\{v_j - v_0\}$ to get a frame. Stabilizer of e_1, e~~

Make $\Gamma_{1,n}$ on the affine space W of linear functionals w on A^{n+1} ~~which carry~~ with $w(e_1)=1$. Form the simplicial complex Q whose i-simplices are families $\{w_0,\dots,w_i\}$ $w_i \in W \ni$ $w_j - w_0$ $j=1,\dots,i$ is an i-frame in the kernel of evaluation on e_1. For large n, $\Gamma_{1,n}$ acts transitively on Q_i with stabilizer of $\{\hat{e}_1, \hat{e}_1+\hat{e}_2, \dots, \hat{e}_1+\hat{e}_{i+1}\}$ which is $\Gamma_{i,n-i}$. So again get spectral sequence with E^1

$$H_q^{\Gamma_{1n}}(\mathbb{Z}) \xleftarrow{\text{edge}} H_q^{GL_n} \xleftarrow{0} H_q^{\Gamma_{1,n-1}} \xleftarrow{\sim} H_q^{\Gamma_{2,n-2}} \xleftarrow{0} ,$$

for n large by earlier ~~[illegible]~~ situation

$$\Gamma_{1n} \quad \left(\begin{array}{c|c} 1 & * \\ \hline 0 & * \end{array}\right) \quad \text{fixes} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

stabilizer of $\hat{e}_1 = (1,0,\dots,0)$ in Γ_{1n} is

$$GL_n \quad \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & * \end{array}\right)$$

and stabilizer of $\hat{e}_1, \hat{e}_1+\hat{e}_2, \dots, \hat{e}_1+\hat{e}_{i+1}$ is

$$i\left\{ \left(\begin{array}{c|cc} 1 & 0 & \\ \hline 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & 0 \\ \hline & * & \end{array}\right) \right. \cong \Gamma_{i,n-i}$$

Conclusion: If A is a finitely-generated algebra over an infinite field k, then for each i

$$H_i(GL_n A) \xrightarrow{\sim} H_i(GL_{n+1} A)$$

for all sufficiently large n. The same argument ought to work with twisted homology with coefficients in any $K_1 A$-module. By Bass one knows that

$$GL_n A / E_n A \xrightarrow{\sim} K_1(A)$$

for n large, consequently we see that for n large

$$BGL_n(A)^+ \longrightarrow BGL_{n+1}(A)^+$$

induces isomorphism on the fundamental group and twisted homology in a range, hence it must induce isomorphism on all homotopy groups in a range increasing with n.

Remaining problem: What to do about ~~A=~~ finite field?

August 18, 1971: On $H^*(GL_3(\mathbb{F}_2), \mathbb{Z}/2)$.

$GL_3(\mathbb{F}_2) = SL_3(\mathbb{F}_2)$ has order $(2^3-1)(2^3-2)(2^3-2^2)$ $= 7.3.8 = 168$ and is simple. In fact one knows (Artin's works p. 400) it is isomorphic to

$$L_2(7) = SL_2(\mathbb{F}_7) / \pm id$$

~~hence~~

$$\cancel{H^q(SL_3(\mathbb{F}_2)) = \begin{cases} 0 & q=1 \\ \mathbb{Z}/2 & q=2 \end{cases}}$$

~~the last coming from exact sequence~~

$$\cancel{0 \to H^1(SL_3(\mathbb{F}_2)) \to \underset{0}{H^1(SL_2(\mathbb{F}_7))} \to H^1(\mathbb{Z}_2) \to H^2(SL_3(\mathbb{F}_2)) \to \underset{0}{H^2(SL_2(\mathbb{F}_7))}}$$

Now it is possible to compute the mod 2 cohomology of $SL_2(\mathbb{F}_7)$. One knows that SL_2 has no 2-torsion and

$$H^*(BSL_2) = \mathbb{Z}/2[c] \qquad \deg c = 4$$

so

$$gr\, H^*(SL_2(\mathbb{F}_7)) = \Lambda[c''] \otimes S[c']$$

$$\deg c'' = 3$$
$$\deg c' = 4$$

In particular if we consider the spectral sequence associated to the central extension

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SL_2(\mathbb{F}_7) \longrightarrow SL_3(\mathbb{F}_2) \longrightarrow 1$$

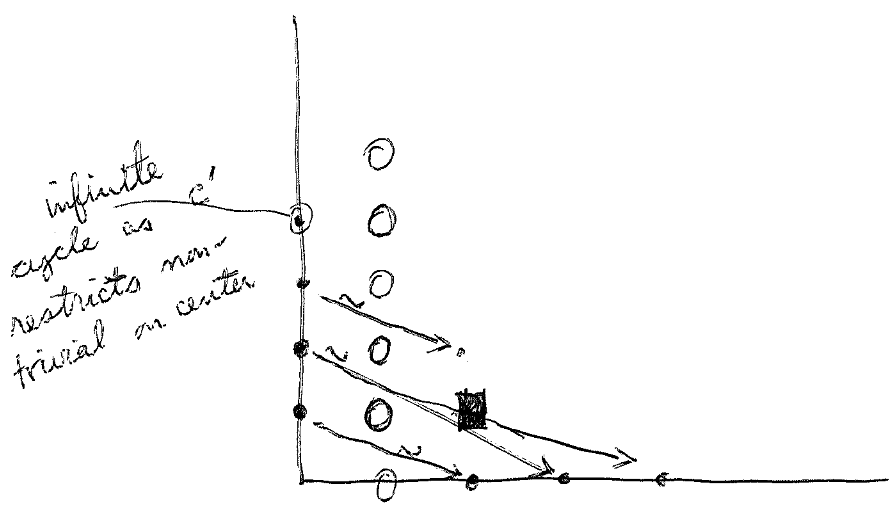
we get an exact sequence

$$0 \longrightarrow H^1(SL_3(\mathbb{F}_2)) \longrightarrow \overset{0}{\overset{\|}{H^1(SL_2(\mathbb{F}_7))}} \longrightarrow H^1(\mathbb{Z}_2) \longrightarrow H^2(SL_3(\mathbb{F}_2)) \longrightarrow \underset{0}{\underset{\|}{H^2(SL_2(\mathbb{F}_7))}}$$

showing that

$$H^q(SL_3(\mathbb{F}_2)) = \begin{cases} 0 & q = 1 \\ \mathbb{Z}/2 & q = 2 \end{cases}$$

Beginning of spectral sequence



$SL_3(\mathbb{F}_2)$ has dihedral group of order 8 for its Sylow subgroup, and one knows its cohomology is detected by elementary abelian 2-subgroups. Thus any non-zero element of its cohomology is a non-zero divisor. Thus denoting by $\alpha \in H^2(SL_3(\mathbb{F}_2))$ the non-zero elt, we have

$$E_3 = H^*(SL_3(\mathbb{F}_2))/(\alpha) \otimes \mathbb{Z}/_2[z^2]$$

$z \in H^1(\mathbb{Z}/2)$ being the generator of the fibre. Since abutment has nothing in degree 2, $d_3(z^2) \neq 0$. Let β represent $\tau(z^2)$. Since $H^3(SL_2(\mathbb{F}_7)) \neq 0$, there is another element $\gamma \in H^3(SL_3(\mathbb{F}_2))$ going into c'. multiplication by c' gives periodicity and since only c' remains in the strip $0 \leq q < 4$, one must have that β is a non-zero divisor in E_3. Thus α, β regular sequence in $H^*(SL_3(\mathbb{Z}_2))$ and $\gamma^2 \in (\alpha, \beta)$. Additively therefore

$$H^*(SL_3(\mathbb{Z}_2)) = \mathbb{Z}/2[\underset{2}{\alpha}, \underset{3}{\beta}] \otimes \Lambda \underset{3}{\gamma}$$

where α is the non-zero element of degree 2, β is the unique non-zero element of degree 3 killed by the map $SL_2(\mathbb{F}_7) \longrightarrow SL_3(\mathbb{Z}_2)$, and γ any element of degree 3 $\neq \beta$. The multiplicative structure can be determined in principle, since the cohomology is detected by the two elementary abelian 2-subgps.

Thus if P is the Sylow group $\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$

we have

$$H^*(P) = \mathbb{Z}/2\,[\underset{1}{x}, \underset{1}{y}, \underset{2}{e}]/(xy)$$

(e = Euler class of repn. of dihedral group P on $\mathbb{R}^2$)

All elements of order 2 are conjugate in $G = SL_3(\mathbb{F}_2)$ (Jordan can. form), hence given $\varphi(x,y,e) \in H^2(G)$ we must have

$$\varphi(t,0,0) = \varphi(0,t,0) = \varphi(0,0,t^2)$$

considering separately the elts. $\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$.

Thus $\alpha = x^2 + y^2 + e$

and

$$\beta = Sq^1(\alpha) = Sq^1(e) = (x+y)e$$

(Note that e is w_2 of the repn of the dihedral group on $\mathbb{R}^2$, ~~have~~ and $Sq^1(w_2) = w_1 \cdot w_2$.). Now $H^3(P)$ has a basis x^3, y^3, xe, ye and all of these vanish for $x \mapsto 0$, $y \mapsto 0$, $e \mapsto t^2$ ~~[illegible]~~, hence

$$\gamma = xe.$$

Thus the relation is $\gamma\beta = \gamma^2$, so

$$H^*(GL_3(\mathbb{F}_2)) \cong \mathbb{Z}/2[\underset{2}{\alpha}, \underset{3}{\beta}, \underset{3}{\gamma}] / (\gamma\beta - \gamma^2).$$

Another attempt at this computation goes as follows. We use that $PSL_2 = SO_3$ is good for the prime 2, ~~[illegible]~~ and

$$H^*(BSO_3) = \mathbb{Z}/2[w_2, w_3].$$

Thus your general theorems furnish an additive isom.

$$H^*(SO_3(\mathbb{F}_q)) = \Lambda[v_1, v_2] \otimes S[w_2, w_3].$$

Unfortunately $SO_3(\mathbb{F}_8)$ is not what we want because

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \overset{\mathrm{Spin}_3}{\overset{\shortparallel}{SL_2}} \longrightarrow SO_3 \longrightarrow 1$$

leads in Galois coh. to

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SL_2(\mathbb{F}_8) \longrightarrow SO_3(\mathbb{F}_8) \xrightarrow{\delta} \underset{\mathbb{Z}_2}{\underset{\shortparallel}{H^1(\mathbb{F}_8, \mathbb{Z}_2)}} \longrightarrow 0$$

where δ is the <u>spinorial</u> <u>norm</u>. Thus it is necessary to cut down to the commutator subgroup $SL_3(\mathbb{F}_2)$ of $SO_3(\mathbb{F}_7)$ and get rid of v_1. It would have been nice if v_1 were a non-zero divisor in $H^*(SO_3(\mathbb{F}_7))$ for then the cohomology of the group $SL_3(\mathbb{F}_2)$ would be easily computable. Unfortunately v_1 dies on any elementary abelian 2 subgroup of $SL_3(\mathbb{F}_2)$ and there are two of these.

(review of Thompson thms.

August 24, 1971: p odd:

Lemma: Let θ be an autom. of an abelian p group B such that $\theta^p = 1$ and $\theta = 1$ on $\Omega_p(B)$. Then $(\theta - 1)B \subset \Omega_p(B)$.

Proof: Let A be the dual of B. Then $\theta = 1 + N$ where $NA \subset pA$. To prove $pN = 0$.

$$\theta^p = 1 + pN + \binom{p}{2} N^2 + \cdots + N^p = 1$$

Let ν be such that $NA \subset p^\nu A$ but $\not\subset p^{\nu+1} A$, so that $\nu \geq 1$. Then

$$\binom{p}{i} N^i A \subset p \cdot p^{\nu i} A \subset p^{\nu+2} A \qquad 2 \leq i < p$$

$$N^p A \subset p^{\nu p} A \subset p^{\nu+2} A$$

(for the last $\nu p \geq \nu + 2 \iff \nu(p-1) \geq 2$ so need $p \geq 3$), so from the equation $\theta^p = 1$ we get

$$pN \cdot A \subset p^{\nu+2} A.$$

But $NA \not\subset p^{\nu+1} A$,

$$\begin{array}{c} A \xrightarrow{N} p^\nu A / p^{\nu+1} A \\ \downarrow p \\ p^{\nu+1} A / p^{\nu+2} A \end{array}$$

So

$$pNA \subset pN^2 A + NpA$$
$$\subset p^2 NA + p^2 N^{p-2} A$$
$$\subset$$

$$N^2A \subset N(pA) = pNA$$

$$N^pA = N^{p-2}(N^2A) \subset N^{p-2}(p^2A) \subset p^2NA$$

if $p \geq 3$. Thus from

$$-pN = \binom{p}{2}N^2 + \cdots + N^p$$

we have

$$pNA \subset pN^2A + N^pA$$
$$\subset p^2(NA)$$

so $pNA = 0$. qed.

Prop: A maximal normal $[p]$-subgroup of a p-group $P \Longrightarrow A$ maximal $[p]$-subgroup.

Proof: Let B be a max. normal ab. subgroup of P containing A. (One knows B is maximal abelian since otherwise the inverse image of a cyclic group in the center of $\mathrm{Cent}(B)/B$ would be normal in P, abelian, and $> B$.) Then $A = \Omega_p(B)$. Let $x \in P$ be an element of order p centralizing A; to show $x \in A$. x normalizes B and induces an autom. of order p trivial on $\Omega_p(B)$, hence by lemma $(x, B) \subset A$.

Claim $\Omega_1 \mathrm{Cent}(A)$ of exponent p. Assume not, and let x, y be two elements of order p centralizing A such that (x,y) is not of order p, and such that $\langle x,y \rangle$ is minimal. Then $\langle x,y \rangle$ not cyclic $\Longrightarrow$ ~~[illegible]~~ $\langle x, yx^{-1}y^{-1} \rangle < \langle x,y \rangle$, hence by minimality

$$x.\, yx^{-1}y^{-1} = (x,y)$$

is of order p. But x,y stabilize $B \supset A \supset 1$, so (x,y) centralizes B; as B maximal abelian, $(x,y) \in B$ $\Longrightarrow (x,y) \in A$. Thus (x,y) belongs to the center of $\langle x,y \rangle$ and $\langle x,y \rangle$ is a group of order p^3 so that every element has order p, because x,y do (p odd again).

If $\Omega_1 \operatorname{Cent}(A) > A$, then taking the inverse image of a ~~[illegible]~~ cyclic subgroup of $\Omega_1\operatorname{Cent}(A)/A$ centralized by P we obtain an abelian group $A' > A$ normal in P, and of exponent p. This isn't possible so $\Omega_1 \operatorname{Cent}(A) = A$, which finishes the proof.

Prop. $\Omega_1 P \subset Z(P) \Longrightarrow P/\Omega_1 P$ has same property.

Proof. Let ~~[illegible]~~ $A = \Omega_1 P$ and let B/A be a maximal normal elem. ab. p-subgroup of P/A. ~~[illegible]~~ Given x in P, x acts on B centralizing A. Since p-th power map $B/A \longrightarrow A$ is injective, it follows that x centralizes B/A, ~~[illegible]~~ showing that B/A is in the center of P/A. By above, B/A is maximal elem. abelian so it contains every element of $\Omega_1(P/A)$. done.