Need a formula for $f: EG \times_G (BG)^c \longrightarrow BG$. In this simplicial setup G acts on the left of $EG \times X$, where

$$\begin{tikzcd} (EG\times X)_p = G^{p+1}\times X \arrow[d, "\pi"] & (g_0,\dots,g_p,x) \arrow[d, mapsto] \\ (EG\times_G X) = G^p \times X & (g_0^{-1}g_1,\dots,g_{\nu-1}^{-1}g_\nu, g_\nu^{-1}x) \end{tikzcd}$$

Now I consider $EG \xrightarrow{\pi} BG$ as being a principal (left) G-bundle which is equivariant for the right action and note that there is a classifying map

$$EG \times_G (EG)^2 \xrightarrow{\tilde f} EG$$ given by

$$\begin{tikzcd} ((g_0,\dots,g_\nu);(g_0',\dots,g_\nu'))_G \arrow[r, mapsto] \arrow[d, "\wr"] & (g_0'g_0,\dots,g_\nu'g_\nu) \\ (g_0^{-1}g_1,\dots,g_{\nu-1}^{-1}g_\nu;\ g_0'g_\nu,\dots,g_\nu'g_\nu) \in BG_\nu \tilde\times (EG)_\nu \arrow[ur, mapsto, "\tilde f"'] & \end{tikzcd}$$

Put in slightly nicer form using $g_0=1$, $x_1 = g_1$, $x_2 = g_1^{-1}g_2$ or

$$\begin{aligned} g_0 &= 1 \\ g_1 &= x_1 \\ g_2 &= x_1x_2 \\ g_\nu &= x_1x_2\cdots x_\nu \end{aligned}$$

we have

$$\tilde f(x_1,\dots,x_\nu; \overset{x_1\cdots x_\nu}{g_0'},\dots,\overset{x_1\cdots x_\nu}{g_\nu'}) = (g_0', g_1'x_1,\dots,g_\nu'x_1x_2\cdots x_\nu)$$

~~The induced map $f: BG \tilde{\times} BG \longrightarrow BG$ is given by~~

or better replacing $\quad g_i'' = g_i' x_1 \cdots x_\nu$

$$\tilde{f}(x_1, .. x_\nu ; g_0'', .., g_\nu'') = (g_0'' x_\nu^{-1} .. x_1^{-1}, g_1'' x_\nu^{-1} .. x_2^{-1}, \ldots, g_\nu'')$$

The induced map $f: BG \tilde{\times} BG \longrightarrow BG$ is given by

$$(x_1, .., x_\nu, g_0''^{-1} g_1'', .., g_{\nu-1}''^{-1} g_\nu'') \longmapsto (x_1 .. x_\nu g_0''^{-1} g_1 x_\nu^{-1} .. x_2^{-1}, \ldots, x_\nu g_{\nu-1}''^{-1} g_\nu'')$$

or

$$\boxed{f(x_1, .., x_\nu ; x_1', .., x_\nu') = (x_1 .. x_\nu x_1' x_\nu^{-1} .. x_2^{-1}, x_2 .. x_\nu x_2' x_\nu^{-1} .. x_3^{-1}, .., x_\nu x_\nu')}$$

Now suppose given a cocycle $u \in I(G^n)$. Then

$$(f^* u)(g_1, .., g_n ; x_1, .., x_n) = u(g_1 .. g_n x_1 g_n^{-1} .. g_2^{-1}, g_2 .. g_n x_2 g_n^{-1} .. g_3^{-1}, \ldots, g_n x_n)$$

is a cocycle on $EG \times_G BG$, which we want to restrict to $EG \times_G (\Sigma G)$. Thus we want $b_{n-1,1}$, which should be (up to sign)

$$(b_{n-1,1} u)_{(g_1, .., g_{n-1}; x)} = \sum_{j=1}^{n} (-1)^{j+1} (f^* u)(g_1, .., g_{j-1}, 1, g_j, .., g_{n-1} ; 1, .. 1, \underset{\substack{\uparrow \\ j\text{th}}}{x}, 1 ..)$$

$$= \sum_{j=1}^{n} (-1)^{j+1} u(g_1, .., g_{j-1}, g_j .. g_{n-1} x g_{n-1}^{-1} .. g_j^{-1}, g_j, .., g_{n-1})$$

Examples: $\underline{n=1}$:

$$b_{01}(x) = u(x)$$

$\underline{n=2}$:

$$b_{11}(g,x) = u(gxg^{-1},g) - u(g,x).$$

$\underline{n=3}$:

$$\begin{aligned} b_{21}(g_1,g_2,x) = {} & u(g_1g_2xg_2^{-1}g_1^{-1},g_1,g_2) \\ & - u(g_1,g_2xg_2^{-1},g_2) \\ & + u(g_1,g_2,x) \end{aligned}$$

Are these cocycles for the vertical d; equivalently is $\delta b_{n-2,2} = 0$? Yes:

Suppose u is a normalized cocycle, i.e. $u(g_1,\dots,g_n)$ is zero if ~~the~~ some $g_i = 1$. Then the same is true of $b_{n-1,1}$ ~~[illegible]~~ as a cochain in x, hence $db_{n-1,1}$ is normalized so its restriction from $I(G^{n-1}\times G^2)$ to $I(G^{n-1}\times V^2G)$ is zero. Thus $b_{n-1,1} \mid G^{n-1}\times G$ will give an element of $H^{n-1}_G(G)$ provided we know that $\delta b_{n-1,1} = 0$. However this is the case because we know by general nonsense that $\delta b_{n-1,1} = \pm\, d b_{n,0}$ which is zero as d is the difference of the maps $G \rightrightarrows pt$. We will now check things for $n=2$:

$$\begin{aligned} (b_{21}\,\delta u)(g_1,g_2,x) &= \delta u(g_1g_2xg_2^{-1}g_1^{-1},g_1,g_2) - \delta u(g_1,g_2xg_2^{-1},g_2) + \delta u(g_1,g_2,x) \\ &= \begin{array}{lll} u(g_1,g_2) & -u(g_2xg_2^{-1},g_2) & +u(g_2,x) \\ -u(g_1g_2xg_2^{-1},g_2) & +u(g_1g_2xg_2^{-1},g_2) & -u(g_1g_2,x) \\ +u(g_1g_2xg_2^{-1}g_2^{-1},g_1g_2) & -u(g_1,g_2x) & +u(g_1,g_2x) \\ -u(g_1g_2xg_2^{-1}g_2^{-1},g_1) & +u(g_1,g_2xg_2^{-1}) & -u(g_1,g_2) \end{array} \end{aligned}$$

$$(\delta b_{11} u)(g_1, g_2, x) = (b_{11}u)(g_2, x) - (b_{11}u)(g_1g_2, x) + (b_{11}u)(g_1, g_2xg_2^{-1})$$
$$= u(g_2xg_2^{-1}, g_2) - u(g_1g_2xg_2^{-1}g_1^{-1}, g_1g_2) + u(g_1g_2xg_2^{-1}g_1^{-1}, g_1)$$
$$- u(g_2, x) + u(g_1g_2, x) - u(g_1, g_2xg_2^{-1})$$

and so we see that

$$b_{21}\delta u = -\delta b_{11} u$$

for any ~~complex~~ cochain $u \in I(G^2)$. We conclude

Proposition: Let $u \in I(G^n)$ be an $n-$cocycle and set

$$(bu)(g_1, \ldots, g_{n-1}; x) = \sum_{j=1}^{n} (-1)^{j-1} u(g_1, \ldots, g_{j-1}, g_j \cdots g_{n-1} x g_{n-1}^{-1} \cdots g_j^{-1}, g_j, \ldots)$$

Then $bu \in I(G^{n-1} \times G)$ is an $n-1$ cocycle for G acting on itself by conjugation; actually $\delta bu = \pm b\delta u$ so that b induces a map

$$b : H^n(BG) \longrightarrow H^{n-1}_G(G)$$

Thus we have succeeded in understanding the map b for G discrete.

Let G^c denote the object of T_G which is the group G with conjugation action. Then G^c is a group in T_G permitting us to define its classifying topos $(T_G)_{G^c}$. An object of $(T_G)_{G^c}$ is thus a sheaf I endowed with a G-action $\mu : G \times I \longrightarrow I$, ~~making~~ giving its structure as an object of T_G forgetting its G^c ~~acting~~ action, together with another action $\nu : G \times I \longrightarrow I$ satisfying

$$\nu\left(gxg^{-1}, \mu(g,a)\right) = \mu(g, \nu(x,a))$$

for $g, x \in G(U)$, $a \in I(U)$ for any U. Now ~~defines an action of~~ the pair (μ,ν) is the same thing as an action of $G \tilde{\times} G$, the semi-direct product with multiplication given by $(g_1,g_2)(g_1',g_2') = (g_1g_1', g_1'^{-1}g_2g_1'g_2')$. But there is an isomorphism

$$(*) \qquad \begin{array}{ccc} G \times G & \longrightarrow & G \tilde{\times} G^c \\ (g_1,g_2) & \longmapsto & (g_1, g_1^{-1}g_2) \end{array}$$

since

$$\begin{array}{ccc} (g_1,g_2) \overset{\times}{} (g_1',g_2') & \longmapsto & (g_1, g_1^{-1}g_2) \overset{\times}{} (g_1', g_1'^{-1}g_2') \\ \downarrow & & \downarrow \\ (g_1g_1', g_2g_2') & \longmapsto & (g_1g_1', g_1'^{-1}g_1^{-1}g_2g_2'), \end{array}$$

and so we get an action of $G \times G$ on I by

$$\boxed{(g_1, g_2)\,a = g_1\,\nu(g_1^{-1}g_2, a) = \nu(g_2 g_1^{-1}, g_1 a)}$$

__Check:__

$$\begin{aligned}(g_1,g_2)\big[(g_1',g_2')a\big] &= g_1\,\nu(g_1^{-1}g_2,\ g_1'\,\nu(g_1'^{-1}g_2', a)) \\ &= g_1\,\nu(g_1^{-1}g_2,\ \nu(g_2'g_1'^{-1}, g_1'a)) \\ &= g_1\,\nu(g_1^{-1}\underbrace{g_2 g_2'}g_1'^{-1}, g_1'a) \\ &= \nu(g_2g_2'(g_1g_1')^{-1}, g_1g_1'a) \\ &= (g_1g_1', g_2g_2')\,a.\end{aligned}$$

Therefore we obtain an equivalence of categories

$$(T_G)_{G^c} \simeq T_{G\times G}\,.$$

The point of this calculation is to define the trangression map

1)
$$\begin{array}{c}\mathrm{Ker}\{H^*((T_G)_{G^c}) \longrightarrow H^*(T_G)\} \\ \downarrow \tau \\ H^{*-1}(T_G/G^c)\end{array}$$

thereby obtaining the map b of before. If you forget the G action it should be compatible with what was defined before:

2)
$$\begin{array}{c}\mathrm{Ker}\ \{H^*(T_{G^c}) \longrightarrow H^*(T)\} \\ \downarrow \\ H^{*-1}(G^c)\end{array}$$

Here's how it should go: Start with an injective $G \times G$ resolution of k, $I^\cdot$. To compute the map 1) on page 30, we first note that the restriction map for $T_G \longrightarrow (T_G)_{G^c}$ when translated through the isomorphism $G \times G \simeq G \tilde{\times} G^c$ is the restriction from $G \times G$ to ΔG. So an element x of $\operatorname{Ker}\{H^*((T_G)_{G^c}) \to H^*(T_G)\}$ is represented by a cocycle $\dot{x} \in I^\delta(pt)$ which is invariant under $G \times G$ and which is a boundary $\dot{x} = dy$ where $y \in I^{\delta-1}(pt)$ is invariant under ΔG i.e.

$$(*) \qquad (g,g)\, y = y$$

Now applying δ^c, the boundary for G^c, we get

$$(\delta^c y)(g) = (e,g)y - y \quad \in I^{\delta-1}(G)$$

which is an invariant class under the ΔG action, i.e.

$$\left[(g_1,g_1)\delta^c y\right](g_2) = (g_1,g_1)\left[\delta^c y(g_2)\right] - (\delta^c y)(g_1 g_2 g_1^{-1})$$

$$= (g_1,g_1)\left[(e,g_2)y - y\right] - (e, g_1 g_2 g_1^{-1})y + y$$

$$= (g_1, g_1 g_2)y - y - (g_1, g_1 g_2)y + y = 0$$

where we have used $(*)$.

Thus $\delta^c y \in \Gamma_G(G, I^{8-1})$ represents an element of $H_G^{8-1}(G)$. If $dy = 0$ so that y represents an element c of $H^{8-1}(T_G)$, then $\delta^c y$ represents the image of c under $d_1 : H^*(T_G) \longrightarrow H^*(T_G/G^c)$ which is zero since it is the difference of the maps coming from $G^c \rightrightarrows pt$. (Thus if $y \in I^{8-1}(pt)$ is ΔG-invariant and if $dy = 0$, then $\delta^c y \in I^{8-1}(G)$ is a boundary $\delta^c y = dz$ where $z \in I^{8-1}(G)$ is invariant for conjugation

$$(g,g)\, z(x) = z(gxg^{-1}). \quad)$$

This takes care of 1) on page 30 and compatibility with 2) on page 30 is clear. What now has to be done is to bring in the map $f : (T_G)_{G^c} \longrightarrow T_G$. Consider the homomorphism pr_2

$$\begin{tikzcd} & (x, xy) \quad G \times G \arrow[dr, "pr_2"] & \\ G \tilde{\times} G^c \ (x,y) \arrow[ur] \arrow[rr] & & G \quad xy \end{tikzcd}$$

~~It has the property that if you forget G^c, i.e. compose with $G \xrightarrow{\Delta} G \times G$ you get the identity, so pr_2 doesn't induce f. What I am looking for is a map which associates to a G-sheaf a $G \times G$ sheaf on which ΔG acts trivially.~~

From the boxed formula on page 26 on sees that $f(g,1)^{\sim} = f(1,g)^{\sim} = g$, hence $f : G \tilde{\times} G \to G$ is $f(g_1,g_2) = g_1 g_2$ and so $f = pr_2 : G\times G \longrightarrow G$. Then the situation is:

$$
\begin{tikzcd}
H^*(BG) \arrow[r, "f^* = pr_2"] \arrow[dr, "{(id,\, \tau_G)}"'] & H^*(BG\times BG) = H^*_G(BG) \arrow[d] \\
& H^*(BG) \oplus H^{*-1}_G(G) \cong H^*_G(\Sigma G)
\end{tikzcd}
$$

So to realize the map τ_G you take $x \in H^*(BG)$ and lift to $pr_2^* x - pr_1^* x \in H^*(BG\times BG) = H^*((T_G)G^c)$ which restricts to zero under Δ and hence by 1) on page 30 defines an element of $H^{*-1}_G(G^c)$.

April 8, 1970

Let $G \hookrightarrow G'$ be an inclusion of groups in a topos T. Given $x \in H^q(BG')$ which restricts to zero on G, represent x by a G-invariant cocycle $\dot{x} \in I^q(pt)$, where $I^\cdot$ is an ~~[illegible]~~ resolution of the constant sheaf ~~[illegible]~~ k by injective G'-modules. Then by hypothesis $\dot{x} = dy$ where $y \in I^{q-1}(pt)$ is G-invariant. Then $u: g'G \mapsto g'y - y$ is an element of $I^{q-1}(G'/G)$ which is a cocycle since

$$(du)(g') = g'dy - g'y = g'\dot{x} - \dot{x} = 0$$

and also is G-invariant since

$$u(gg'G) = gg'y - y = g(g'y - y) = g\,u(gG).$$

Hence u ~~[illegible]~~ represents an element of $H^{q-1}_G(G'/G)$. Altering y by a G-invariant cocycle z, u changes by $v(g') = g'u - u$. Thus if we consider the map

$$\alpha: H^{q-1}(G) \longrightarrow H^{q-1}_G(G'/G)$$

which is the difference of the maps

$$(*) \qquad H^q(BG) \underset{pr_2}{\overset{pr_1^*}{\rightrightarrows}} H^{q-1}_{G\times G}(G') \simeq H^{q-1}_G(G'/G),$$

Then we obtain a well-defined map

$$\boxed{\operatorname{Ker}\{H^q(BG') \to H^q(BG)\} \longrightarrow H^{q-1}_G(G'/G)/\alpha H^{q-1}(BG).}$$

Actually $(*)$ might better be replaced by the two maps

$$\begin{tikzcd} H^*_{G\times G}(G') \arrow[r, phantom, "\cong"] & H^*_G(G'/G) & \\ \arrow[r, phantom, "\cong"] & H^*_G(G\backslash G') & H^*_G \arrow[ul] \arrow[l] \end{tikzcd}$$

<u>Example:</u> Consider $G \xrightarrow{\Delta} G\times G$. Then

$$\begin{array}{c} H^*_G(G'/G) = H^*_G(G^c) \\ \| \\ H^*_{G\times G}(G\times G) \quad \text{where } (g_1,g_2)\cdot(x_1,x_2) = (g_1x_1g_2^{-1}, g_1x_2g_2^{-1}) \end{array}$$

so there are two maps

$$H^*(BG) \rightrightarrows H^*_G(G^c)$$

defined by

$$H^*_{G\times G} \longrightarrow H^*_{G\times G}(G\times G).$$

~~This map factors~~

~~I think these maps are the same but this isn't so clear~~
~~the factoring~~

$$\cancel{H^*_{G\times G}(pt) \longrightarrow H^*_{G\times G}(G') \longrightarrow H^*_{G\times G}(G\times G)}$$

$$\cancel{x_1x_2^{-1} \longleftarrow\!\shortmid (x_1,x_2)}$$

~~showing that g_2 acts trivially, but it also factors the other way $(x_1,x_2) \mapsto x_1^{-1}x_2$ showing that g_1~~

α should be zero here?

σ endomorphism of G. Let I° be an injective resolution of the constant sheaf k by injective (σ, G)-sheaves, i.e. sheaves I endowed with a G action and $\sigma : I \to I$ such that

$$\sigma(gu) = \sigma(g)\cdot\sigma(u)$$

Restricting to G^σ, this endom. σ must be homotopic to the identity so $\exists\, h$ G^σ-equivariant $\ni$

$$id - \sigma = dh + hd.$$

Let $x \in H^8(BG)^\sigma$, and represent x by $\dot{x} \in I^8(pt)$. Then

$$\dot{x} - \sigma\dot{x} = dy$$

where $y \in I^{8-1}(pt)$ is G-invariant. So

$$dy = \dot{x} - \sigma\dot{x} = dh\dot{x}$$

and $h\dot{x} - y \in I^{8-1}(pt)$ is a G^σ-invariant cycle, which represents $\Phi(x)$.

Now define $u \in I^{8-1}(G/G^\sigma)$ by

$$u(gG^\sigma) = g(h\dot{x} - y) - (h\dot{x} - y)$$
$$= gh\dot{x} - h\dot{x}$$

since y is G-invariant.

On the other hand we want to compute $\tau x \in H^{8-1}(G)$.

So choose $z \in I^{8-1}(pt)$ with $dz = \dot{x}$; since $x \mapsto 0$ in $H^*(BG^\sigma)$ one can suppose z is G^σ-equivariant. Then τx is represented by $v \in I^{8-1}(G)$

$$v(g) = gz - z$$

Finally we consider

$$\theta : G/G^\sigma \longrightarrow G$$
$$gG^\sigma \longmapsto g(\sigma g)^{-1}$$

and compute $\theta^* \tau x$. Clearly it is represented by

$$(\theta^* v)(gG^\sigma) = v(g(\sigma g)^{-1}) = g(\sigma g)^{-1}z - z.$$

So our problem now is to prove that

$$gG_\sigma \longmapsto g(\sigma g)^{-1}z - z \quad \text{and} \quad gh\dot{x} - h\dot{x}$$

are cohomologous. Now

$$\begin{aligned} gh\dot{x} - h\dot{x} &= g(hdz) - (hdz) \\ &= g(z - \sigma z - dhz) - (z - \sigma z - dhz). \\ &= (gz - z) - (g(\sigma z) - \sigma z) - d[g(hz) - hz], \end{aligned}$$

and $gG^\sigma \mapsto g(hz) - hz$ is well defined element of $I^{8-2}(G/G^\sigma)$, so consequently we are reduced to relating the cocycles

$$g(\sigma g)^{-1}z - z \quad \text{and} \quad gz - z - (g(\sigma z) - \sigma z) \quad \text{of } I^{8-1}(G/G^\sigma)?$$

Why if univ. arrow $kF_q \longrightarrow [\ ,B]$ exists then $H_*(B)$ finite.

April 8, 1970.

Let $\tilde{k}(X)$ be the reduced Grothendieck group of $\mathbb{F}_q$-vector bundles over X, and suppose there is a universal map $\tilde{k}(X) \longrightarrow [X,B]$ of k to a representable functor. I know B is a homotopy commutative, associative, H-space with inverses. Here's the proof: Fix $y \in k(Y)$, then you get $\tilde{k}(X) \longrightarrow \tilde{k}(X\times Y) \longrightarrow [X\times Y, B] = [X, B^Y]$, so by universality a unique map in the homotopy category $B \xrightarrow{\varphi_y} B^Y$ such that

$$\begin{tikzcd}
\tilde{k}(X) \arrow[r, "x \mapsto x \boxtimes y"] \arrow[d] & \tilde{k}(X\times Y) \arrow[d] \\
{[X,B]} \arrow[r, "(\varphi_y)_*"] & {[X\times Y, B]}
\end{tikzcd}$$

commutes for all X. Thus for each y, Y, get $Y \longrightarrow B^B$ hence a unique map $B \longrightarrow B^B$ or $\mu: B\times B \longrightarrow B$ such that

$$\begin{tikzcd}
\tilde{k}(X) \times \tilde{k}(Y) \arrow[r] \arrow[d] & \tilde{k}(X\times Y) \arrow[d] \\
{[X,B] \times [Y,B]} \arrow[r, "\mu"] & {[X\times Y, B]}
\end{tikzcd}$$

commutes for all X, Y. Similarly μ must be commutative, etc. up to homotopy.

I want to show $H_n(B)$ is finite for each n. I know that $H_*(B) \otimes \mathbb{Q} = 0$ and $H_n(B, \mathbb{Z}/l)$ is finite for each n by calculation of characteristic classes, for l prime to p, and that $H_*(B, \mathbb{Z}/p) = 0$.

Since $$0 \to H_n(B) \otimes \mathbb{Z}/\ell \to H_n(B, \mathbb{Z}/\ell) \to \mathrm{Tor}_1(H_{n-1}(B), \mathbb{Z}/\ell) \to 0.$$

one sees that $H_n(B)_{(\ell)}$ is a ℓ-torsion group with finite "socle", hence $\mathrm{Hom}(H_n(B), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is a finite type $\mathbb{Z}_{\hat{\ell}}$-module and so

$$H_n(B)_{(\ell)} = (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r + \text{finite } \ell\text{-group}.$$

Next I note that the characteristic class calculation shows that

$$H^n(BGL_N(\mathbb{F}_q)^2, \mathbb{Z}/\ell) \xleftarrow{\sim} H^n(BGL_{N+1}(\mathbb{F}_q)^2, \mathbb{Z}/\ell)$$

for $N >> n$, ~~[illegible]~~. By 5-lemma this must be true for all finite coefficient groups of order prime to p, hence for all torsion groups with torsion prime to p. But if A is an arbitrary abelian group without torsion, there is an exact sequence

$$0 \to A[p^{-1}] \to A \otimes \mathbb{Q} \to (A \otimes \mathbb{Q})/A[p^{-1}] \to 0,$$

hence one sees that for any abelian group Γ

$$H^n(BGL_N(\mathbb{F}_q)^2, \Gamma) \longleftarrow H^n(BGL_{N+1}(\mathbb{F}_q)^2, \Gamma)$$

is an isomorphism modulo p-torsion abelian groups. Next note that if Γ ~~[illegible]~~ arbitrary, then

$$H^*(B, \Gamma) \hookrightarrow \varprojlim_N H^*(BGL_N(\mathbb{F}_q)^2, \Gamma)$$

since ~~[illegible]~~ any element of $k(X)$ is of the form $[E] - [F]$ ~~[illegible]~~ Fixing the degree H^n & taking N large we have for $\Gamma = \mathbb{Q}_\ell/\mathbb{Z}_\ell$

$$\begin{array}{ccc} H^n(B,\Gamma) & \hookrightarrow & H^n(BGL_N(\mathbb{F}_q)^2, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ \| & & \text{finite} \\ \mathrm{Hom}(H_n(B), \mathbb{Q}_\ell/\mathbb{Z}_\ell) & & \end{array}$$

hence $H_n(B)_{(\ell)}$ must be finite. q.e.d.

Outline: (coh. of finite groups of rational points)

§1. G group in a topos T
Eilenberg-Moore spectral sequence
suspension homomorphism
~~good situa~~ Borel's condition. + the good situation!

§2. G endom. σ
map Φ, compatible with suspension
derivation property strongly formulated i.e. $S(P)^\sigma \xrightarrow{\partial}$ —
~~...~~ computation on 1+2 dim'l classes.

§3. main theorem in abstract form.
i) G Kunneth + Borel's condition
ii) σ preserves P given
$\Longrightarrow$ (a) $E^2 = E^\infty$ $= S(P_\sigma) \otimes \Lambda(P^\sigma)$
(b) $H^*_\sigma(G^t)$ free over $S(P_\sigma)$ with simple generating subspace $\Phi(P^\sigma)$

Cor: ℓ odd $\Longrightarrow$ $H^*_\sigma(G^t) \cong S(P_\sigma) \otimes \Lambda \Phi(P^\sigma)$.

§4. Apply to algebraic groups
~~Borel's condition~~ Künneth
Borel's condition $\Leftrightarrow$ true for compact form
Main theorem: G alg. gp over $k = \bar{k}$ satisfying Borel's condition (e.g. when ℓ odd equiv. to G has no ℓ-torsion) + let P gen. subs. for BG stable under σ. Then

$$H^*_G(pt) = \quad \text{———}$$

§5. Examples of $GL_n(\mathbb{F}_q)$.
1. torus

2. $GL_n(\mathbb{F}_q)$ detecting by the appropriate torus.

[illegible] multiplicative structure [illegible] (esp. $\ell=2$)

§6. $O_n(\mathbb{F}_q)$, $Sp_n(\mathbb{F}_q)$, $U_n(\mathbb{F}_q)$ possibly Suzuki groups ?

? §7. Theorem on restriction to centralizer of ~~the group~~ a maximal abelian ℓ-subgroup.

~~stable homotopy of the classical~~

difficulties:

~~a)~~ §2 still presents technical problems

~~when it's~~ computation of E^2 in §3 will be a pain

~~[illegible]~~

Summary of paper on the cohomology of finite groups of rational points.

1/ Let G be a group in a topos T and let X be an object in T_G, then there is a spectrla sequence $E_1^{pq} = H^q(G^p x X) \Longrightarrow H_G^{p+q}(X)$, which is multiplicative. It is necessary to ~~make~~ set up this spectral sequence, and to explicit the d_1. Then define the ~~transgr~~ suspension $H^*(BG) \text{ -- } H^*(G)$, and the refined suspension, and the map Φ in case there is an endomorphism θ.

2. Now suppose that G satisfies the Kunneth theorem, i.e. $H^*(G)$ is finite dimensional and $H^*(GxX) = H^*(G) \otimes H^*(X)$ for any object X. Then we can identify the E_2 term with $\mathrm{Ext}^p_{H^*(G)}(k, H^*(X))$. First example X= pt and G has no ℓ-torsion, so $H^*(G)$ is an exterior algebra with odd degree generators as an algebra, hence also as a Hopf algebra, so the E_2 is a polynomial ring with even degree generators, the spectral sequence collapses and the image of the suspension homomorphism is the subspace of primitive elements. Refinement in case of characteristic 2. Second example: Take ~~X=G/G^o~~ $X = G^t$, assume that one can ~~show~~ lift P to $H^*(BG)$ so as to be invariant under θ σ and that we are in the situation of Borel. Then compute the E_2 term and you find it is the tensor product of $E_2^{0*} = U(P^\sigma)$ and ~~E_2^{1*}~~ E_2^{1*} the symmetric algebra generated by $E_2^{1*} = P_\sigma \cdot \sigma$. Then by your map Φ you produce elements of $H^*_G(G^t)$ which restric ~~xxxxxx~~ to P^o and by the obvious map $H^*(BG) \text{-- } H^*_G(G^t)$ you get the elements of P_o. Therefore the spectral sequence degeneraties and so you conclude your first theorem ~~xxxx~~ in abstract form. ~~xxxxxxx~~

3. Now you apply ~~xxxxxxxxxxxxxx~~ your theorem in the case where T is the etale topos of sheaves on schemes of finite type over an algebraically ~~xxx~~ closed field and where G is an algebraic group endowed with an endomorphism σ $\ni (1-\sigma)G = G$. I ~~xxxx~~ guess it will be desirable ~~xxxxxxxxx~~ to do the proof of the Kunneth formula and the proof that ~~x~~ Borels condition holds for a reductive group iff it holds for the compact form, or maybe one should avoid that latter complexity. Give ~~a simple~~ examples

5. Localization theorem and the second theorem on the restriction to the centralizer of a maximal elementary abelian ℓ-subgroup. You need to go over this theorem to see exactly how much can be done in the topos and what has to be done geometrically. It seems that one has to prove the existence of A, the goodness of Z when $\ell \neq 2$,

This requires work because you want now to understand the relation with Φ.

4. Include your theorem 3 about the computation of the cohomology of $H^*(BG)$ from the various finite groups, independent of ~~the~~ assumptions of goodness.

Examples: What the map Φ does in terms of the restriction to the centralizer. Formulas for the general linear, orthogonal and symplectic groups.

weak points: Second theorem, third theorem, Computation of Φ for a torus.

$$(g, x) \longmapsto (g, gx)$$

$\boxed{\Phi \text{ again}}$

$$g g g_1^{-1}, g_1 x \qquad g_1 g g_1^{-1}, g_1 g x$$

$$h : G \times I \longrightarrow I$$

$$dh(g, x) + h(g, dx) = gx - x$$

$$\underset{\text{trivial action}}{G \times I} \longrightarrow \underset{\text{non-trivial action.}}{G \times I}$$

$$(g, x) \longmapsto (g, gx).$$

I injective $G \times G$-resolution

$$G \underset{\Gamma_\sigma}{\overset{\Delta}{\rightrightarrows}} G \times G$$

$(I^\bullet)^\Delta$ $\quad$ $(I^\bullet)^{\Gamma_\sigma}$ are homotopy equivalent complexes because both are injective complexes. Hence $\exists$

$$\varphi : I^{\bullet\Delta} \longrightarrow I^{\bullet\Gamma}$$

$$(g_1, \sigma g_1)\, \varphi(x) = \varphi((g_1, g_1) x)$$

and for some reason, not too clear I've never used this before so assume $\theta(IJ) = 0$ known.

$$SP \longrightarrow H(BG)$$

equivariant, where σ acts on P.

$$\begin{tikzcd}
S(P)\otimes S(P) \arrow[r] \arrow[d] & {} \\
H^*(BG\times BG)
\end{tikzcd}$$

$$S(P)\otimes S(P).$$

$$I\cap J/I\cdot J \cong S(P_\sigma)\otimes P^\sigma$$

__Lemma:__ σ endom. of P. $I, J \subset S(P\oplus P) = S(P)\otimes S(P)$ as above. Then $\exists$ comm. triangle

$$\begin{tikzcd}
& f \quad S(P)^\sigma \arrow[dl, "d(f\otimes 1 - 1\otimes f)"'] \arrow[d, "d"] \\
I\cap J/IJ & (S(P)\otimes \Lambda P)^\sigma \arrow[d] \\
& S(P_\sigma)\otimes \Lambda P^\sigma \arrow[ul, "\simeq"] \arrow[ul, bend left]
\end{tikzcd}$$

module for $S(P+P)/I+J = S(P_\sigma)$

$U(X) = C.q>0U2q(X),$

verification of the conjecture.

first relate ∂ to $I \cap J / IJ$.

stages: 1) define $\Phi : I \cap J / IJ \longrightarrow$ ______

then Φ

$$
\begin{tikzcd}
H(BG)^{\sigma} \arrow[rr, "\alpha \mapsto pr_1^*\alpha - pr_2^*\alpha"] \arrow[d, equal] & & I \cap J / IJ \arrow[rr] \arrow[d, leftrightarrow] & & H(\ \ ) \\
S(P)^{\sigma} \arrow[rr, "\partial"] \arrow[dr, "d"'] & & S(P_{\sigma}) \otimes \Lambda P^{\sigma} \arrow[urr, bend right] \arrow[dr] & & \\
& (S(P) \otimes \Lambda P)^{\sigma} \arrow[ur] \arrow[rr] & & S(P_{\sigma}) \otimes \Lambda P &
\end{tikzcd}
$$

Proof of the conjecture.

defn. of Θ.

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why is $(g,e)b$ homologous to b over G^t if $db=0$?

$$G^t\times I^\circ \longrightarrow G^t\times I^\circ$$

$$(g,x) \longmapsto (g,(g^{-1}e)^{-1}x)$$

G acts through $g \circ \sigma g$ on I

$$\left(g_1 g(\sigma g_1)^{-1}, (g_1,\sigma g_1)x\right) \longmapsto \left(g_1 g(\sigma g_1)^{-1}, (g_1 g(\sigma g_1)^{-1},e)(g_1,\sigma g_1)x\right)$$

$$(\sigma g_1 g^{-1}\,\cancel{g_1}, \sigma g_1)x$$

$$g g(\sigma g_1)^{-1}, (g_1 g^{-1}, \sigma g_1)x$$

$$G^t\times I^\Delta \longrightarrow G^t\times I^\Gamma$$

$$(g,x) \longmapsto (g, \cancel{(g,x)}\ \varphi(g,x))$$

$$\Big\downarrow\ g_1(\sigma g_1)^{-1}\ \Big\downarrow$$

$$\left(g_1 g(\sigma g_1)^{-1}, (g_1,g_1)x\right) \longmapsto \left(g_1 g(\sigma g_1)^{-1}, \varphi(g_1 g(\sigma g_1)^{-1}, (g_1,g_1)x)\right)$$

$$(g_1,\sigma g_1)\,\varphi(g,x) = \varphi(g_1 g(\sigma g_1)^{-1}, (g_1,g_1)x)$$

~~m(qpx - x) = ψ1( ) + y ( ) in U*(S2m-1/ ℤp × X)(p).~~

original proof

$$G^t \quad \Gamma^*(I^\cdot) \qquad\qquad G^t \quad \Delta^*(I^\cdot)$$

$$(G\times G)\times_G I = {}^{G}I$$

$$T_G/G^t \longrightarrow T_{G\times G}/G^s \xleftarrow{\ \sim\ } T_G$$

$$G \xrightarrow{\ \Gamma\ } G\times G$$

Start with a $G\times G$-sheaf I

$$G^t \times \Gamma^*(I) \qquad \Gamma^*(\cancel{G\times G}\times_{G} I)$$

$$\Gamma^*(\cancel{G\times G}\times_{G} I) \longrightarrow G^s \times I$$

$$g_1(g,x) = (g_1 g\sigma g_1)^{-1}, g_1\sigma g_1) x)$$

$$(g_1,g_2)(g,e,x)$$

$$= + (g_1 g, g_2,$$

$$\begin{array}{ccc} G^s\times I & \longleftarrow & \Delta^* I \\ \downarrow & & \downarrow \\ G^s & \longleftarrow & pt \end{array}$$

Situation: T torus, σ endo of T

Problem: Compute $\Phi : H^*(BT)^\sigma \longrightarrow H_T^{2*-1}(T^t)$.

Recall definition of Φ: to see if its range can be extended. Start with $\alpha \in H^*(BG)$ $\ni$ $\sigma^*\alpha = \alpha$. Then

$$\beta = pr_1^*\alpha - pr_2^*\alpha \in \underbrace{\mathrm{Ker}\,(\Delta^*)}_{I} \cap \underbrace{\mathrm{Ker}\,(\Gamma_\sigma)^*}_{J}$$

where $\Delta : G \longrightarrow G\times G$ + $\Gamma_\sigma = (id, \sigma) : G \longrightarrow G \times G$.

Hence if we represent β by a cocycle a, then

$$a = db \qquad (\Delta G)b = b$$
$$= dc \qquad (\Gamma_\sigma G)\, c = c$$

so the difference $b-c$ is stable under G^σ.

Indeterminacy is: can modify $b-c$ by b' ~~Im $H^*(BG) \to H^*$~~

$$\begin{tikzcd}
H^*(BG^\sigma) & H^*(BG) \arrow[l] \\
H^*(BG) \arrow[u] & H^*(BG\times BG) \arrow[l, "\Gamma_\sigma^*"] \arrow[u, "\Delta^*"]
\end{tikzcd}$$

So the point is that Φ is the composition

$$H^*(BG)^\sigma \xrightarrow{\alpha \mapsto pr_1^*\alpha - pr_2^*\alpha} I \cap J \xrightarrow{\Theta} H^{*-1}(BG^\sigma)/H^{*-1}(BG)$$

another point is that $\Theta(IJ) = 0$. In effect if a rep. α in I a' rep. α' in J, and so

$$a = db \qquad (\Delta G)b = b$$
$$a' = dc \qquad (\Gamma_\sigma G)c = c$$

then $aa' = db \cdot a' = d(ba')$ $\qquad \Delta G\,(ba') = ba'$

and so $\quad aa' = a\,dc = \cancel{\ } (-1)^{\deg a} d(ac) \qquad (\Gamma_\sigma G)(ac) = ac.$

$\theta(\alpha\alpha')$ rep. by $\cancel{ba' \mp ac}$

$\cancel{= bdc \mp db\cdot c}$

$\cancel{= d(bc}$

$$ba' - (-1)^{\deg a} ac$$

$$= b\,dc - (-1)^{\deg a}(db)c$$

$$= (-1)^{\deg b}\left\{(-1)^{\deg b} b(dc) + (db)c\right\}$$

$$= (-1)^{\deg b}\, d(b\cdot c)$$

and $b\cdot c$ is invariant under G^σ. ✓ | must be rewritten with G^t instead of G^σ

Now $\theta : \dfrac{I\cap J}{IJ} \longrightarrow H(BG^\sigma)/H(BG)$

$$\frac{I\cap J}{IJ} = \mathrm{Tor}_1^R(R/I, R/J)$$

Here $R = H^*(BG)\otimes H^*(BG)$ and I is the diagonal, so Tor_* ~~[illegible]~~ can be computed via the Koszul complex

April 9, 1970:

Let T be a topos, ~~and~~ let G be a group in T, and let T_G be its classifying topos. If X is an object of T_G we have the covering $G\times X \xrightarrow{pr_2} X$ giving rise to the simplicial object of T_G

(1) $$G\times G\times X \rightrightarrows G\times X \longrightarrow X$$

and to the spectral sequence

(2) $$E_1^{pq} = H^q(G^p\times X) \Longrightarrow H_G^{p+q}(X),$$

where $H_G^*(X)$ denotes the cohomology of the topos T_G/X (coefficients are $\mathbb{Z}/\ell$ always).

~~When X = pt, the [illegible] of [illegible] of T_G, we have~~

One of the edge homomorphisms of this spectral sequence is the ~~$H_G^*(pt) \to H^*(pt) \to [illegible]$~~ map

(3) $$H_G^n(X) \longrightarrow H^n(X)$$

which forgets the G-action. Its kernel $F_1 H_G^n(X)$ maps:

(4) $$F_1 H_G^n(X) \longrightarrow H^{n-1}(G\times X)/\delta H^{n-1}(X)$$

We now describe these in terms of cocycles.

So suppose $I^\cdot$ is an injective resolution of the constant sheaf $\mathbb{Z}/\ell$. Applying $I^\cdot$ to the simplicial object (1) of T_G we obtain a double complex

$$K^{pq} = I^q_G(G^{p+1} \times X)$$

which using the standard formulas for getting unnormalized cochains can be written

(5) $$K^{pq} = I^q(G^p \times X)$$

where the vertical differential $d: I^q \longrightarrow I^{q+1}$ comes from the fact that $I^\cdot$ is a complex, and where the horizontal differential is given by the standard formula

(6)
$$\begin{aligned}(\delta f)(g_1, \dots, g_{q+1}, x) = {} & g_1 f(g_2, \dots, g_{q+1}, x) \\ & + \sum_{j=1}^{q} (-1)^j f(\dots, g_j g_{j+1}, \dots, x) \\ & + (-1)^{q+1} f(g_1, \dots, g_q, g_{q+1} x).\end{aligned}$$

(Meaning of the notation: If $I, X \in \mathrm{Ob}\, T$, then $I(X) = \mathrm{Hom}_T(X,T)$. An element $f \in I(X)$ gives rise to a map $x \mapsto f(x)$ from $X(U)$ to $I(U)$ as U runs over T, and by Yoneda's lemma f is determined by $x \mapsto f(x)$. Thus the elements g_i, x in the above formula are to be interpreted and elements of $G(U)$, $X(U)$ for variable U in T.)

The double complex gives rise to a spectral sequence

$$E_0^{pq} = I^q(G^p \times X) \qquad \text{with } d_0 = d$$

$$E_1^{pq} = H^q(G^p \times X) \qquad \text{with } d_1 = \delta$$

and it is easy to see that the spectral sequence from E_1 on is independent of the choice of the resolution $I^\cdot$. Since the I^q are injective objects of T_G^{ab}, and since (1) is acyclic (in fact contractible when the G action is ignored), it is known that the double complex (5) is horizontally acyclic. Therefore the abutment of the spectral sequence is

$$H^n\left(\mathrm{Ker}\left\{I^\cdot(X) \xrightarrow{\delta} I^\cdot(G\times X)\right\}\right)$$

$$\wr\|$$

$$H^n\left(I_G^\cdot(X)\right) = H_G^n(X)$$

(need to explain notation I_G)

~~We can now compute (3) + (4). Thus let $x \in H_G^n(X)$ be represented by $\dot{x} \in I^{n-1}(X)$ which is a cocycle: $d\dot{x} = 0$ and invariant under G: $g\dot{x} = \dot{x}$. Then $x \mapsto cl(\dot{x}) \in H^n(I(X)) = H^n(X)$ is the map (3). Suppose that $cl(\dot{x}) = 0$, i.e. $\exists\ y \in I^{n-1}(X)$ such that $dy = \dot{x}$. Then consider the element $u \in I^{n-1}(G\times X)$ given by~~

~~$$u(g) = gy - y$$~~

We can now compute (3) + (4). Thus let $\alpha \in H^n_G(X)$ be represented by an element $a \in I^n(X)$ ~~which is~~ such that $da = 0$ and $\delta a = 0$, i.e. $ga = a$ for any g. Forgetting this last condition we get an element $cl(a) \in H^n(X)$ and the map $\alpha \mapsto cl(a)$ is (3).

If $cl(a) = Res^G_{pt}\, \alpha = 0$, then $\exists\, b \in I^{n-1}(X)$ such that $db = a$. Then δb is the element of $I^{n-1}(G\times X)$ given by

$$(\delta b)(g,x) = g\,b(x) - b(gx)$$

This is a cocycle since $db = a$ and a is invariant. Altering b by a cocycle b' changes $cl(\delta b) \in H^{n-1}(G\times X)$ by

$$cl(\delta b') = \delta\, cl(b')$$

hence there is a well-defined map (4) given by $\alpha \mapsto cl(\delta b)$. (It is desirable to understand the map $\delta : H^n(X) \longrightarrow H^n(G\times X)$, which should be $pr_1^* - \mu^*$. For this to be so it is necessary that

$$g, x \longmapsto g\,b(x) - b(x) \qquad \in I^{n-1}(G\times X)$$

is a coboundary when $b \in I^{n-1}(X)$ is a cycle. However over G, we have a map of resolutions of $\mathbb{Z}/\ell$

$$I^{\cdot} \longrightarrow I^{\cdot}$$

which for a Y over G is given by

$$I^{\cdot}(Y) \longrightarrow I^{\cdot}(Y)$$

$$b \longmapsto g\cdot b,$$

and this map induces the identity on $\mathbb{Z}/\ell$, hence is homotopic

better: $G\times I^{\cdot} \longrightarrow G\times I^{\cdot}$

$G(Y)\times I^{\cdot}(Y) \longrightarrow G(Y)\times I^{\cdot}(Y)$

$(g, y) \mapsto (g, gy)$

induces the identity map on homology.

warning: h maybe not quite right. be careful

to the identity, i.e. $\exists$ $h : G \times I^{\cdot} \longrightarrow I^{\cdot - 1}$ such that

$$g b - b = d h(g, b) + h(g, d b) \qquad \begin{matrix} g \in G(Y) \\ b \in I^{\cdot}(Y) \end{matrix}$$

for $b \in I(Y)$. Taking $Y = G \times X$ we get a universal formula for all X $\ni$

$$g b(x) - b(x) = d h(g, b(x)) + h(g, db(x)), \qquad ?$$

which proves what we want.)

Example: $X = pt$. Then $\delta = p^* - p^* : H(pt) \longrightarrow H(G)$ is zero. ~~...~~ We can write

$$H^n_G(pt) \simeq H^n(pt) \oplus \tilde{H}^n_G(pt)$$

using the map $H^n(pt) \longrightarrow H^n_G(pt)$ furnished by the homomorphism $G \longrightarrow \{1\}$. The edge homomorphism (4) goes

$$\text{susp} : \tilde{H}^n_G(pt) \longrightarrow H^{n-1}(G)$$

and is called the suspension homomorphism.

In terms of cochains, one starts with $\alpha \in \tilde{H}^n_G(pt)$, represents α by $a \in I^n(pt)$, $da = 0$, $ga = a$. Then as α dies in $H^n(pt)$ we have $a = db$ with $b \in I^{n-1}(pt)$. Then the $\text{susp}(\alpha) \in H^{n-1}(G)$ is represented by $\delta b \in I^{n-1}(G)$

$$(\delta b)(g) = g \cdot b - b.$$

Again recall $g \in G(U)$ and b is lifted into $\mathbb{I}(U)$ by the map $U \to pt$. Since

$$g \cdot b - b = d\,h(g,b) + h(g,db),$$

This cohomology class in $H^{n-1}(G)$ is zero if $db = 0$; thus there is no indeterminacy!

For later purposes we shall need ~~the following~~ a factorization of the suspension homomorphism

$$\begin{tikzcd}
\tilde{H}^n_G(pt) \arrow[rr, dashed] \arrow[dr, "susp"'] & & H^{n-1}_G(G^c) \arrow[dl, "res^G_{\{1\}}"] \\
& H^{n-1}(G) &
\end{tikzcd}$$

where G^c denotes the object of T_G furnished by G with the conjugation action. Let $\alpha \in \tilde{H}^n_G(pt)$ and consider the element $pr_1^*\alpha - pr_2^*\alpha \in H^n_{G\times G}(pt)$. Represent this by a cocycle $a \in I^n(pt)$, where $I^\cdot$ is an injective resolution of the constant sheaf $\mathbb{Z}/\ell$ in $T_{G\times G}$. Note that upon restriction to a subgroup $I^\cdot$ remains injective (ref.!!), hence as $pr_1^*\alpha - pr_2^*\alpha$ dies on restriction to ΔG we have that $\exists\ b \in I^{n-1}(pt)$ with $db = a$ and $(g,g)b = b$ for all g. Then let $u \in I^{n-1}(G)$ be given by

$$u(g) = (g,e)b - b.$$

Then u is invariant for the conjugation action since

$$u(g_1 g g_1^{-1}) = (g_1 g g_1^{-1}, e)\, y - y$$
$$= (g_1 g, g_1)\, y - (g_1, g_1)\, y$$
$$= (g_1, g_1)\, u(g)$$

Therefore u defines an element of $H^{n-1}_G(G^c)$. ~~For the same reasons as before we are going to have a formula of the type~~

$$\cancel{(g_1, g_2) b - b = d h(g_1, g_2, b) + h(g_1, g_2, db)}$$

~~where $h: (G\times G)\times_G I^\cdot \longrightarrow I^\cdot$ is a homotopy operator. Thus if $db = 0$~~

$$\cancel{u(g) = (g, e) b - b = d h(g, e, b)}$$

~~and~~

$$\cancel{h(g_1 g g_1^{-1}, e, b) =}$$

Now I want to prove the indeterminacy is zero and argue that there must be a G-invariant homotopy h

$$\begin{tikzcd} G^c \times I^\cdot \arrow[rr, "{(id,\, h)}"] \arrow[dr] & & G^c \times I^\cdot \arrow[dl] \\ & G^c & \end{tikzcd}$$

such that

$$g\cdot b - b = d h(g, b) + h(g, db),$$

because the map $G^c \times \mathbb{Z}/l \longrightarrow G^c \times \mathbb{Z}/l$ induced on homology is the identity. Thus if b is a cocycle $g\cdot b - b = dh(g,b)$ and $g \mapsto h(g,b) \in I^{n-1}(G^c)$ will be equivariant since

$$(g_1,g_1)\,h(g,b) = h(g_1 g g_1^{-1}, (g_1,g_1)b) = h(g_1 g g_1^{-1}, b).$$

This proves that there is no indeterminacy for the dotted arrow on page 6.

(This is a bad proof. The good proof consists in noting that $(T_G)_{G^c} \simeq T_{G\times G}$ with

$$\begin{tikzcd}
\operatorname{Ker}\left\{H^*_{G\times G}(pt) \xrightarrow{\Delta^*} H^*_G(pt)\right\} \arrow[r, phantom, "\simeq"] \arrow[d, "\text{as above}"] & \tilde{H}^*((T_G)_{G^c}) \arrow[d, "\sigma \text{ for } G^c"] \\
H^{*-1}_G(G^c) \arrow[r, phantom, "\simeq"] & H^{*-1}(T_G/G^c)
\end{tikzcd}$$

hence there is no indeterminacy.)

The fact that the triangle on p. 6 commutes is because we could have taken $I^\cdot$ at bottom p. 5 to be the restriction via the map $in_1 : G \to G\times G$. Equivalently use compatibility with the morphism $T \to T_G$:

$$\begin{tikzcd}
\tilde{H}^*((T_G)_{G^c}) \arrow[r] \arrow[d, "\sigma \text{ for } G^c \text{ in } T_G"] & H^*((T)_G) \arrow[d, "\sigma \text{ for } G \text{ in } T"] \\
H^{*-1}(T_G/G^c) \arrow[r] & H^{*-1}(T/G)
\end{tikzcd}$$

Let σ be an endomorphism of G. We consider the ~~maps~~ map $(id, \sigma) : G \longrightarrow G \times G$ and let the latter group act on G by $(g_1, g_2)\, g = g_1 g g_2^{-1}$. The resulting $G \times G$ object will be denoted G^s; the ~~map~~ G object induced from G^s by the map (id, σ) will be denoted G^t. The action of G on G^t is given by $g_1(g) = g_1 g (\sigma g_1)^{-1}$ and will be called the <u>twisted conjugation</u> action. Diagram:

$$
\begin{tikzcd}
H^*(G) & & H^*(G) \arrow[ll, "\simeq"', "id"] \\
H^*_G(G^t) \arrow[u, "i^*"] & & H^*_{G\times G}(G^s) \arrow[ll] \arrow[u] & \cong & H^*_G(pt) \\
H^*_G(pt) \arrow[u, "p^*"] & & H^*_{G\times G}(pt) \arrow[ll, "(id,\sigma)^*"'] \arrow[u] \arrow[urr, "\Delta^*"']
\end{tikzcd}
$$

The isomorphism at the right is a special case of the isomorphism $H^*_G(G/H) \cong H^*_H(pt)$ where H is a subgroup of G, and it results from the isomorphism of topi

$$T_G / (G/H) \cong T_H$$

(which we shall undoubtedly have to establish when we show that the restriction homomorphism $T_G \to T_H$ preserves injectives)

We want perhaps to know that the horizontal arrow $H^*_G(pt) \longrightarrow H^*_G(G^t)$ coincides with the vertical one, which follows in this way. The horizontal arrow comes from

the map of topoi whose inverse image sends a G-sheaf S into $(G\times G)\times_G S$ viewed as a G-set through (id,σ).
Define

$$G\times S \xrightarrow{\varphi} (G\times G)\times_G S$$
$$(g, s) \longmapsto (g, 1, g^{-1}s)$$

Then given g_1, we have $g_1\cdot(g,1,g^{-1}s) = (g_1 g, \sigma g_1, g^{-1}s)$ $= (g_1 g(\sigma g_1)^{-1}, 1, (\sigma g_1)g^{-1}s) = \varphi(g_1 g(\sigma g_1)^{-1}, g_1 s)$. Thus we see that $(G\times G)\times_G S$ is the same as $G^t\times S$, which produces the vertical arrow.

Start by defining the map

$$(*)\qquad \Phi: H^*_G(pt)^\sigma \longrightarrow H^{*-1}_G(G^t)/H^{*-1}_G(pt)$$

as follows. Given $\alpha\in H^n_G(pt)$ invariant under σ, consider $\beta = pr_1^*\alpha - pr_2^*\alpha \in H^n_{G\times G}(pt)$, and represent it by a $G\times G$-invariant cocycle $a\in I^n(pt)$. Then as $\Delta^*\beta = 0$ and $(id,\sigma)^*\beta = 0$ there are elements $b, c\in I^{n-1}(pt)$ with $db = dc = a$ such that

$$(g,g)b = b \qquad (g,\sigma g)c = c \qquad \text{all } g.$$

Then let $v\in I^{n-1}(G)$ be defined by

$$v(g) = (g,1)b - c.$$

Then
$$(dv)(g) = (g,1)a - a = 0$$
and
$$v(g_1 g(\sigma g_1)^{-1}) = (g_1 g(\sigma g_1)^{-1}, 1)b - c$$
$$= (g_1 g, \sigma g_1)b - c = (g_1,\sigma g_1)\, v(g)$$

and so ψ defines an element of $H^{n-1}_G(G^t)$. Changing b by a cocycle alters the element by something in the image of the horizontal map $H^*_G(pt) \longrightarrow H^*_G(G^t)$, which changing c alters the element by ~~[illegible]~~ something in the image of the vertical map p^*. These two maps are equal as we have seen before so Φ as in (*) is well-defined.

Claim that

$$\begin{tikzcd} H^*_G(pt)^\sigma \arrow[r, "\Phi"] \arrow[dr, "\text{susp}"'] & H^{*-1}_G(G^t)/H^{*-1}_G(pt) \arrow[d, "i^*"] \\ & H^{*-1}(G)/H^{*-1}(pt) \end{tikzcd}$$

commutes. Indeed $i^*\Phi(\alpha)$ is represented by the cocycle $\psi(g) = (g,1)b - c$, while $\text{susp}(\alpha)$ is represented by $u(g) = (g,1)b - b$ hence the difference is represented by $b - c$ which ~~[illegible]~~ lands in $H^{*-1}(pt)$.

Derivation property of Φ: Since $I^\cdot \otimes I^\cdot$ is a resolution of the constant sheaf, there is a map of complexes

$$\mu : I^\cdot \otimes I^\cdot \longrightarrow I^\cdot \quad , \text{ denoted } \mu(x,y) = xy$$

whose effect on cohomology gives the cup product. Given $\alpha, \alpha' \in H^*_G(pt)$

$$pr_1^*(\alpha\alpha') - pr_2^*(\alpha\alpha') = (pr_1^*\alpha - pr_2^*\alpha)\, pr_1^*\alpha' + pr_2^*\alpha\,(pr_1^*\alpha' - pr_2^*\alpha')$$

I suppose a, a', b, b' chosen as in definition of $\mathrm{susp}\,\alpha$, $\mathrm{susp}\,\alpha'$, and let z represent $pr_1^*\alpha'$, w represent $pr_2^*\alpha$. Then $pr_1^*(\alpha\alpha') - pr_2^*(\alpha\alpha')$ is represented by

$$az + wa' = d(bz + (-1)^{\deg w} wb')$$

and so $\mathrm{susp}(\alpha\alpha')$ is represented by

$$g \longmapsto (g,e)[bz \pm wb'] - [bz \pm wb']$$

$$= [(g,e)b - b]z \pm w[(g,e)b' - b']$$

hence

$$\boxed{\mathrm{susp}(\alpha\alpha') = \mathrm{susp}(\alpha)\cdot\varepsilon\alpha' \pm \varepsilon\alpha\cdot\mathrm{susp}(\alpha').}$$

where $\varepsilon : H_G^*(pt) \longrightarrow H^*(pt)$ is the augmentation. Note also that on restriction $\Delta^* : H_{G\times G}^*(pt) \longrightarrow H_G^*(pt)$, $pr_1^*\alpha'$ and $pr_2^*\alpha$ go into α, α'; hence

$$\boxed{\mathrm{refsusp}(\alpha\alpha') = \mathrm{refsusp}(\alpha)\cdot\alpha' \pm \alpha\cdot\mathrm{refsusp}(\alpha')}$$

where the refined suspension is the map $H_G^*(pt) \longrightarrow H_G^*(G^c)$. In addition if c, c' are as in the definition of Φ, then

$$az + wa' = d(cz \pm wc')$$

so $\Phi(\alpha\alpha')$ is represented by the cocycle

$$(g,e)[bz \pm wb'] - [cz \pm wc'] = [(g,e)b - c]z \pm w[(g,e)b' - c']$$

in $I^\cdot$ with G-action induced via (id, σ). ~~[illegible] restrict to~~ Since $pr_2^* \alpha$, $pr_1^* \alpha'$ restrict to α and α' under (id, σ^*) we have

$$\boxed{\Phi(\alpha\alpha') = \Phi\alpha \cdot \alpha' \pm \alpha \cdot \Phi\alpha'}$$

Digression to see if the following might be clearer: The point is to define Φ and then deduce susp as special case where $\sigma = id$. Start with $I^\cdot$ an injective G-resolution, construct $\varphi : (I^\cdot)^\sigma \to I^\cdot$ with ~~[illegible]~~ $g\varphi(x) = \varphi(\sigma g \cdot x)$. Then over G^t have equivariant map

$$\begin{array}{ccc} G^t \times I^\cdot & \xrightarrow{\Gamma} & G^t \times I^\cdot \\ (g, x) & \longmapsto & (g, \varphi(g^{-1}x)) \end{array}$$

of complexes

$$\begin{aligned} g_1\Gamma(g,x) = g_1(g, \varphi(g^{-1}x)) &= (g_1 g(\sigma g_1)^{-1}, \varphi(\sigma g_1 \cdot g^{-1}x)) \\ &= (g_1 g(\sigma g_1)^{-1}, \varphi(\sigma g_1 \cdot g^{-1} \cdot g_1^{-1} \cdot g_1 x)) \\ &= \Gamma(g_1 g(\sigma g_1)^{-1}, g_1 x) \end{aligned}$$

which induces the identity on $(\mathbb{Z}/\ell)_{G^t}$, hence $\exists\ h(g,x)$

$$x - \varphi(g^{-1}x) = dh(g,x) + h(g,dx)$$

$$h(g_1 g(\sigma g_1)^{-1}, g_1 x) = g_1 h(g,x).$$

Now you define Φ as follows: Given $\alpha \in H_G^*(pt)^\sigma$ represent it by $a \in I(pt)$, $Ga = a$, $da = 0$. Then $\sigma^*\alpha = \alpha$ implies $\exists b$ with ($Gb = b$ and) $a - \varphi(a) = db$; but over G^t we have $a - \varphi(a) = dh(g,a)$ so we get a cocycle $u(g) = h(g,a) - b$ with

$$g_1 u(g) = g_1\left(h(g,a) - b\right) = h(g_1 g (\sigma g_1)^{-1}, a) - b$$

$$= u(g_1 g (\sigma g_1)^{-1})$$

and so u represents an element of $H^*_G(G^t)$.

~~If one forgets the G-action one gets $u(g) = h(g,a) - b$.~~ ~~$dh(g,b) = \varphi(b)$ — $\varphi(b) - dh(g,b)$ and if $a = dx$ then modulo stuff coming from $H^*(pt)$, b can be replaced by $x - \varphi x$ and so~~ ?

~~$u(g) \sim h(g, dx) - x + \varphi x = d$~~

From this point of view it is unclear why $u(g) = h(g,a) - b$ coincides with $\operatorname{susp}(\alpha)$ which is represented by $gx - x$ if $x = da$, not to mention problems with non-uniqueness of h.

(It seems better to define the suspension as the map $H^*_G(pt) \longrightarrow H^{*-1}(G)$ which first removes the constant component and then takes the edge homomorphism. Then the refined suspension $H^*_G(pt) \longrightarrow H^{*-1}_G(G^c)$ appears as the special case of the map Φ with $\sigma = $ identity.)

Calculation of Φ on one and two dimensional classes:

Assume first of all that we know that

$H^1_G(pt)$ = isom classes of $\mathbb{Z}/\ell\mathbb{Z}$-torsors in T_G, i.e. a $\mathbb{Z}/\ell\mathbb{Z}$-torsor $X \to pt$ endowed with a left G-action.

Then the candidate for Φ is as follows. Let X be (such) a torsor representing an invariant element $\alpha \in H^1_G(pt)$. By invariance there an isomorphism $\varphi : X^\sigma \to X$ where X^σ has G-action $g * x = (\sigma g)x$, and that $\varphi(\sigma g \cdot x) = g\varphi(x)$. Lift this up over G^t and define an automorphism

$$G^t \times X \xrightarrow{\Gamma} G^t \times X$$

$$(g, x) \longmapsto (g, \varphi g^{-1} x)$$

which is G-equivariant by the calculation on page 13. This automorphism must be given by an invariant section of $\mathbb{Z}/\ell\mathbb{Z}$ over G^t. Thus we get an element of $H^0_G(G^t)$ which ought to be $\Phi(\alpha)$. Observe that φ can be altered by a section of $\mathbb{Z}/\ell\mathbb{Z}$ over a point so the indeterminacy is $H^0(pt) = H^0_G(pt)$.

? Let us also assume known that

$$\tilde{H}^2_G(pt) = \text{isom. classes of extensions of groups}$$

$$1 \to \mathbb{Z}/\ell\mathbb{Z} \to E \to G \to 1$$

in T. Then if we represent $\alpha \in \tilde{H}^2_G(pt)$ by this extension, say, and it is invariant under σ we get a map of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/\ell\mathbb{Z} & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\sigma} & & \\ 1 & \longrightarrow & \mathbb{Z}/\ell\mathbb{Z} & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

Now lift this up to G^t and define Γ by

$$G^t \times E \longrightarrow G^t \times E \quad ?$$

? it won't be an isomorphism since σ needn't be; thus we have wrong interpretation of H^2.

more K-theory
arithmetic Chern classes.

April 11, 1970.

I want to ~~[illegible]~~ write out carefully the computation of $K_i(\mathbb{F}_q)$. I recall that I have defined $k(X)$ to be the Grothendieck group of $\mathbb{F}_q$-vector bundles over a space X, $\bar{k}(X)$ the reduced group, and that

$$(1) \qquad \bar{k}(X) \longrightarrow [X,B] = \bar{K}(X)$$

is a universal map of $\bar{k}$ into a representable functor on the homotopy category ~~[illegible]~~. I proved that B has a unique H-space structure such that (1) is a natural transformation of abelian groups; ~~[illegible]~~ and propose now to calculate what B must be. I know B is connected.

Since B is a ~~[illegible]~~ H-space, $H_*(B)$ (homology with coefficients in a field k) is a ring with Pontryagin product. Following Grothendieck's lesson ~~[illegible]~~ we think of this as

$$(2) \qquad \mathrm{Hom}_{\mathrm{rgs}}(H_*(B), R) = \mathrm{Hom}_{\mathrm{ab}}(\bar{K}(?), H^*(?,R)^{\times})$$

where R runs over the category of graded k-algebras and $H^*(X,R)^{\times}$ is the set of units of degree 0, i.e. of form $\sum x_i r_i$ with $x_i \in H^a(X)$ and $r_i \in R_a$. By the universal nature of (1), the right side of (2) may be rewritten

$$\mathrm{Hom}_{\mathrm{ab}}(\bar{k}(?), H^*(?,R)^{\times})$$

and an element φ of this is a multiplicative characteristic class for $\mathbb{F}_q$-vector bundles, i.e. to each such E over an X, there is associated an element $\varphi(E) \in H(X,R)^\times$ in a natural way such that $\varphi(1)=1$ and such that $\varphi(E)=\varphi(E')\varphi(E'')$ whenever there is an exact sequence $0\to E'\to E\to E''\to 0$.

Proposition 1: If A is an abelian group which is a $\mathbb{Z}_{(p)}$ module then $\tilde{H}^*(B,A)=0$.

Proof: Let $u\in H^n(B,A)$, $n\geq 0$; then u is represented by a map in the homotopy category $B\to K(A,n)$, hence by universal nature of B may be identified with a natural transformation $\tilde{k}(?)\longrightarrow H^n(?,A)$. Given $x\in\tilde{k}(X)$ it can be represented as the difference $x=[E]-[E_1]$ of two $\mathbb{F}_q$ bundles over X. By passage to a covering $f: Y\to X$ of E and E_1 of order prime to p one may reduce the structural groups to the Sylow p-subgroups, hence E, E_1 will have flags and so $f^*x=0$. But f^* is injective on $H^*(?,A)$ since the degree of the covering acts invertibly on A. Thus $f^*\varphi(x)=\varphi(0)$, so φ is determined by its restriction to a pt. so $\varphi=0$ if $\varphi\in\tilde{H}^*(B,A)$. qed.

This proposition shows that $H_*(B,k)=0$ if char $k=p$ or 0. Suppose that $k=\mathbb{Z}/l\mathbb{Z}$ where l is a prime number $\neq p$, and let $d=[\mathbb{F}_q(\mu_l):\mathbb{F}_q]$, i.e. d is the least positive number such that l divides q^d-1. Choose an isomorphism $\mathbb{F}_q(\mu_l)\simeq\mathbb{F}_q^d$ and let

$$\mathbb{F}_q(\mu_l)^*\longrightarrow GL_d(\mathbb{F}_q)$$

the resulting representation of $\mathbb{F}_q(\mu_l)^*$ be denoted

η. Set $C = \mathbb{F}_q(\mu_\ell)^* = \mathbb{F}_{q^d}^*$; it is a cyclic group of order $q^d - 1$ and its cohomology is

(3)
$$H^*(BC) = \mathbb{Z}_\ell[v, u]$$

where v, u are generators of H^1 and H^2, respectively, and $v^2 = u$ only if ($\ell = 2$ and) $\nu_2(|C|) = 1$, otherwise $v^2 = 0$.

Now η defines an $\mathbb{F}_q$-bundle over BC so

(4)
$$\varphi(\eta) = \sum_{i \geq 0} r_i u^i + \sum_{i \geq 1} r_i' v u^{i-1}$$

where $r_i \in R_{2i}$, $r_i' \in R_{2i-1}$. Also $r_0 = 1$. The Galois group of $\mathbb{F}_q(\mu_\ell)$ over $\mathbb{F}_q$ is cyclic of order d generated by Frobenius $x \longmapsto x^q$, hence the representation $\eta \circ q$

$$\begin{tikzcd}
C \arrow[r, "q"] \arrow[d, "\eta"] & C \arrow[d, "\eta"] \\
\mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_q(\mu_\ell)) \arrow[r, "\mathrm{Frob}"] & \mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_q(\mu_\ell))
\end{tikzcd}$$

is isomorphism over $\mathbb{F}_q$ to η. Thus $\varphi(\eta)$ must be invariant under the transf. of $H^*(BC)$ produced by $C \xrightarrow{q} C$; since $q^* u = qu$ and $q^* v = qv$, it follows that

$$(q^i - 1)\, r_i = 0$$
$$(q^i - 1)\, r_i' = 0$$

and hence $r_i = r_i' = 0$ ~~unless~~ for $i \not\equiv 0\ (d)$. Therefore we see that (4) can be rewritten

$$(5) \qquad \varphi(\eta) = \sum_{i\geq 0} r_i\, u^{di} + \sum_{i\geq 1} r_i'\, v u^{di-1} \qquad r_0 = 1$$

~~Theorem 1: Associating to each multiplicative characteristic class φ over R the elements $\{r_i, r_i'\}_{i\geq 1}$ where $r_i \in R_{2id}$, $r_i' \in R_{2id-1}$ gives a bijection of the set of such φ with the set of $\{r_i, r_i'\}$. Equivalently $H_*(B, \mathbb{Z}/l\mathbb{Z})$~~

~~Now I want to prove that the map $\varphi \longmapsto \{r_i, r_i'$~~

where $r_i \in R_{2id}$ and $r_i' \in R_{2id-1}$. I am going to show that the map

$$\varphi \longmapsto \{r_i, r_i'\}_{i\geq 1}$$

is injective and determine the image. (When l is odd, the image will be those sequences with $(r_i')^2 = 0$; the same should be true when $l = 2$ and $v_2(q-1) \geq 2$, but if $v_2(q-1) = 1$, then I expect more complicated relations.)

First suppose l is odd; this case I can handle in a completely elementary fashion (no etale cohomology). We begin with the computation of $H^*(BGL_n(\mathbb{F}_q))$. Write $n = dm + e$ $\quad 0 \leq e < d$, and let

$$(6) \qquad C^m \longrightarrow GL_n(\mathbb{F}_q)$$

be the embedding obtained from composition

$$C^m \longrightarrow GL_d(\mathbb{F}_q)^m \hookrightarrow GL_n(\mathbb{F}_q).$$

Let $\pi = \mathbb{Z}/d\mathbb{Z}$ act on C with the generator action as raising to the qth power. Then this embedding factors

$$C^m \longrightarrow \Sigma_m \tilde{\times} (\pi \tilde{\times} C)^m \hookrightarrow GL_n(\mathbb{F}_q)$$

and the group $\Sigma_m \wr (\pi \tilde{\times} C) \times GL_e(\mathbb{F}_q)$ is the normalizer of C^m in $GL_n(\mathbb{F}_q)$.

$$|GL_n(\mathbb{F}_q)| = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} (q^i - 1)$$

$$|\Sigma_m \wr C| = m!\ (q^d - 1)^m$$

By number theory this is prime to l. C has its mod l cohomology detected by abelian groups of exponents $q-1$, hence also for $\Sigma_m \wr C$, and these are conjugate to a subgp of C^m in $GL_{dm}(\mathbb{F}_q)$ as one can easily seen by the structure of irreducible representations of such an abelian gp. over $\mathbb{F}_q$. Consequently as in the Adams conjecture paper the map on cohomology induced by (6) is injective.

(7) $$H^*(BGL_n(\mathbb{F}_q)) \hookrightarrow H^*(BC^m)^W$$

where $W = \Sigma_m \wr \pi$. Now we've seen that

$$H^*(BC)^\pi = \mathbb{Z}_l[vu^{d-1}, u^d]$$

and since ℓ is odd, the theorem of symmetric functions which you proved shows that

$$H^*(BC)^W = \mathbb{Z}_\ell[\partial c_1, \dots, \partial c_m; c_1, \dots, c_m]$$

where the c_i are the elementary symmetric functions of $u_1^d, \dots, u_m^d$ and where ∂ is the derivation with $\partial u_i^d = v_i u_i^{d-1}$. (Since $d \mid \ell - 1$ $d \not\equiv 0 \bmod \ell$ so ∂ could also be defined by $\partial u_i = v_i$, which is perhaps better). (It's only at this last step where we use the symmetric functions result that we need ℓ is odd. Before this everything works for $\ell = 2$ if $4 \mid q-1$.)

Now we prove (7) is an isomorphism by exhibiting the required elements. Recall that there are arithmetic Chern classes

$$c_i \in H^{2i}\left(B(GL_{n, \mathrm{Spec}\, \mathbb{F}_q}), \mu_\ell^{\otimes i}\right)$$

which when pulled back to $B\,GL_n(\bar{\mathbb{F}}_q)$ give elements

(8) $$c_i \in H^{2i}\left(\mathrm{Spec}\, \mathbb{F}_q \times BGL_n(\bar{\mathbb{F}}_q), \mu_\ell^{\otimes i}\right)$$

where the product means the classifying topos of the constant sheaf over $\mathrm{Spec}\, \mathbb{F}_q$ associated to $GL_n(\bar{\mathbb{F}}_q)$. There is a spectral sequence

$$E_2^{pq} = H^p(\pi, H^q(BGL_n(\bar{\mathbb{F}}_q), \mu_\ell^{\otimes i})) \Rightarrow \text{above}$$

which in this case $\pi = \hat{\mathbb{Z}}$ gives an exact sequence

$$0 \longrightarrow H^{2i-1}(BGL_n(\mathbb{F}_q), \mu_\ell^{\otimes i})_\pi \longrightarrow H^{2i}(\mathrm{Spec}\,\mathbb{F}_q \times BGL_n(\mathbb{F}_q), \mu_\ell^{\otimes i}) \longrightarrow H^{2i}(BGL_n(\mathbb{F}_q), \mu_\ell^{\otimes i})^\pi \to 0$$

which is canonically split since

$$\begin{tikzcd} \mathrm{Spec}\,\bar{\mathbb{F}}_q \times BGL_n(\mathbb{F}_q) \arrow[r] \arrow[rr, bend left, "\simeq"] & \mathrm{Spec}\,\mathbb{F}_q \times BGL_n(\mathbb{F}_q) \arrow[r] & BGL_n(\mathbb{F}_q) \end{tikzcd}$$

since π only acts through the $\mu_\ell^{\otimes i}$, the arithmetic Chern classes (8) give rise to elements

$$\begin{aligned} c_i &\in H^{2i}(BGL_n(\mathbb{F}_q), \mathbb{Z}/\ell\mathbb{Z}) \qquad && i \equiv 0 \ (d) \\ c_i' &\in H^{2i-1}(\quad " \quad) && \quad " \end{aligned}$$

assuming an isomorphism of $\mathbb{Z}/\ell\mathbb{Z} \xrightarrow{\sim} \mu_\ell^{\otimes d}$ is given. It seems reasonable that c_i' is related to $\Phi(c_i)$.

<u>Philosophy</u>: Suppose given a map $f : X \longrightarrow S$, let Γ be the space of its sections, i.e.

$$\mathrm{Hom}(U, \Gamma) = \mathrm{Hom}_{/S}(S \times U, X)$$

Given a cohomology class $x \in H^*(X)$ it may be pulled back to $S \times U$, whence by Kunneth it defines a homomorphism $H_*(S) \longrightarrow H^*(U)$. Therefore taking U to be Γ we have a canonical map

$$H^*(X) \otimes H_*(S) \longrightarrow H^*(\Gamma)$$

Example: Take $S = BC$ where C is a cyclic group ~~of~~ of order l. Let $X = (Z^l)_C$ and $x = P_{ext} z$. Then for U we can take the fixpoints Z of the action and the map defined above by x:

$$H^*_*(BC) \longrightarrow H^*(Z)$$

is what one uses to define the Steenrod operations for the space Z.

Question: Is there any chance that Γ is a generalized loop space and that its cohomology can be computed from a bar or cobar type construction?

Example of multiplicative characteristic classes: Suppose R is a ring over $\mathbb{Z}[l^{-1}]$. Then for any group G which acts on a finitely generated projective R-module P, we have "arithmetic" Chern classes

$$c_i(P,G) \in H^{2i}(\operatorname{Spec} R, G; \mu_l^{\otimes i})$$

from which can be constructed multiplicative characteristic classes, e.g.

$$c_{\underline{t}}(P,G) = \sum t^\alpha c_\alpha(P,G)$$

$$c_\alpha(P,G) \in H^{2|\alpha|}(\operatorname{Spec} R, G; \mu_l^{\otimes|\alpha|})$$

Now let us suppose that $H^n(G, \mathbb{Z}/l\mathbb{Z})$ is finite type for all n, then

$$H^*(\operatorname{Spec} R, G; \mu_\ell^{\otimes i}) \simeq H^*(\operatorname{Spec} R, \mu_\ell^{\otimes i}) \otimes H^*(G, \mathbb{Z}/\ell\mathbb{Z})$$

by a Kunneth formula. (G may be replaced by a space X.) Therefore we get by using the arithmetic Chern classes a homomorphism

$$H_*(B) \longrightarrow \bigoplus_{i \geq 0} H^*(\operatorname{Spec} R; \mu_\ell^{\otimes i}).$$

It is necessary to be a bit more explicit about the degrees.

$$c_\alpha \in H^{2|\alpha|}(\operatorname{Spec} R, X; \mu_\ell^{\otimes|\alpha|}) = \bigoplus_i H^i(X) \otimes H^{2|\alpha|-i}(\operatorname{Spec} R, \mu_\ell^{\otimes|\alpha|}).$$

so

$$\underline{c_t} = \sum t^\alpha c_\alpha \in H^*(X) \otimes H^*(\operatorname{Spec} R, \bigoplus_j \mu_\ell^{\otimes j})[t_1, \ldots]$$

and $H^i(X)$ occurs with $\sum t^\alpha H^{2|\alpha|-i}(\operatorname{Spec} R, \mu_\ell^{\otimes|\alpha|})$ of the sort that

$$\begin{array}{ccc} H_*(B) & \longrightarrow & H^*(\operatorname{Spec} R, \bigoplus_j \mu_\ell^{\otimes j})[t] \\ & \searrow & \cup \\ & & H^*(\operatorname{Spec} R, \bigoplus_\alpha t^\alpha \mu_\ell^{\otimes|\alpha|}) \end{array}$$

Hopefully one can work this out for a finite field and produce the unknown multiplicative structure this way.

April 12, 1970: Arithmetic Chern classes and algebraic K-theory.

Let $(X, \mathcal{O}_X)$ be a scheme. Given a topos I one can consider the relative scheme $I \times X$ over I. A vector bundle over $I \times X$ will be called a family of vector bundles on X parameterized by I.

For example suppose G is a group and I is its classifying topos. Then a vector bundle over X parametrized by I is the same thing as a vector bundle E over X endowed with an action of G.

This example is typical. In effect recall that any locally connected topos I has a "fundamental groupoid" topos $\pi_1 I$ whose objects are the locally constant objects of I. If E is a vector bundle over $I \times X$, then it should be locally constant with respect to the map $I \times X \longrightarrow X$ hence the inverse image of a bundle on $\pi_1 I \times X$.

We let $k(I, X)$ be the Grothendieck group of vector bundles over X parametrized by I. (This is just the naive K-group of the ringed topos $I \times X$, but psychologically X is fixed and I varies. In fact I think of I as being a simplicial set.) It follows from the fact that $k(I, X) \xleftarrow{\sim} k(\pi_1 I, X)$, that $I \mapsto k(I, X)$ is a functor on the homotopy category of simplicial sets. The basic problem is to prove the existence of a universal map

$$k(I, X) \longrightarrow K(I, X)$$

where $K(I, X) = [I, B_X]$ is a representable functor on the homotopy category. (Perhaps this is to much to ask for and what one needs only is a pro-representable functor.)

Let Λ be a coefficient ring for cohomology, which for simplicity I suppose is a field, say $\mathbb{Z}/\ell\mathbb{Z}$. If I is a topos, let $H^*(I)$ be the cohomology ring of I with coefficients in Λ. By a multiplicative characteristic class for families of vector bundles over X with values in $H^*(\ )\otimes_\Lambda R$ (R is a Λ-algebra, say $R = \bigoplus_{n\geq 0} R_n$) I mean a natural transformation

$$\varphi : k(I, X) \longrightarrow \left(H^*(I)\otimes_\Lambda R\right)^\times$$

of abelian groups. Here $(H^*(I)\otimes_\Lambda R)^\times$ is the group of units of degree 0.

Example 1: The total Chern class (etale mod ℓ)

$$c_t : k(I\times X) \longrightarrow \prod_i H^{2i}(I\times X, \mu_\ell^{\otimes i})t^i$$

where $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$ and ℓ^{-1} exists on X.
Suppose $H^*(I)$ is finite dimensional, e.g. I a finite simplicial set, then by Künneth

$$H^{2i}(I\times X, \mu_\ell^{\otimes i}) = \bigoplus_{j=0}^{2i} H^j(I)\otimes H^{2i-j}(X, \mu_\ell^{\otimes i}).$$

Thus c_t lies in

$$\bigoplus_{j\geq 0}\left\{H^j(I)\otimes \prod_{2i\geq j} H^{2i-j}(X, \mu_\ell^{\otimes i})t^i\right\}$$

hence we get a multiplicative characteristic class

$$c_t : k(I \times X) \longrightarrow \left(H^*(I) \otimes R\right)^\times$$

where $R = \bigoplus_{j \geq 0} R_j$ and

$$R_j = \prod_i H^{2i-j}(X, \mu_\ell^{\otimes i}) t^i \qquad \deg t = -2$$

<u>Example 2</u>: The gigantic Chern class (etale mod ℓ)

$$c_{\underline{t}} : k(I \times X) \longrightarrow \left(H^*(I) \otimes R\right)^\times \quad \text{where}$$

$$R_j = \prod_\alpha H^{2|\alpha|-j}(X, \mu_\ell^{\otimes |\alpha|})\, t^\alpha$$

The following examples are special cases of the above. Take $X = \operatorname{Spec} k$ where k is a field, of the sort that the etale cohomology coincides with Galois cohomology.

3.) If k is separably closed, then

$$H^+(X, \mu_\ell^{\otimes i}) = 0$$
$$H^0(X, \mu_\ell^{\otimes i}) = \mu_\ell^{\otimes i}$$

so the Chern class maps

$$c_t : k(I \times X) \longrightarrow \prod_i H^{2i}(I, \mu_\ell^{\otimes i}) t^i$$

corresponding to

$$H_*(B) \longrightarrow \bigoplus_{i \geq 0} \mu_\ell^{\otimes i} t^i = S[\mu_\ell t] \quad \left(\deg t = -2\right)$$

The gigantic Chern class goes

$$c_t : k(I \times X) \longrightarrow \prod_\alpha H^{2|\alpha|}(I, \mu_\ell^{\otimes|\alpha|})t^\alpha$$

and corresponds to a ring homomorphism

$$H_*(B) \longrightarrow \bigoplus_\alpha \mu_\ell^{\otimes|\alpha|} t^\alpha$$

$$= S[\mu_\ell^{\otimes 1} t_1 \oplus \mu_\ell^{\otimes 2} t_2 \oplus \cdots\cdots] \qquad \text{degree } t_i = -2i$$

These are what one gets from geometric Chern classes.

4.) Suppose $X = \operatorname{Spec} k$ where k is a finite field with q elements.

$$H^*(X, \mu_\ell^{\otimes *}) = \mathbb{Z}/\ell\mathbb{Z}[\xi, \sigma]$$

where

$$\begin{array}{ll} \sigma & \text{generates } H^1(X, \mathbb{Z}/\ell\mathbb{Z}) \\ \xi & \text{generators } H^0(X, \mu_\ell^{\otimes d}) \end{array}$$

Thus the total Chern class is a map

$$c_t : k(I \times X) \longrightarrow (H^*(I) \otimes R)^\times$$

where

$$R = \mathbb{Z}[t^d\xi, \sigma]$$

degree $\xi = 0$
deg $\sigma = 1$
deg $t = +2$

Also the gigantic Chern class is a map

$$c_t : k(I\times X) \longrightarrow (H^*(I)\otimes R)^\times$$

where $R = \bigoplus_j R_j$ and

$$R_j = \prod_\alpha H^{2|\alpha|-j}(X, \mu_\ell^{\otimes|\alpha|})t^\alpha$$

Thus $R_0 = \prod_\alpha H^{2|\alpha|}(X, \mu_\ell^{\otimes|\alpha|})t^\alpha = \mathbb{Z}/\ell\mathbb{Z}$
and if $\ell | q-1$ it would appear that

$$R_j \cong \prod_{2|\alpha|=j} (\mathbb{Z}/\ell\mathbb{Z})t^\alpha \times \prod_{2|\alpha|=j+1} (\mathbb{Z}/\ell\mathbb{Z})t^\alpha$$

Thus

$$R \cong S[\mu_\ell t_1 + \cdots + \mu_\ell^{\otimes i} t_i + \cdots] \otimes \mathbb{Z}/\ell\mathbb{Z}[\sigma] \qquad \begin{matrix}\sigma^2=0\\ \deg\sigma = 1.\end{matrix}$$

---

5). Let $X = \operatorname{Spec} K$ where K is a field (for simplicity) and consider the Chern classes in absolute De Rham cohomology

$$c_t : k(I\times X) \longrightarrow H^*_{DR}(I\times X)[t]$$

Then

$$H^*_{DR}(I\times X) \cong H^*(I)\otimes H^*_{DR}(K)$$

$$\cong H^*(I)\otimes H^*(\Omega^\cdot)$$

where $\Omega^\cdot = \Omega^\cdot_{K/\mathbb{Z}}$ is the absolute De Rham complex of K.
One can also consider the Hodge Chern classes.

April 14, 1970: Computation of Φ for a torus.

It appears that more must be done to compute Φ. The problem arises from the fact that $S(P^\sigma) \longrightarrow H^*(BG)^\sigma$ is not an isomorphism, consequently even though we know $\Phi(x)$ for $x \in P^\sigma$ we don't know yet how to compute Φ on a general element of $H^*(BG)^\sigma$.

First we discuss what we think ought to happen. Let $V = P^\vee$ and let σ be the transpose of σ^* on P. $S(P)$ is thus the ring of polynomial functions on V. If $f \in S(P)$ is invariant under σ note that for $v \in V$, $x \in V^\sigma$

$$\bigl(i(\sigma v - v)\, df\bigr)(x) = \lim_{\varepsilon\to 0} \frac{f(x+\varepsilon\sigma v) - f(x+\varepsilon v)}{\varepsilon} = 0,$$

or more accurately

$$\varepsilon\,\underbrace{\bigl(i(\sigma v - v)\, df\bigr)(x)}_{=0} = f(x+\varepsilon\sigma v) - f(x+\varepsilon v) \quad \operatorname{mod} \varepsilon^2$$

Therefore df defines an element of

$$S\bigl((V^\sigma)^*\bigr) \otimes (V_\sigma)^* = S(P_\sigma) \otimes P^\sigma.$$

which we shall denote by ∂f. Here's how ∂f is obtained. One starts with

$$df \in S(P) \otimes P \qquad \text{which is } \sigma\text{-invariant},$$

and takes its image in $S(P_\sigma) \otimes P$; since σ acts trivially on the first factor, the image ∂f lies in $S(P_\sigma) \otimes P^\sigma$.

Conjecture: The following commutes

$$\begin{tikzcd}
H^*(BG)^\sigma \arrow[rr, "\Phi"] & & H^{*-1}_G(G^t)/H^{*-1}_G(G) \\
S^*(P)^\sigma \arrow[u, "\cong"] \arrow[r, "\partial"] & S^*(P_\sigma)\otimes P^\sigma \arrow[ur] &
\end{tikzcd}
\qquad
g\otimes x \mapsto g\cdot\Phi(x)$$

Example (from April 13, p. 11). Here $P = ({}_\ell T)^\vee \cong (\mathbb{Z}/\ell\mathbb{Z})^d$ and σ acts by

$$\sigma^*(pr_i) = \begin{cases} q\cdot pr_{i-1} & 1 < i \le d \\ q\cdot pr_d & i = 1 \end{cases}$$

I am interested in computing Φ of c_d of standard representation ρ of T. Now

$$c_d(\rho) = u_1 \cdots u_d \qquad \text{where } u_i = pr_i$$

so

$$\partial c_d(\rho) = \sum u_1 \cdots \hat{u}_j \cdots u_d \cdots du_j$$

~~and~~ $S(P_\sigma) \xrightarrow{\sim} \mathbb{Z}/\ell\mathbb{Z}\,[u]$

$u_i \longmapsto q^i u$,

~~thus~~ $\partial c_d(\rho) = (-u)^{d-1} \sum_{j=1}^{d} du_j\, q^{-j}$.

~~Now it should be so that $\Phi(\sum_i du_j)$ is dv when restricted to $T^\sigma = C$; this is an explicit computation in the~~

~~case of a 2-dimensional class. Therefore if the conjecture is correct we find that~~

$$\cancel{\Phi\left(c_d(\rho)\right) = du^{d-1}.v}$$

Restricting to $T^\sigma = C$ gives the map

$$S(P) \twoheadrightarrow S(P_\sigma) \hookrightarrow H^*(BC)$$

$$u_i \longmapsto \rho^{i-1}u .$$

Hence

$$\partial c_d(\rho) = \left(\prod_{i=0}^{d-1}\rho^i\right) u^{d-1}\cdot d\sum_{i=1}^{d} \rho^{-i+1}u_i$$

and note

$$\sigma\left(\sum \rho^{-i+1}u_i\right) = \sum \rho^{-i+1}\rho\, u_{i-1}$$

showing the sum is an invariant element of P. Now to compute

$$\Phi\left\{\sum_{i=1}^{d} \rho^{-i+1}u_i\right\}$$

~~we just have to look at the homomorphism~~ $\cancel{C \to \mu_\ell = \mathbb{Z}/\ell\mathbb{Z}}$ ~~produced by this character. But if~~ $\cancel{\zeta \in C}$

$$\cancel{\prod_{i=1}^{d}\left(\zeta\rho^{i-1}\right)^{\rho^{-i+1}} = \zeta^d}$$

~~so~~

$$\cancel{\Phi\left\{\sum_{i=1}^{d} \rho^{-i+1}u_i\right\} =}$$

we proceed as follows. The element $\sum_{i=1}^{d} \rho^{-i+1}u_i$ is the Euler class of the character

$$T^\sigma \xrightarrow{\chi} S^1$$

$$(z_1,\dots,z_d) \longmapsto \prod_{i=1}^{d} z_i^{\rho^{d-i+1}}$$

and hence represents the pullback by ψ of the extension

$$0 \to \mathbb{Z}/\ell\mathbb{Z} \to S^1 \xrightarrow{\ell} S^1 \to 0$$

By assumption this pullback and its tranform by σ are isomorphic, meaning that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/\ell\mathbb{Z} \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, dashed] & T \arrow[r] \arrow[d, "\sigma"] & 0 \\
0 \arrow[r] & \mathbb{Z}/\ell\mathbb{Z} \arrow[r] & E \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

hence there is a dotted arrow

$$\begin{tikzcd}
 & & & T \arrow[dll, dashed] \arrow[d, "\psi - \psi\circ\sigma"] & \\
0 \arrow[r] & \mathbb{Z}/\ell\mathbb{Z} \arrow[r] & S^1 \arrow[r, "\ell"] & S^1 \arrow[r] & 0
\end{tikzcd}$$

.

Now restricting this dotted arrow to $C = T^\sigma$ gives a map $C \to \mathbb{Z}/\ell\mathbb{Z}$. The dotted arrow, γ say, satisfies

$$\gamma(z_1,\dots,z_d)^\ell = \psi(z_1,\dots,z_d) / \psi\sigma(z_1,\dots,z_d)$$

$$= \prod_{i=1}^{d} z_i^{q^{(d-i+1)}} \cdot \left(\prod_{i=1}^{d} (\sigma z)_i^{q^{(d-i+1)}} \right)^{-1}$$

$$\left(z_d^{q^{(d+1)}} \prod_{i=2}^{d} z_{i-1}^{q^{(d-i+2)}} \right)^{-1}$$

$$\left(z_d^{q^{(d+1)}} \prod_{i=1}^{d-1} z_i^{q^{d-i+1}} \right)^{-1}$$

$$= z_d^{q - q^{d+1}} = \left(z_d^{-q}\right)^{q^d - 1}$$

So can take

$$\gamma(z_1,\dots,z_d) = \left(z_d^{-q}\right)^{\frac{q^d-1}{\ell}}$$

and so restricting to C where $z_i = \eta(c)^{q^{i-1}}$, have

$$z_d^{-q} = \left(\eta(c)^{q^{d-1}}\right)^{-q} = \eta(c)^{-1}.$$

So therefore it seems that

$$\Phi\left\{ \sum_{i=1}^{d} q^{-i+1} u_i \right\} = -V \qquad \left(\text{recall } V(c) = \eta(c)^{\frac{q^d-1}{\ell}}\right)$$

implying that

$$\Phi c_d(\rho) = (-1)^{d} u^{d-1} (-V)$$

$$= (-1)^{d-1} u^{d-1} V$$

I now take up the proof of the conjecture on page 2.

Set
$$I = \operatorname{Ker}\left\{H^*_{G\times G}(pt) \xrightarrow{\Delta^*} H^*_G(pt)\right\}$$
$$J = \operatorname{Ker}\left\{H^*_{G\times G}(pt) \xrightarrow{(\Gamma_\sigma)^*} H^*_G(pt)\right\}$$
where $\Gamma_\sigma = (id, \sigma)$. I claim that in defining Φ I construct a map

$$\Theta : I\cap J / IJ \longrightarrow H^*_G(G^t)/H^*_G(pt)$$

lowering degrees by 1. Indeed recall that given $\alpha \in H^*_{G\times G}$ with $\Delta^*\alpha = 0$ and $\Gamma_\sigma^*\alpha = 0$, representing α by a, we ~~get~~ choose b, c such that
$$a = db \qquad (\Delta G)\, b = b$$
$$a = dc \qquad (\Gamma_\sigma G)\, c = c$$
and that
$$u(g) = (g,e)\, b - c$$
represents an element of $H^*_G(G^t)/H^*_G$ independent of the choices of b and c. This defines $\Theta\alpha$ for $\alpha \in I\cap J$. If $\alpha \in I$, say
$$a = db \qquad (\Delta G)\, b = b$$
and $\alpha' \in J$, say
$$a' = dc \qquad (\Gamma_\sigma G)\, c = c,$$
then
$$aa' = d(ba') \qquad \Delta G\,(ba') = ba'$$
$$= (-1)^{\deg a} d(ac) \qquad \Gamma_\sigma G\,(ac) = ac$$

so $\Phi(\alpha\alpha')$ is represented by

$$u(g) = (g,e)\,ba' - (-1)^{\deg a}\,ac.$$

It is necessary to prove this element comes from $H^*(BG)$:

Special situation: instead of landing in $H^*_G(G^t)$ let's land in $H^*_{G^\sigma}(pt)$. Here it should be so that the element representing $\Phi(\alpha)$, where a, b, c are as above, is $b-c$, which is a cycle invariant under G^σ; note G^σ is embedded in $G\times G$ in one way. Then if $\alpha \in I$ is represented by $a = db$, $(\Delta G)\, b = b$ and if $\alpha' \in J$ is represented by $a' = dc$, $(\Gamma_\sigma G)\, c = c$, then

$$aa' = d(ba') \qquad ba' \text{ inv. under } \Delta G$$
$$= (-1)^{\deg a} d(ac) \qquad ac \text{ ——— } \Gamma_\sigma G$$

so $\Phi(\alpha\alpha')$ is represented by

$$ba' - (-1)^{\deg a} ac = b\,dc + (-1)^{\deg b}\, db\cdot c$$
$$= (-1)^{\deg b}\, d(bc)$$

and bc is G^σ-invariant. This shows that $\Phi(\alpha\alpha') = 0$.

I would like to generalize this proof to the G^t situation. It is first necessary to understand why the formula $u(g) = (g,e)b - c$ works, and for this why $EG\times_G G^t$ is the homotopy-equalizer of Δ, Γ_σ.

Let $\Gamma^*(I^\cdot)$ be the injective G-resolution of $\mathbb{Z}/\ell\mathbb{Z}$ furnished by $I^\cdot$ with G-action $(g_1, \sigma g_1)x$, and define $\Delta^*(I^\cdot)$ similarly. Then there is a homotopy equivalence

$$\varphi : \Gamma^*(I^\cdot) \longrightarrow \Delta^*(I^\cdot)$$

i.e. a map $\ni$ $\varphi((g_1, \sigma g_1)x) = (g_1, g_1)\varphi(x)$. Now given a cohomology class α in $H^*_{G\times G}$ to compare $\Delta^*\alpha$ and $\Gamma^*\alpha$ we compare a and $\varphi(a)$ where a represents α.

~~When φ is lifted to EG^t it becomes [illegible]~~

If $EG \times_G G^t$ is the homotopy equalizer, then certainly we must have a canonical boundary for $a - \varphi(a)$ over G^t. Thus first

?

April 16, 1970:

Here is how to formulate the derivation property of Φ properly:

Suppose G group in a topos T, σ endom. of G, H^* denotes cohomology with coefficients in some ring A. Suppose given an A-module P ~~on which σ~~ with an endomorphism, denoted σ^*, and a map $P \longrightarrow H^*_G$ compatible with the action of σ^*. Then this extends to a map of A-algebras $S(P) \longrightarrow H^*_G$ (assume P is graded of entirely even dimension unless $2A = 0$.) Since there is a commutative diagram

$$\begin{tikzcd} H^*_G \arrow[dr, "i^*"] \arrow[d, "\sigma^*"'] & \\ H^*_G \arrow[r, "i^*"'] & H^*_G(G^t) \end{tikzcd}$$

it follows that the natural $S(P)$-module structure of $H^*_G(G^t)$ factors through the surjection $\pi: S(P) \longrightarrow S(P_\sigma)$, induced by the surjection $\pi: P \longrightarrow P_\sigma$ $(= P/(\sigma - id)P)$.

Define ∂ to be the composition

$$S(P)^\sigma \xrightarrow{d} (S(P) \otimes P)^\sigma \xrightarrow{\pi \otimes id} (S(P_\sigma) \otimes P)^\sigma = S(P_\sigma) \otimes P^\sigma$$

Then the extended derivation property of Φ asserts commutativity of the square

$$\begin{tikzcd} S(P)^\sigma \arrow[r] \arrow[d, "\partial"] & (H_G^*)^\sigma \arrow[d, "\Phi"] \\ S(P_\sigma) \otimes P^\sigma \arrow[r] & H_G^{*-1}(G^t)/H_G^{*-1} \end{tikzcd}$$

$$x \otimes y \longmapsto x \cdot \Phi y$$

Observe that as ∂ is a derivation, this implies that Φ is a derivation, at least when restricted to elements of H_G^{ev} because we can take $P = H_G^{ev}$.

Proof: Recall notation: $\Delta^*, \Gamma^* : H^*_{G\times G} \longrightarrow H^*_G$ given by Δ and graph of σ. Let I, J be similarly defined ideals in $S(P) \otimes S(P)$. Then using cochains we have defined a map

$$(*) \qquad \frac{\operatorname{Ker}\Delta^* \cap \operatorname{Ker}\Gamma^*}{\operatorname{Ker}\Delta^* \cdot \operatorname{Ker}\Gamma^*} \xrightarrow{\ \int\ } H_G^{*-1}(G^t)/H_G^{*-1}$$

which is an H_G^*-module homomorphism (one that the left is annihilated by $\operatorname{Ker}\Delta^*$ and so is an H^*_G module in a natural way. Finally Φ is obtained by composing $\int$ with the map

$$(H_G^*)^\sigma \longrightarrow \frac{\operatorname{Ker}\Delta^* \cap \operatorname{Ker}\Gamma^*}{\operatorname{Ker}\Delta^* \cdot \operatorname{Ker}\Gamma^*}$$

induced by sending α to $pr_2^*\alpha - pr_1^*\alpha$. Now the product map

$$S(P) \otimes S(P) \longrightarrow H_{G\times G}$$

carries $I \cap J / IJ$ into the left side of $*$, and so what we must prove is the commutativity of

$$\begin{tikzcd}
S(P)^\sigma \arrow[d, "\partial"] \arrow[dr, "u"] & \\
S(P_\sigma)\otimes P^\sigma \arrow[r, "v"] & I\cap J/IJ
\end{tikzcd}$$

where $u(f) = 1\otimes f - f\otimes 1 \mod IJ$ and $v(f\otimes g) = f\cdot u(g)$. However this follows immediately from the diagram

$$\begin{tikzcd}
 & S(P)^\sigma \arrow[r, phantom, "\simeq"] \arrow[d, "d"] \arrow[dl, "\partial"'] \arrow[dr, phantom, "\text{defn of } d"] & S(P)^\sigma \arrow[d, "f \mapsto f\otimes 1 - 1\otimes f"] \arrow[ddr, "u"] & \\
 & (S(P)\otimes P)^\sigma \arrow[r, "\sim"] \arrow[d, "\pi\otimes id"] & (I/I^2)^\sigma \arrow[d] \arrow[r, phantom, "\text{defn of } u"] & \\
S(P_\sigma)\otimes P^\sigma \arrow[r, hook] \arrow[ur, phantom, "\text{defn of } \partial"] \arrow[rrr, bend right, "v"'] & (S(P_\sigma)\otimes P)^\sigma \arrow[r, phantom, "\simeq"] & (I/I^2+IJ)^\sigma & I\cap J/IJ \arrow[l]
\end{tikzcd}$$

whose commutativity is straight-forward from the definitions, provided one knows that

$$I\cap J/IJ \hookrightarrow I/I^2+IJ.$$

So I need:

<u>Lemma</u>: Let P be a vector space endowed with an endomorphism σ, let I and J be the ideals in $R = S(P)\otimes S(P)$ with generators $1\otimes x - x\otimes 1$ and $1\otimes x - \sigma x\otimes 1$ respectively where x runs over P. Let $I\cap J/IJ$

module in the natural way. Then there is a diagram

$$\begin{tikzcd} S(P)^\sigma \arrow[d, "\partial"] \arrow[dr, "u"] & \\ S(P_\sigma)\otimes P^\sigma \arrow[r, "\cong", "v"'] & I\cap J/IJ \end{tikzcd}$$

where $\quad u(f) = 1\otimes f - f\otimes 1 \mod IJ$
and $v(f\otimes g) = f\cdot u(g)$.

<u>Proof</u>: Idea: One knows that

$$I\cap J/IJ \cong \mathrm{Tor}_1^R(R/I, R/J)$$

and that if $x_1, \dots, x_n$ is a basis for P, then

$$\mathrm{Tor}_*^R(R/I, R/J)$$

$\cong$ Koszul homology of R/J with respect to the sequence $1\otimes x_i - x_i\otimes 1 \quad 1\le i\le n$

$\cong$ Koszul homology of $S(P)$ with respect to the sequence $\sigma x_i - x_i \quad 1\le i\le n$

$$\cong S(P_\sigma)\otimes \Lambda P^\sigma$$

so there is an isomorphism

$$I\cap J/IJ \cong S(P_\sigma)\otimes P^\sigma$$

and the lemma should result after making all of these canonical isomorphisms explicit.

Let P be a vector space of finite dimension over a field K endowed with an endomorphism σ. Let P^{σ} be the subspace of elements invariant under σ and let $\pi: P \to P_{\sigma}$ be the quotient space of P by the image of $id - \sigma$. Denote also by $\pi: S(P) \to S(P_{o})$ the induced map on symmetric algebras (over K). Set $R = S(P) \otimes S(P)$ and consider the maps of rings

$$R \underset{v}{\overset{u}{\rightrightarrows}} S(P) \xrightarrow{\pi} S(P_{o})$$

where $u(f \otimes g) = fg$ and $v(f \otimes g) = f.\sigma(g)$. This diagram is clearly a cokernel diagram in the category of rings, i.e. if I and J are the kernels of u and v respectively, then $I + J$ is the kernel of $\pi u = \pi v$. We wish to compute the algebra $Tor_{*}^{R}(R/I,R/J)$. ~~and in particular its elements of degree 1 which is well-known to be isomorphic to I+J/I.~~ For this we can use the resolution of R/I as an R-module which is furnished by the Koszul complex $R \otimes \Lambda^{*} P$ with differential given by the derivation d with $d(r \otimes x) = r(1 \otimes x - x \otimes 1)$. Then tensoring this with R/J which we identify with $S(P)$ via the map v, we obtain an isomorphism

$$Tor_{*}^{R}(R/I,R/J) \cong H_{*}(S(P) \otimes \Lambda^{*} P, d')$$

where d' is the derivation with $d(f \otimes x) = f(\sigma x - x)$. Choose complementary subspaces $P = P^{\sigma} \oplus W$ and $P = Im \oplus V$, where Im is the image of $id - \sigma$; then the complex whose homology we are trying to compute is the tensor product of two complexes

$K_1 = S(W) \otimes \Lambda\, Im$ differential = derivation with $f \otimes x \mapsto f(\sigma x - x)$

$K_2 = S(V) \otimes \Lambda\, P^{\sigma}$ with zero differential.

As $id - \sigma : W \to Im$ is an isomorphism it is well-known that the first complex is acyclic. Therefore we conclude that there is an isomorphism of algebras over $S(P_{\sigma})$

$$S(P_{\sigma}) \otimes \Lambda^{*} P^{\sigma} \xrightarrow{\sim} H_{*}(S(P) \otimes \Lambda^{*} P, d')$$

where the map of P^{σ} to H_1 sends x to $1 \otimes x$.

~~It will be necessary to~~

It remains to make explicit in dimension 1 the isomorphism that we have just obtained. Recall that there is a canonical isomorphism

$$\mathrm{Tor}_1^R(R/I,R/J) \cong I\cap J/IJ$$

which using the Koszul resolution of R/I to compute the left side does the following. Suppose given an element f of $I\cap J$; writing $f = d(s)$ with s in $R\otimes P$, the element $a = (v\otimes \mathrm{id}_P)s$ satisfies $d'a = 0$ and hence represents an element z of Tor_1. Associating to f the element z gives the above isomorphism. Clearly if $x\in P^\sigma$ and if $f = 1\otimes x - x\otimes 1$, then we can take $s = 1\otimes x$, hence we see that the diagram

$$\begin{tikzcd} & P^\sigma \arrow[dl] \arrow[dr] & \\ I\cap J/IJ \arrow[rr, "\sim"] & & H_1(S(P)\otimes \Lambda P, d') \end{tikzcd}$$

commutes. Thus we have proved

<u>Lemma</u>: The map which associated to an element x of P^σ the element $1\otimes x - x\otimes 1$ mod IJ of $I\cap J/IJ$ extends to an isomorphism of $S(P_\sigma)$-modules

$$S(P_\sigma)\otimes P^\sigma \xrightarrow{\sim} I\cap J/IJ$$

Let us denote by ∂ the composition

$$S(P)^\sigma \xrightarrow{d} (S(P)\otimes P)^\sigma \xrightarrow{\pi\otimes 1} (S(P_\sigma)\otimes P)^\sigma \xrightarrow{=} S(P_\sigma)\otimes P^\sigma$$

where d denotes differentiation.

<u>Corollary</u>: The triangle

$$\begin{tikzcd} & S(P)^\sigma \arrow[dl, "\partial"'] \arrow[dr, "\alpha"] & \\ S(P_\sigma)\otimes P^\sigma \arrow[rr, "\beta"] & & I\cap J/IJ \end{tikzcd}$$

is commutative, where $\alpha(f) = 1\otimes f - f\otimes 1$ mod IJ and where β is the $S(P_\sigma)$-linear extension of $\alpha|P^\sigma$.

April 18, 1970:

I want to compute Φ on two-dimensional classes. Let G be a group in a topos T and let A be an abelian group. Let $\mathcal{E}$ be ~~as~~ central extension

$$1 \to A \to E \to G \to 1$$

and assume $\sigma^* \mathcal{E} \cong \mathcal{E}$. Choosing an endomorphism $\varphi : E \to E$ covering σ and inducing id on A we set

$$u(g) = e \cdot \varphi(e)^{-1} \qquad \text{where} \qquad \pi(e) = g.$$

If $g \in G^\sigma$, then $u : G^\sigma \to A$ is a homomorphism. Eventually I hope to be able to identify $\mathcal{E} \mapsto u$ with $\Phi : H^2_G(pt; A)^\sigma \to \tilde{H}^1_{G^\sigma}(pt, A)$.

Next we want to replace G^σ by G^t. So we consider

$$1 \to A \to E \to G \to 1$$

as being an equivariant ~~torsor~~ torsor for A over G^t ~~with~~ where G acts by

$$g.e = \tilde{g}\; e\, \varphi(\tilde{g})^{-1} \qquad \pi\tilde{g} = g.$$

~~[illegible]~~ Suppose that $\varphi^{\#}(e) = \varphi'(e) \cdot \psi(e)$ where $\psi : G \to A$ is a homomorphism. Then ψ defines an equivariant torsor for A over a pt, namely $A \to pt$. Letting this to G^t gives $G^t \times A$ with right G action. The product with the above torsor is

$$0 \to A \times A \longrightarrow E \times A \longrightarrow G^t \to 0$$

$$g.(e,a) = \left(\tilde{g}\, e\, \varphi(\tilde{g})^{-1}, (\psi g) a\right)$$

and the Baer sum of torsors is obtained by dividing out by the action of A, e.g making the identifications $(ea_1, a) = (e, a, a)$. Thus one gets the torsor

$$A \longrightarrow E \longrightarrow G^t$$

with action $g.(e) = \tilde{g}\, e\, \varphi(\tilde{g})^{-1} \psi(g)$. This shows that the indeterminacy comes from $\mathrm{Hom}(G, A)$.

<u>Summary</u>: If $\mathcal{E}: \quad 1 \to A \to E \longrightarrow G \longrightarrow 1$ is a central extension of G by A, invariant under G so that there exists a $\varphi : E \to E$ covering $\sigma : G \to G$ and inducing identity on A, then on defining a G action on E by the formula $g.e = \tilde{g}\, e\, \varphi(\tilde{g})^{-1}$ we obtain a G-equivariant torsor for A over G^t. Changing φ by a homomorphism $\psi : G \longrightarrow A$ alters this torsor by the inverse image under $G \to pt$ of the G-equivariant torsor associated to A, ψ; this latter is A itself with G-action $g.a = \psi(g) + a$. (Thus we should get a map $\tilde{H}^2(BG, A)^\sigma \longrightarrow H^1_G(G^t, A)/H^1_G(pt, A)$

<u>Example 1</u>: If $\sigma = id$, then the extension $\mathcal{E}$ gives rise to the G-torsor E over G^c (G^c has conjugation action), where G acts on E by $g.e = \tilde{g}\, e (\tilde{g})^{-1}$. This is well-defined since A is in the center of E. (This is the refined suspension $H^2_G \to H^1_G(G^t)$.)

Example 2: Suppose T is the topos of sheaves for the étale topology on some category of schemes and that G comes from a group scheme. Let $\chi : G \longrightarrow \mathbb{G}_m$ be a character and let E be the extension of G by μ_ℓ (ℓ some number prime to the residual characteristics) which is the pull-back via χ of the canonical extension

$$1 \longrightarrow \mu_\ell \longrightarrow \mathbb{G}_m \xrightarrow{\ \ell\ } \mathbb{G}_m \longrightarrow 1$$

Thus $E = \mathbb{G}_m \times_{\mathbb{G}_m} G$ and a point of E is a pair (z, g) such that $z^\ell = \chi(g)$. Suppose $\sigma^* E \cong E$, i.e. that there is a map $\varphi : E \to E$ covering σ and inducing the identity on the subgroup $(z, 1)$ with $z \in \mu_\ell$. Then there is a unique homomorphism $\chi_1 : G \longrightarrow \mathbb{G}_m$ such that

$$\varphi(z, g) = (z\,\chi_1(g), \sigma g).$$

Indeed locally given a point g of G, we can find a z such that (z, g) is a point in E. As $\varphi / \mu_\ell = \mathrm{id}$ one sees that the element $\chi_1(g)$ defined by the above equation is independent of the choice of z. This defines χ_1 locally hence by the sheaf property, globally also. It is clear that χ_1 is a homomorphism and that

$$\chi(\sigma g) = \chi(g)\, \chi_1(g)^\ell.$$

Conversely any χ_1 with $\sigma^* \chi = \chi \cdot \chi_1^\ell$ defines a φ.

The corresponding G-equivariant torsor over G^t is E with G-action

$$g_1(z,g) = (z_1,g_1)(z,g)\,\varphi(z_1,g_1)^{-1} \qquad z_1 \text{ chosen } \ni z_1^\ell = \chi(g_1)$$

$$= (z_1,g_1)(z,g)(z_1\chi_1(g_1), \sigma g_1)^{-1}$$

$$= (z\,\chi_1(g_1)^{-1}, g_1 g(\sigma g_1)^{-1})$$

Restricting to $pt \longrightarrow G^t$ we get the μ_ℓ torsor $\mu_\ell \times \{1\}$ with G^σ action

$$g_1(z,1) = (z\,\chi_1(g_1)^{-1}, 1)$$

(Note that as $\sigma g_1 = g_1$ $\quad \chi_1(g_1)^\ell = 1$)

<u>Conclusion:</u> If $\mathcal{E}$ is the "first Chern class mod ℓ" of $\chi: G \longrightarrow G_m$, then $\Phi(\mathcal{E})$ as ~~[illegible]~~ a G^σ-equivariant torsor for μ_ℓ is ~~[illegible]~~ μ_ℓ with G^σ action ~~[illegible]~~ obtained from $\chi_1^{-1}|G^\sigma : G^\sigma \longrightarrow \mu_\ell$ where $\chi_1^\ell = \sigma^*\chi \cdot \chi^{-1}$.

Now suppose G is a torus endowed with an endomorphism σ such that G^σ is finite, or equivalently $(id-\sigma)G = G$. Let M be the character group of G. By assumption $id-\sigma^*$ on M is injective and there is an exact sequence

$$0 \longrightarrow M \xrightarrow{id-\sigma^*} M \longrightarrow M_\sigma \longrightarrow 0.$$

where $M_\sigma = \mathrm{Hom}(G^\sigma, G_m)$. G^σ being a finite abelian group one knows that its mod l cohomology (l a prime no.) is given by

$$H^*(BG^\sigma) = \Lambda\,(G^\sigma/l)^\vee \otimes S({}_lG^\sigma)^\vee$$

$\vee$ = dual as a $\mathbb{Z}/l$ vector space.

except possibley when $l=2$ and ${}_lG^\sigma \longrightarrow G^\sigma/l \neq 0$ when the multiplicative structure is slightly different. In any case we have generating subspaces

$$H^1(BG^\sigma) = \mathrm{Hom}\,(G^\sigma, \mathbb{Z}/l)$$

$$H^2(BG^\sigma) \supset \mathrm{Ext}^1(G^\sigma, \mathbb{Z}/l) \cong \mathrm{Hom}({}_lG^\sigma, \mathbb{Z}/l)$$

~~It [illegible] that the [illegible] exact sequence~~

$$\cancel{0 \to \mathrm{Hom}(G^\sigma/l, M_l) \to M_0 \xrightarrow{l} M_0 \to \mathrm{Ext}^1(G^\sigma, \mathbb{Z}}$$

$$\cancel{0 \to \mathrm{Hom}(G^\sigma/l, \mu_l)}$$

$$\cancel{0 \to \mathrm{Hom}(G^\sigma, \mu_l) \to M_0 \xrightarrow{l} M_0 \to \mathrm{Ext}^1(G^\sigma, \mu_l) \to 0}$$

But also we know that

$$H^*(BG) = S(P)$$

where

$$P = M \otimes \mu_l^\vee$$

and I want now to make all of these things explicit.

Define a map

$$P = M \otimes \mu_l^\vee \longrightarrow H^2(BG, \mathbb{Z}/l)$$

by associating to a character $\chi \in M$ and an ~~...~~ a hom. $\mu_\ell \to \mathbb{Z}/\ell$ ~~...~~ the image of the first Chern class $c_1(\chi) \in H^2(BG, \mu_\ell)$ ~~...~~ under the homomorphism. It is known that this map induces isomorphism

$$S(P) \xrightarrow{\sim} H^*(BG).$$

(All of this works for ℓ any integer prime to residual characteristics.) Indeed by the E-M spectral sequence one is reduced to computing ~~...~~ $H^*(\mathbb{G}_m)$ and showing that $H^1(\mathbb{G}_m)$ is a free $\mathbb{Z}/\ell$-module of rank 1 with generator obtained by suspending the element of $H^2(B\mathbb{G}_m)$ represented by the extension

$$0 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{\ell} \mathbb{G}_m \to 0$$

& some given isom. $\mu_\ell \simeq \mathbb{Z}/\ell$.

I now propose to compute

(*) $$\Phi : H^2(BG)^\sigma \longrightarrow H^1(BG^\sigma).$$

Start with $\alpha \in H^2(BG)^\sigma$ and represent it by a character $\chi \in M$ (suppose $\mu_\ell \simeq \mathbb{Z}/\ell$ given). Then $\sigma^*\alpha = \alpha$ implies that $\chi = \sigma^*\chi \cdot \chi_1^\ell$ for some χ_1 in M. $\Phi(\alpha)$ is then represented by the homomorphism $\chi_1 | G^\sigma$ as we saw on page 4. From general results we know (*) is an isomorphism, but would like a more elementary way of seeing this.

So given a homomorphism $u : G^\sigma \longrightarrow \mathbb{Z}/\ell\mathbb{Z} \simeq \mu_\ell$, u can be extended to a character $\chi_1 : G \to \mathbb{G}_m$. Then χ_1^ℓ kills G^σ and so extends to $G/G^\sigma \xrightarrow{\sim} G$, i.e. there is a unique $\chi : G \to \mathbb{G}_m$ with

$\chi \cdot (\sigma^* \chi)^{-1} = \chi_1^{\ell}$. It follows that $u = \Phi(\alpha)$ where α is the class in $H^2(BG)$ of χ. It is easy to check that this gives an inverse to $(*)$, so $(*)$ is an isomorphism.

(Comments for the rewriting: $\mathbf{E}$ for extension is unfortunate, since e should be final object.

For calculating Φ (first Chern class) you should always work with $A = \mu_\ell$ since the multiplicative structure is not used at this stage.)

April 19, 1970. Torsors, extensions, and cohomology.

G group in a topos T, A a G-module. I propose to interpret $\tilde{H}^1_G(e,A)$ and $\tilde{H}^2_G(e,A)$ in terms of extensions generalizing the ~~usual~~ standard situation when $T=$ sets. Let $0 \to A \to I^0 \to I^1 \to \cdots$ be an injective resolution in T_G.

First suppose that X is an object of T_G. Given an A-torsor $P \xrightarrow{\pi} X$ in T_G, I claim the extended I^0-torsor $I^0 \times^A P$ (action on left) splits. (In effect if A were injective one would know that

$$A_G(X) \longrightarrow A_G(P) \rightrightarrows A_G(P \times_X P) \equiv A_G(P \times_X P \times_X P)$$

is exact. Define $f \in A_G(P \times_X P)$ by $f(p_1,p_2)\cdot p_2 = p_1$. Then $\delta f = 0$ so by exactness of the above sequence $\exists\ g \in A(P)$ with $f(p_1,p_2) = g(p_2) - g(p_1)$. Then one obtains a section s of P by $s(\pi p) = g(p)\cdot p$.)

Returning to the situation at hand, let $s : X \longrightarrow I^0 \times^A P$ be a section. Define $v : X \longrightarrow I^1$ $dv=0$ as follows: Given $x \in X$ choose p with $\pi(p)=x$, write $\lambda s(x) = p$ with $\lambda \in I^0$, and set $v(x) = d\lambda$. Note that $g(\lambda)s(gx) = g(p)$ so $v(gx) = g\,v(x)$, i.e. $v \in I^1_G$. We can recover P and s from v, namely

$$P \longrightarrow \left\{(\lambda,x) \in I^0 \times X \;\middle|\; d\lambda = v(x)\right\}$$

Indeed if $p = \lambda s(x)$ then $d\lambda = v(x)$.

Therefore we see that if $v \in I^1_G(X)$ satisfies $dv=0$

we get an A-torsor P over X ~~such that~~

$$P = \{ (\lambda, x) \in I^0 \times X \mid d\lambda = v(x) \}.$$

Following Grothendieck one introduces the category $\mathcal{A}$ with objects $v \in Z^1_G$ and morphisms given by elements of $I^0_G(X)$. One gets by the above procedure a functor

$$\mathcal{A} \longrightarrow A\text{-torsors over } X$$

which is easily seen to be fully faithful and which is essentially surjective as we have seen, since I^0 is injective. Thus we obtain the usual isomorphism

$$H^1_G(X, A) \cong \begin{array}{l}\text{Isom. classes of } A\text{-torsors} \\ \text{over } X \text{ in } T_G.\end{array}$$

The special case of interest to me is where $X = e$ and the torsors under attention are provided with a section not equivariant. Then $P \cong A$ and the G action defines a map $u : G \longrightarrow A$ by the formula

$$g(a) = u(g)a$$

Then $\quad u(g_1 g_2)a = g_1(g_2 a) = u(g_2)^{g_1} u(g_1)a$,
hence $u \in A(G)$ is a 1-cocycle.

__Extensions:__ Suppose given in the topos T an extension of G by A:

$$(*) \qquad 1 \longrightarrow A \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1$$

Claim if A is an injective G-module, then π admits a section s which is a homomorphism. Indeed we've already seen that the A-torsor over G is trivial, hence there exists a section s. Then the usual formula describes the extension in terms of the cocycle $f \in A(G \times G)$ defined by

$$f(g_1, g_2)\, s(g_1 g_2) = s(g_1) s(g_2).$$

But we've seen that the complex

$$A(e) \xrightarrow{\delta} A(G) \xrightarrow{\delta} A(G \times G) \xrightarrow{\delta} A(G \times G \times G)$$

is exact for A injective, hence $f = \delta h$, $h \in A(G)$ and so modifying s by h, we obtain a section homomorphism.

As a consequence the extension obtained from $(*)$ by the homomorphism $A \longrightarrow I^0$, namely $E' = I^0 \times^A E$ splits. Choosing a section homomorphism $s : G \longrightarrow E'$. Then writing $s(\pi y) = \lambda y$ $\quad y \in E$, $\lambda \in I^0$ and setting $v(\pi y) = d\lambda$ we obtain a map $v : G \longrightarrow I^1$ satisfying $dv = 0$ and $\delta v = 0$. The last comes from s being a homomorphism:

$$s(g_1) = \lambda_1 y_1 \qquad s(g_2) = \lambda_2 y_2 \qquad g_i = \pi y_i$$

$$s(g_1 g_2) = s(g_1) s(g_2) = \lambda_1 y_1 \lambda_2 y_2 = \lambda_1 \lambda_2^{g_1} y_1 y_2$$

$$\begin{aligned} v(g_1 g_2) &= d(\lambda_1 \lambda_2^{g_2}) \\ &= v(g_1) + g_2 v(g_1). \end{aligned}$$

Moreover E can be recovered from v by

$$E \xrightarrow{\sim} \{(\lambda, g) \in I^0 \times G \mid d\lambda + v(g) = 0\}$$

Indeed if $y \in E$ and $y = \lambda s(g)$ in $I^0 \overset{A}{\times} E$, then $s(g) = \lambda^{-1} \cdot y$, so $v(g) = -d\lambda$.

So again one introduces the **category** obtained from the complex

$$\{ {}_\delta I^0(G) \xrightarrow{d} {}_\delta Z^1(G) \}$$

and maps it to the category of extensions by the above formula for E in terms of v. We've just proved the essentially surjective part of the theorem

$$\mathrm{Cat}\{ {}_\delta I^0(G) \xrightarrow{d} {}_{\delta,d} I^1(G) \} \xrightarrow[\text{equiv.}]{\sim} \underline{\mathrm{Ext}}(G; A)$$

Having gotten this far the rest proceeds through quasi-isomorphisms of complexes. Thus since $I^1(G^\cdot)$ is acyclic we have an equivalence of categories

$$\mathrm{Cat}\{ I^0(e) \times_{I^0_G(e)} I^1_G(e) \xrightarrow{d\delta + d} {}_{d\delta} I^1(e) \} \xrightarrow[\text{equiv.}]{\delta} \mathrm{Cat}\{ {}_\delta I^0(G) \xrightarrow{d} {}_{d,\delta} I^1(G) \}$$

$I^0(e) \times^{I^0_G(e)} I^1_G(e)$

and a functor

$$\Big\downarrow d$$

$$\mathrm{Cat}\{ I^1_G(e) \xrightarrow{d} {}_d I^2_G(e) \}$$

with essential image those elements of $H^2_G(e, A)$ which die on forgetting the G-action.

It is necessary to be careful; thus we obtain a functor

$$\underline{\mathrm{Ext}}(G;A) \longrightarrow \mathcal{C}at\left\{ I^1_G(e) \xrightarrow{\ \ } {}_d I^2_G(e) \right\}$$

~~which is fully faithful $\iff H^1_G(e;A) \neq \emptyset$ and an equivalence $\iff A(e)\ H^1_G(e;A) = H^2_G(e,A) = 0$. For our purposes it is enough to know this is a functor~~ which is not usually an equivalence, however just the functor itself should be enough for our purposes.

So at the moment we have worked out the relation between low dimensional cohomology and extensions. Now I need to use this relation to compute Φ, which recall is defined when A has trivial G-action. The problem arises from the fact that Φ has been defined using $G \times G$, Γ, Δ with a specific cochain formula.

April 20, 1970. More on Φ.

Given two homomorphisms $\sigma_1, \sigma_2 : G' \rightrightarrows G$ of groups in the topos T and an abelian group A of T, we wish to understand the difference homomorphism

$$\Phi : \{\alpha \in H^*_G(e;A) \mid \sigma_1^*\alpha = \sigma_2^*\alpha\} \longrightarrow H^{*-1}_{G'}(G_t;A) / H^{*-1}_{G'}(e;A)$$

Here G_t denotes the object of $T_{G'}$ which is G with action $g'(g)_t = (\sigma_1 g')g(\sigma_2 g')^{-1}$.

Method 1 (not using $G\times G$). Let I', I be injective resolution of A in $(T_{G'})_{ab}$ and $(T_G)_{ab}$ respectively. Let $\sigma_i^* I$ be the resolution of A in $(T_{G'})_{ab}$ which is I and with G' acting through σ_i. Then there are maps of complexes

$$\varphi_i : \sigma_i^* I \longrightarrow I' \qquad i = 1,2$$

unique up to homotopy in $(T_{G'})_{ab}$. Associating a a cocycle $a \in I(e)$ the cocycle $\varphi_i(a)$ in $I'(e)$, induces the restriction homomorphism

$$\sigma_i^* : H^*_G(e;A) \longrightarrow H^*_{G'}(e;A)$$

Now there is a triangle of resolutions of $G_t \times A$ in $(T_{G'}/G_t)_{ab}$

$$(g, a) \longmapsto (g, ga)$$

$$\begin{tikzcd}
G_t \times \sigma_2^* I \arrow[rr, phantom, "\cong"] \arrow[dr, "id\times\varphi_2"'] & & G_t \times \sigma_1^* I \arrow[dl, "id\times\varphi_1"] \\
& G_t \times I' &
\end{tikzcd}$$

which is not commutative, ~~but all these maps~~ however these maps reduce the identity on $G_t \times A$, hence as $G_t \times I'$ is injective in $(T_{G'}/G_t)_{ab}$ the triangle is homotopy commutative. Thus there is a homotopy operator

$$\begin{aligned}
G_t \times \sigma_2^* I &\longrightarrow G_t \times I' \\
(g, a) &\longmapsto (g, h(g, a))
\end{aligned}$$

satisfying

$$\varphi_1(ga) - \varphi_2(a) = d\, h(g, a) + h(g, da)$$

$$g'\, h(g, a) = h((\sigma_1 g') g (\sigma_2 g')^{-1}, (\sigma_2 g') a)$$

~~Suppose~~ We are now ready to define Φ. Given $\alpha \in H^q_G(e, A)$ represent it by $a \in I^q(e)$ satisfying $da = 0$, $Ga = a$. Then $\sigma_i^* \alpha$ is represented by $\varphi_i(a)$. Suppose $\sigma_1^* \alpha = \sigma_2^* \alpha$; Then

$$\varphi_1(a) - \varphi_2(a) = db$$

with $b \in I^{q-1}(e)$ satisfying $G'b = b$. On the other hand from the above we have

$$\varphi_1(a) - \varphi_2(a) = d\, h(g, a)$$

with $h(g,a) \in (I')^{\delta-1}(G)$ satisfying

$$g'\,h(g,a) = h((\sigma_1 g')g(\sigma_2 g')^{-1}, a).$$

Thus the element

$$h(g,a) - b \quad \in (I')^{\delta-1}(G_t)$$

is a cocycle invariant for the action of G' and hence represents an element of $H^{\delta-1}_{G'}(G_t; A)$. Modulo the ~~...~~ choices of the φ_i and h, which ~~...~~ it will turn out don't matter, this element can be altered by changing b by a cocycle. This means that we get a well-defined map

$$\Phi : \{\alpha \in H^{\delta}_{G}(e;A) \mid \sigma_1^*\alpha = \sigma_2^*\alpha\} \longrightarrow H^{\delta-1}_{G'}(G^t; A)/\mathrm{Im}\, H^{\delta-1}_{G'}(e;A)$$

Method 2: Let $\Gamma = (\sigma_1, \sigma_2) : G' \longrightarrow G\times G$ and let $\Delta : G \longrightarrow G\times G$ be the diagonal. Let J be an injective $G\times G$ resolution of A, and let

$$\varphi_\Gamma : \Gamma^* J \longrightarrow I'$$

be a map of G'-resolutions. If $\beta \in H^{\delta}_{G\times G}(e; A)$ satisfies $\Delta^*\beta = 0$, then representing β by $a \in J(e)$, $(G\times G)a = a$, $da = 0$ since $\Delta^* J$ is an injective G-resolution we know that $a = db$ where $(\Delta G)b = b$. b gives rise to a section $u \in J(G_s)$ by

$$u(g) = (g, e)\,b$$

which is invariant under $G\times G$, hence $\varphi_\Gamma(u) \in I'(G_t)$ is invariant under G'. Suppose also that $\Gamma^*\beta = 0$, that is $\varphi_\Gamma(a) = dc$ with $c \in I'(e)$ invariant under G'. Then

(*)
$$g \longmapsto \varphi_\Gamma((g,e)b) - c$$

defines an element of $I'(G_t)$ invariant under G' which is also a cocycle, hence represents an element z of $H^{g-1}_{G'}(G_t; A)$. Changing c alters z by something coming from $H^{g-1}_{G'}(e; A)$. We now investigate the effect of altering b by a cocycle.

We want to show that if $b \in J$, $db = 0$, $(\Delta G)b = b$, then the element of $H'_G(G^t; A)$ represented by $g \mapsto (g,e)b$ is the same as the element coming from $H'_G(e; A)$ which one gets by using J as an injective $\Delta G = G$ resolution + then say either σ_1^* or σ_2^*. So to obtain the element of $H_G(e; A)$ represented by b choose

$$\varphi_\Delta : \Delta^* J \longrightarrow I.$$

So we want to compare $g \mapsto \varphi_2 \varphi_\Delta(b) \in I'_{G'}(G_t)$ and $g \mapsto \varphi_\Gamma((g,e)b) \in I'_{G'}(G_t)$. Consider the triangle of G'-resolutions

$$\begin{tikzcd}
G^t \times \sigma_2^* \Delta^* J \arrow[rr, "\cong", "(g,\, b) \mapsto (g,\, (g,e)b)"'] \arrow[dr, "\mathrm{id} \times \varphi_2 \varphi_\Delta"'] & & G^t \times \Gamma^* J \arrow[dl, "\mathrm{id} \times \varphi_\Gamma"] \\
& G^t \times I' &
\end{tikzcd}$$

of $G_t \times A$ in $(T_{G'}/G_t)_{ab}$. Check invariance

$$\begin{array}{ccc}
(g, b) & \longmapsto & (g, (g,e)b) \\
\downarrow & & \downarrow \\
((\sigma_1 g')g(\sigma_2 g')^{-1}, (\sigma_2 g', \sigma_2 g')b) & \longmapsto & ((\sigma_1 g')g(\sigma_2 g')^{-1}, ((\sigma_1 g')g, \sigma_2 g')b)
\end{array}$$

$$\longmapsto ((\sigma_1 g')g(\sigma_2 g')^{-1}, (\sigma_1 g' g, \sigma_2 g')b)$$

As this induces the identity on $G_t \times A$ one gets a homotopy operator $\bar{h}: G_t \times \sigma_2^* \Delta^* J \longrightarrow I'$ such that

$$\varphi_\Gamma((g,e)b) - \varphi_2\varphi_\Delta(b) = d\,\bar{h}(g,b) + \bar{h}(g, db)$$

When $db = 0$ and $(\Delta G)b = b$, $g \mapsto \bar{h}(g,b) \in I'_{G'}(G_t)$ so the two elements $g \longmapsto \varphi_\Gamma((g,e)b)$ and $\varphi_2\varphi_\Delta(b)$ represent the same element in $H_{G'}(G_t)$.

So the indeterminacy of the element represented by $*$ on page 4 always lies in the image of $H^*_{G'}(e;A)$ and so we have a well-defined map

$$\Phi : \operatorname{Ker}\Delta^* \cap \operatorname{Ker}\Gamma^* \longrightarrow H^*_{G'}(G_t;A) / H^*_{G'}(e;A)$$

(Just in case we may need it later the analogous isomorphism with φ_1 instead of φ_2 is

$$\begin{array}{ccc} G_t \times \sigma_1^*\Delta^* J & \simeq & G_t \times \Gamma^* J \\ (g,\ b) & \longmapsto & (g,\ (e,g^{-1})b)\ .\ ) \end{array}$$

~~The next step is to compose~~

The second method for defining Φ is to compose Φ with the map

$$\begin{array}{ccc} \operatorname{Ker}\{\sigma_1^*, \sigma_2^* \text{ on } H^*_G(e;A)\} & \longrightarrow & \operatorname{Ker}\Delta^* - \operatorname{Ker}\Gamma^* \\ \alpha & \longmapsto & pr_1^*\alpha - pr_2^*\alpha. \end{array}$$

The next step is to compare the two definitions of Φ.

~~For simplicity we suppose that $\varphi_\Delta : \Delta^* J \longrightarrow I$ is the identity so as not to have to put up with extra homotopies~~

$$\begin{tikzcd}
G_t^\sharp \times \sigma_2^* \Delta^* J \arrow[r, "\simeq"] \arrow[dr, "\mathrm{id} \times \varphi_2 \varphi_\Delta"'] & G_t^\sharp \times \Gamma^* J \arrow[r, "\simeq"] \arrow[d, "\mathrm{id} \times \varphi_\Gamma"] & G_t^\sharp \times \sigma_1^* \Delta^* J \arrow[dl, "\mathrm{id} \times \varphi_1 \varphi_\Delta"] \\
& G_t^\sharp \times I' &
\end{tikzcd}$$

$$(g, b) \longmapsto (g, (g,e) b) \longmapsto (g, (g,g) b)$$

Suppose h_2, h_1 make these triangles commute

$$\varphi_\Gamma((g,e)b) - \varphi_2 \varphi_\Delta (b) = d h_2(g,b) + h_2(g, db)$$

$$\varphi_1 \varphi_\Delta ((e,g)b) - \varphi_\Gamma(b) = d h_1(g,b) + h_1(g, db)$$

Then we take

$$h(g,b) = h_1(g, (g,e)b) + h_2(g,b)$$

so

$$\varphi_1 \varphi_\Delta((g,g)b) - \varphi_2 \varphi_\Delta(b) = d h(g,b) + h_2(g, db)$$

and we have the data we need for method 1. The only problem is to express that $\alpha = (pr_1^* - pr_2^*)\dot{\alpha}$. So we introduce the maps

$$\varphi_{pr_i} : pr_i^* I \longrightarrow J$$

which one needs to define $pr_i^* : H_G^*(e; A) \longrightarrow H_{G\times G}^*(e; A)$.
For simplicity suppose that

$$\varphi_\Gamma \varphi_{pr_i} = \varphi_i \quad : \quad \sigma_i^* I \longrightarrow I' \; ;$$

this should offer no problem because its a diagram condition. Then we start with a class $\dot{\alpha} \in H_G^*(e; A)$ $\ni \sigma_1^* \dot{\alpha} = \sigma_2^* \dot{\alpha}$, represent $\dot{\alpha}$ by ~~a cochain~~ a cocycle x; then $\alpha = pr_1^* \dot{\alpha} - pr_2^* \dot{\alpha}$ is represented by $a = \varphi_{pr_1}(x) - \varphi_{pr_2}(x)$

$$\varphi_\Delta(a) = \varphi_\Delta \varphi_{pr_1}(x) - \varphi_\Delta \varphi_{pr_2}(x) = \varphi_{id}(x) - \varphi_{id}(x) = 0$$

(diagram condition) hence as before

$$a = db \qquad b \in J(e) \qquad (\Delta G) b = b.$$

Also

$$\varphi_\Gamma(a) = \varphi_\Gamma \varphi_{pr_1}(x) - \varphi_\Gamma \varphi_{pr_2}(x)$$
$$= \varphi_1(x) - \varphi_2(x)$$
$$= dc \qquad c \in I'(e) \qquad G'c = c.$$

Now for method 1 we must also write

$$\varphi_1(x) - \varphi_2(x) = d\, h(g, x) \qquad h: G_t \times \sigma_2^* \Delta^* J \to I'$$

and consider the element

$$g \longmapsto h(g, x) - c$$

?

April 25, 1970:

Recall your scheme for algebraic K-theory. If S is a ringed topos and I is any topos, let $k(I)$ be the Grothendieck group of the ringed topos $I \times S$. The idea is to find a topos B with the property that there is a map ~~[illegible]~~ $k \longrightarrow [\,,B]$ inducing an isomorphism on cohomology. In other words given a abelian group A, there should be a 1-1 correspondence between cohomology classes of B with coefficients in A and ~~[illegible]~~ ways of assigning to each virtual bundle over $I \times S$ a cohomology class of I with coefficients in A.

Consider the following modification in which you let I be a scheme, whence $k(I)$ is the Grothendieck group of vector bundles on $I \times S$.

Example: Suppose I and S run over nice topological spaces. Then the functor

$$k(I) = K(I \times S) = [I \times S, \mathbb{Z} \times BU] = [I, (\mathbb{Z} \times BU)^S]$$

is representable and its homotopy groups are

$$\pi_i\left((\mathbb{Z} \times BU)^S\right) = K^{-i}(S).$$

This is quite different from the algebraic K-theory groups of S where one knows that there is a surjective map

$$\tilde{K}_1^{alg}(S) \longrightarrow \mathbb{C}(S)^*.$$

Suppose we now consider the category of schemes over a field k, say projective non-singular schemes, and work modulo truth of the theory of motives. Thus take Grothendieck group of $I \times S$ to be numerical equivalence classes of cycles on $I \times S$, or invariants for a Galois-type group. First assume Galois is trivial. Then

$$\text{ch}: \tilde{K}(I) \xrightarrow{\sim} \tilde{H}^{ev}(I \times S). \qquad \text{ring isom.}$$

To give a multiplicative characteristic class $\tilde{K} \to (H(X) \otimes R)^{\times}$ is same as a ring homomorphism $H_*(\tilde{B}) \longrightarrow R$. But homotopy theory in characteristic zero says $H_*(\tilde{B}) = S[\pi_* \tilde{B}]$, ($B$ connected) hence a mult. char class is the same as a linear homomorphism $\pi_*(\tilde{B}) \longrightarrow R$. ~~What it says is that~~ Now by the above isomorphism a mult. char. class is the same thing as a map

$$\theta: \tilde{H}^{ev}(I \times S) \longrightarrow 1 + \sum_{i>0} H^i(I) \otimes R^{-i}$$

which is exponential i.e. sends sums to products. In virtue of the logarithm this is the same as a linear map

$$\tilde{H}^{ev}(I \times S) \longrightarrow \sum_{i>0} H^i(I) \otimes R_i$$

$$\parallel$$

$$\sum_{\substack{i+j \text{ even} \\ i \text{ or } j > 0}} H^i(I) \otimes H^j(S)$$

which is natural in I. But one knows that $\mathrm{Hom}_{ad}(H^i, H^j)$

$= \mathbb{Q}$ if $i \equiv j$ and zero otherwise, so what we get is a family of linear maps

$$\sum_{j \equiv i\,(2)} H^j(S) \longrightarrow R_i$$

Thus $R_1 = H^{odd}(S)$, $R_2 = H^{ev}(S)$ which gives us what we want.

The moral of the above calculation is the following. Working with $\tilde{k}$ and $\tilde{B}$, and over $\mathbb{Q}$ one knows that

$$H_*(\tilde{B}) = S[\pi_*(\tilde{B})] \quad \text{and that}$$

$$\begin{array}{ccc} \mathrm{Hom}_{rgs.}(H_*(\tilde{B}), R) & \cong & \mathrm{Hom}_{ab}(\tilde{k}, 1+(\tilde{H}^*(?)\otimes R)) \\ \wr\| & & \wr\| \\ \mathrm{Hom}_{ab}(\pi_*\tilde{B}, R) & & \mathrm{Hom}_{ab}(\tilde{k}, \tilde{H}^*(?)\otimes R) \end{array}$$

Therefore there is a universal homomorphism

$$\tilde{k}(?) \longrightarrow \left[\tilde{H}^*(?) \otimes \pi_*(\tilde{B})\right]^\circ$$

and probably the ring structure on $\pi_*(B)$ is such that this is a ring homomorphism. Simplest statement is that the representable functor generated by k is

$$[\ ,B] = (H^*(\ ) \otimes \pi_* B)^\circ$$

by structure theory over $\mathbb{Q}$.

~~What~~ happens when Galois is not trivial?

$$\tilde{k}(I) \xrightarrow{\sim} \tilde{H}^{ev}(I \times S)^G$$

I want all additive natural transformations of this to $H^*(I)$. Thus we want

$$\mathrm{Hom}\left(H^{ev}(? \times S)^G, H^*(?)\right)$$

which I don't see how to compute.