March 1970

Problem: Carry out the Brauer lifting explicitly for the standard representation of $Gl_n(\mathbb{F}_q)$.

So the idea is as follows: Do first for C a cyclic group of order $n = p^a m$.

$$\begin{tikzcd}
P_k(C) \arrow[r, "c"] \arrow[dr, "e"'] & R_k(C) \arrow[r, phantom, "\cong"] & R_K(\mathbb{Z}_m) \\
 & R_K(C) \arrow[u, "d"'] \arrow[r, phantom, "\cong"] & R_K(\mathbb{Z}_{p^a}) \otimes R_K(\mathbb{Z}_m) \arrow[u, "\varepsilon \cdot id"']
\end{tikzcd}$$

If we tensor with $\mathbb{Z}[p^{-1}]$, then c becomes an isom. and so there is a section

$$\begin{tikzcd}
R_K(C)[p^{-1}] \arrow[r, "d"] & R_k(C)[p^{-1}] \arrow[l, bend right, dashed, "ec^{-1}"']
\end{tikzcd}$$

d should ~~also~~ be a λ-ring homomorphism. Thus $ec^{-1}d$ is an idempotent operation whose effect on characters is I believe

$$\left[(ec^{-1}d)\chi\right](g) = \chi(g_r)\,\varepsilon(g_s)$$

not quite correct because if g is p-singular then $g_{reg} = 1$ and $\chi(1)$ might be $\neq 0$.

The idea is that $R_K(G) \xrightarrow{d} R_k(G) \xrightarrow{ec^{-1}} R_K(G)[\frac{1}{p}]$ should be some kind of λ-operation. When you have a ~~p-regular~~ an element $g = g_r g_s$ and you have a proj. $A[G]$ module M free over $A[P]$ then $\chi(g) = 0$ if $g_s \neq 1$ because first break up into g_r eigenspaces which each must be $A[P]$-proj. hence g_s has tr 0.

A program for a proof of the conjecture ~~[illegible]~~ for the general linear groups

$$\bigoplus_n H_*(BGl_n(R)) \qquad \text{ring}$$

~~[illegible]~~ missing step is to define Δ_{add}

Involves $$Gl_k \times Gl_{n-k} \longrightarrow Gl_n$$

and the Grassmannians so perhaps possible

If can do then we know that

$$H_*(Gl_n(R)) \hookrightarrow H_*(Gl_{n+1}(R))$$

In the limit we have a map

$$\boxed{Gl_\infty(k) \longrightarrow Gl_\infty(R)}$$

furnished by Brauer theory

$$\begin{array}{ccc} H_*(Gl_n(R)) & \hookrightarrow & H_*(Gl_\infty(R)) \\ \downarrow & & \downarrow \uparrow \\ H_*(Gl_n(k)) & \hookrightarrow & H_*(Gl_\infty(k)) \end{array}$$

$Gl_k \times Gl_{n-k} \longrightarrow Gl_n$

$H^*(BU_m \times BU_m) \longrightarrow H^*(BU_{n+m})$

$\longrightarrow BU_n \times BU_m \longrightarrow BU_{n+m}$

$G_{m,n}$

~~[illegible]~~

---

actually Brauer gives you

$$\cancel{Gl_\infty} \quad Gl_n(k) \longrightarrow Gl_{p^a n}(R)$$

unfortunately can't do better because you haven't brought in the hypo. that R is henselian.

$\cancel{H^*(BU_1 \times BU_n) \to H^*(BU_n)}$

$$Gl_n(R) \longrightarrow Gl_n(k) \longrightarrow Gl_N(R)$$

probably you can show this is ~~multiplication~~ equivalent to the p^a th power?

by explicit formulas (derived from Green) maybe you can see this lifting

$$Gl_n(R) \longrightarrow Gl_n(k) \longrightarrow Gl_N(R)$$

now the idea is that you have applied some λ-operation to the standard complex representation of $Gl_n(R)$ ~~which you should be able to compute~~ whose effect on cohomology you ought to be able to compute in terms of the God-given structure on the family.

now if this is true ~~that~~ then the projection operator we are after is ψ^Q where Q a higher power of p $\ni$

$$Q = p^N \quad \text{and} \quad (i) \ |G|_p \mid p^N$$

$$(ii) \ \frac{|G|}{|G|_p} \mid p^N - 1$$

thus if m is prime to p and $((\psi^p)^N \chi)(g) = \chi(g^{p^N}) = \chi(g)$

if ord $g = m$ & $p^N \equiv 1 \ (m)$.

~~[illegible]~~

$$0 \to G(k) \longrightarrow G \xrightarrow{x \mapsto x(Fx)^{-1}} G \longrightarrow 0$$

is a principal $G(k)$ bundle with basepoints so there should be a hom.

$$\pi_1(G) \twoheadrightarrow G(k)$$

$$G \xrightarrow{F} G$$

$$G \longrightarrow G$$

surjective since G is connected.

so ~~[illegible]~~ we get lots of maps

$$\rho_n : \pi_1(G) \twoheadrightarrow G(\mathbb{F}_{q^n})$$

which don't fit together since there doesn't seem to be any good homomorphism ~~maps of~~

$$G(\mathbb{F}_{q^n}) \longrightarrow G(\mathbb{F}_q)$$

unless G is abelian and we take $\frac{F^n-1}{F-1} = \sum_{i=0}^{n} F^i$

make the group act on itself in a funny way ~~[illegible]~~

$$x \mapsto \cancel{}\ x \cdot Fx^{-1}$$

why

$$\begin{array}{ccc} G^t & \longleftarrow & G \\ \downarrow & & \downarrow \\ EG \times_G G^t & \longleftarrow & EG \\ \downarrow & & \downarrow \\ BG & \longleftarrow & BG \end{array}$$

~~[illegible]~~

$G = SL_2$ defd. over $\mathbb{F}_p$.

$$0 \to B \longrightarrow G \longrightarrow G/B \longrightarrow 0$$

$G/B = P^1$ 1-conn.

$B = \mathbb{G}_m \times \mathbb{G}_a$

$\pi_1 B = \pi_1 \mathbb{G}_m \times \pi_1 \mathbb{G}_a$

$SL_2(\mathbb{F}_q)$ simple except for $\mathbb{Z}_2$

G ~~k~~ $= Sl_2(R)$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad ad - bc = 1.$

$G = (1+\mathfrak{m})^*$. $\quad R/\mathfrak{m}$ char p. $\quad l \neq p$.

why is $H^*(BG, \mathbb{Z}_l) = 0$. My reason is that

$$H^*(BG,\mathbb{Z}_l) = \varprojlim_{A} H^*(BA, \mathbb{Z}_l)$$

where A runs over finitely gen. abelian group

since G is uniquely l-divisible

$$\begin{tikzcd} E\pi \times_\pi S^{n-1} \arrow[d, "\pi"] & \\ B\pi \arrow[r] & BTop \end{tikzcd}$$

~~so~~ get an element in $[B\pi, BG]$ $\quad [B\pi, BTop]$

so get Thom isom in $KO[\tfrac{1}{2}]$-theory.

$$\rho_k(S,\pi) = \frac{\psi^k \Delta}{\Delta} \in 1 + \overline{KO}[2^{-1}](B\pi)$$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & E\pi \times_\pi S^{n-1} \arrow[d] \\ S/\pi \arrow[r] & B\pi \end{tikzcd}$$

thus if π acts freely

$$(1+\mathfrak{m})^\times \big/ \{(1+\mathfrak{m})^\times\}^p \qquad \text{so if unramified !}$$

<u>Question</u>

$$0 \to \mu_p \longrightarrow \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \longrightarrow 0$$

exact sequence of sheaves for flat topology so can ask

$$0 \to \mu_p(R) \longrightarrow R^* \xrightarrow{p} R^* \longrightarrow H^1(R, \mu_p) \to H^1(R, \mathbb{G}_m) \xrightarrow{p} H^1(R, \mathbb{G}_m)$$

$G = \mathbb{G}_m$

$$0 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{\ell} \mathbb{G}_m \to 0$$ exact sequence of sheaves for etale top.

then we ~~can~~ get exact sequence

$$0 \to \mu_\ell(R) \to R^* \xrightarrow{\ell} R^* \to H^1(\mathrm{Spec}\, R, \mu_\ell) \to \mathrm{Pic}\, R \xrightarrow{\ell} \mathrm{Pic}\, R$$
$$\to H^2(\mathrm{Spec}\, R, \mu_\ell) \to \cdots$$

<u>critical problem:</u> G ~~a~~ group in a topos $\mathcal{T}$ can form classifying topos $\mathcal{T}_G \to \mathcal{T}$ over $\mathcal{T}$ and the question is whether you can say anything about the cohomology with coeffs. $\mathbb{Z}/\ell\mathbb{Z}$ of $\Gamma(S,G)$ where S is an object of $\mathcal{T}$.

~~Obvious that $S \mapsto H^*(B\Gamma(S,G), \mathbb{Z}/\ell\mathbb{Z})$ is a contravariant functor etc.~~ Because if $f: S_1 \to S_2$ is a map, then have

$$f^*: \Gamma(S_2, G) \to \Gamma(S_1, G)$$

$$\begin{tikzcd} S_1 \arrow[d] \arrow[dr, dashed] & \\ S_2 \arrow[r] & G \end{tikzcd}$$

hence $B\Gamma(S_2, G) \to B\Gamma(S_1, G)$

so $H^*(B\Gamma(S_2, G)) \leftarrow H^*(B\Gamma(S_1, G))$

so we have the wrong variance

$$G(X) \to G(U)$$
$$H(BG(X)) \leftarrow H(BG(U))$$ <u>no good.</u>

~~So~~ maybe we should go back

March 23, 1970: stable homotopy of symmetric groups

Recall the basic structure theorem for the ring

$$R(X) = \bigoplus_{n\geq 0} H_*(E\Sigma_n \times_{\Sigma_n} X^n)$$

which says that it is a free commutative ring with generators $\mathcal{K} \otimes H_*(X)$, $\mathcal{K}$ being the algebra of KADL operations. On the other hand the KADL theorem asserts that

$$H_*(QX) = \varinjlim_n H_*(\Omega^n S^n X)$$

is a free commutative ring with generators $\mathcal{K} \otimes H_*(X)$ localized by inverting the elements of $\pi_0(X)$. Therefore the natural map

$$\coprod_{n\geq 0} E\Sigma_n \times_{\Sigma_n} X^n \longrightarrow QX$$

which should deloop because by a suitable choice of associative Q we can suppose that QX is associative and that this map is a strict H-map. The induced map on homology only inverts polynomial generators, so gives an isom. on E_2 of Eilenberg-Moore. Thus the map

$$B\left\{\coprod_{n\geq 0} E\Sigma_n \times_{\Sigma_n} X^n\right\} \longrightarrow BQX$$

is a homotopy equivalence. Taking X to be a point we deduce

$$\pi_q(B\Sigma) = \pi_q^s(pt).$$

$(j_0, j_1, j_2, \dots, \dots)$ $\quad j_i \in \mathbb{N}$, $j_0 = 0$, $\sum_i j_i = a$ and the set of
$\beta = (\beta_0, \beta_1, \dots \beta_{a-1})$ $\quad \beta_i \in \mathbb{N}$ $\beta_0 > 0$ such that

$$\beta_0 = \text{card}\{i \mid j_0 + \dots + j_i = 0\}$$
$$\beta_1 = \text{card}\{i \mid j_0 + \dots + j_i = 1\}$$
$$\beta_{a-1} = \text{card}\{i \mid j_0 + \dots + j_i = a-1\}.$$

<u>Check for an odd prime p:</u> Here we know the image of

$$P H^*(B\Sigma_{p^a}) \hookrightarrow H^*(B\mathbb{Z}_p^a)$$

is isomorphic to

$$\mathbb{Z}_p[c_{p^a-p^{a-1}}, \dots, c_{p^a-1}, dc_{p^a-p^{a-1}}, \dots, dc_{p^a-1}] \cdot c_{p^a-1}.$$

Therefore $\bigoplus_{n\geq 0} H_*(B\Sigma_n)$ has a minimal system of generators of degree

$$\sum_{i=0}^{a-1} \beta_i (p^a - p^i) \cdot 2 + \sum_{i=0}^{a-1} \gamma_i [2(p^a - p^i) - 1]$$

where $\beta_0, \beta_1, \dots, \beta_{a-1} \in \mathbb{N}$, $\beta_0 > 0$; $\gamma_0, \dots, \gamma_{a-1} = 0, 1$
and this is for each $a \geq 1$ (+ one extra gen. degree 0).

$H_*(\Omega S^{n+1})$ $\quad$ gen. deg. n even
$H_*(\Omega^2 S^{n+1})$ $\quad$ $p^{j_n} n - 1$, $p^{j_n+1} n - 2$
$H_*(\Omega^3 S^{n+1})$ $\quad$ $p^{j_n} n - 2$, $\underset{\text{ext}}{p^{j_{n-1}+j_n+1} n - 2p^{j_{n-1}} - 1}$, $p^{j_{n-1}+j_n+2} n - 2p^{j_{n-1}+1} - 2$

mess?

Check this ~~[illegible]~~ mod 2, and $X = pt$. I computed that ~~[illegible]~~ for each integer $a \geq 0$, I get generators in $R(pt)$ from the dual of

$$PH^*(B\Sigma_{2^a}) \hookrightarrow H^*(B\mathbb{Z}_2^a)$$

$$\cong \mathbb{Z}_2[w_{2^a-2^{a-1}}, \dots, w_{2^a-1}] \cdot w_{2^a-1}$$

and therefore I get generators ~~for each~~ of degree

$$\sum_{i=0}^{a-1} \beta_i (2^a - 2^i)$$

for each sequence $(\beta_0, \beta_1, \dots, \beta_{a-1})$ of natural nos. with $\beta_0 > 0$.
On the other hand by ~~[illegible]~~ K-A. one knows

$H_*^*(\Omega S^{n+1})$	is a ~~[illegible]~~ poly ring with gen. of deg	n	
$H_*^*(\Omega^2 S^{n+1})$	"	$2^{j_n} n - 1$	$j_n \in \mathbb{N}$
$H_*^*(\Omega^3 S^{n+1})$	"	$2^{j_{n-1}+j_n} n - 2^{j_{n-1}} - 1$	$j_{n-1}, j_n \in \mathbb{N}$
$H_*^*(\Omega^{n+1} S^{n+1})$	"	$2^{j_1+\cdots+j_n} n - 2^{j_2+\cdots+j_n} - \cdots - 2^{j_1} - 1$	

But this no. equals ~~[illegible]~~

$$\underbrace{(2^{j_1+\cdots+j_n} - 1) + (2^{j_1+\cdots+j_n} - 2^{j_n})}_{\beta_0 \text{ of these equal to } 2^a - 1} + \underbrace{\cdots}_{\beta_1 \text{ of these equal to } 2^a - 2} \cdots + \underbrace{(2^{j_1+\cdots+j_n} - 2^{j_2+\cdots+j_n})}_{\beta_{a-1} \text{ of these equal to } 2^a - 2^{a-1}},$$

~~where~~ as $n \to \infty$ here $a = j_1 + \cdots + j_n$. Note that $\beta_0 > 0$ and that ~~[illegible]~~ there is a 1-1 correspondence between the set of $\cancel{j_1 + \cdots + j_n = a}$

so you let the number of steps be arbitrary

what is a $\sum_{i \geq 0} j_i$ $\qquad j_0 = 0.$

number of steps is arbitrary but adds up to $\underline{\underline{n}}$.

~~deg~~

$$r \text{ odd} \longmapsto r-1$$

$$r \text{ even} \longmapsto p^{j} r - 1$$

$$p^{j+1} r - 2.$$

and so a typical thing looks like

$$\overset{\text{odd}}{r} \longrightarrow \overset{\text{even}}{r-1}$$

$$\overset{\text{even}}{r} \xrightarrow{\text{odd}} p^{j} r - 1$$

$$\xrightarrow[\text{even?}]{} p^{j+1} r - 2.$$

each transition ~~...~~ contributes ~~an odd~~

in each step there is a jump

$$j_1, j_2, \cdots, j_n$$

$$j_1 + j_2 + \cdots + j_n = a$$

$$p^{j_{n-2}}\left(p^{j_{n-1}}\left\{p^{j_n} a - \left\{\begin{matrix}1\\2\end{matrix}\right\}\right\} - \left\{\begin{matrix}1\\2\end{matrix}\right\}\right) - \left\{\begin{matrix}1\\2\end{matrix}\right\}$$

poly. generator $\qquad n = 2m.$

requirements are that if you put down a 1 the following j is 0 and must put 1 and if you put down a 2 the j must be > 0.

$$a^2 + b^2 = 1$$

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$(a+ib)(a-ib) = 1$$

$m(X) = [X, M]$ $\qquad$ $M = \coprod_n B\Sigma_n$

also Sing C $\quad$ C cat of finite sets.

and you take Sing C

take the category C of finite sets with its sum operation $\amalg$

~~and then form the map which associates to C~~

try to form a category whose classifying space is what you want?

A topus is a pretty icky thing.

$m(X) \cong [X, M]$ $\qquad$ $\cancel{M \times M} \longrightarrow$

interesting λ-operations on homotopy groups of spheres?

Idea is that $\times$ should give rise to ring structure but what does exponentiation do? ~~ad~~

$$\begin{tikzcd} Z \arrow[d] & \\ X & Z' \arrow[l] \end{tikzcd}$$

form $\mathrm{Hom}_{/X}(Z', Z)$

obviously additive in first variable

if Z is fixed ~~should~~ get definite power operations on

$\cancel{\mathrm{Hom}_{/X}(Z', Z)}$!! $\qquad$ $1 + \bar{k}(X)$

Z.

Thus will get $\quad m(X) \times (\pm 1 + \bar{k}(X)) \longrightarrow \pm 1 + \bar{k}(X)$.

generality: G space X.

G acts on R

then want G ~~B~~ R-bundles / X.

and can ask for representability ~~on~~ on G-spaces.

---

problem:

1) inverses

2)

$$\boxed{\coprod_n B\Sigma_n \cong \mathrm{Sing}\, C \qquad C \text{ cat of finite sets.}}$$

know more or less how to handle 2) but not 1).

special cases of 1): a) M monoid $\Rightarrow T_M$.

b) M category $\Longrightarrow T_M$

---

1) amounts to finding an acceptable method of defining $\Omega B(\coprod_{n\geq 0} B\Sigma_n)$.

vertex for the set $\{1,2,\dots,n\}$ and each n

1-simplex for each n and auto

2-simplex for each n and pair of autos.

q-simplex $(n,\theta_1,\dots,\theta_q)$

$d_0(n,\theta_1,\dots$

think of this as object n plus maps

$$\underbrace{n \xrightarrow{\theta_1} n \xrightarrow{\theta_2} n \longrightarrow \cdots}_{q}$$

started March 23, estimate three weeks for writing!!!

Cohomology of finite groups of rational points

general results.

Thm 1: G connected alg. gp. over k alg. closed field $F: G \longrightarrow G$ radicial surjective (whatever Steinberg says), G_f finite group of fixed points. Assume $H^*(G)$ (etale coh. mod l) admits a simple system of (universally) transgressive generators. Then spectral sequence

$$E_2 = H(BG) \otimes H(G) \Longrightarrow H(BG_f)$$

shows that

$$gr\, H^*(BG_f) \cong H^*(BG)_F \otimes H^*(G)^F$$
$$S[R_F] \otimes \Lambda(P^F)$$

Cor: l odd $\Longrightarrow$ above is an algebra isomorphism. (uses that exterior algebra on odd degree generators is free anti-comm. alg.)

~~Thm 2: Hypotheses as above, let~~

Cor: ~~All maximal elementary abelian l-subgroups of G_f are of same rank (~~ l odd $\Longrightarrow \exists!$ maximal ~~element~~ $[l]$-subgroup of G_f. $l=2 \Longrightarrow$ all max. $[l]$-subgroups of the same rank.

$A \subset G_f$ maximal $[l]$-subgp., Z centralizer of A in G.

$$\Longrightarrow H^*_G(G^t) \hookrightarrow H^*_Z(Z^t)$$

Recall the situation required for your paper:

Let k be an algebraically closed field, let G be an algebraic group over k, let T be the etale topos of schemes of finite type over k, and T_G the classifying topos of G viewed as ~~object~~ a group in T. Then $H^*(BG)$ denotes the cohomology of the final object of T_G with codefficients in Z_ℓ where ℓ is a prime nember different from the characteristic of k. More generally if X is an object of T_G, that is an object of T endowed with an action of G, e.g. a scheme on which G acts, then $H^*_G(X)$ is the ~~equivar~~ cohomology of X with coefficients in Z_ℓ.

<u>Problem</u>; do there exist the standard spectral sequences

$$E_2 = H^*(BG, H^*(X)) \Longrightarrow H^*_G(X)$$

$$E_2 = H^*(X/G, \; Gx \mapsto H^*_G(Gx)) \Longrightarrow H^*_G(X)$$

If so then the first one should result from

[redacted]

1. The spectrum of the cohomology ring of the classifying space of a compact Lie group. Let ℓ be a prime number and let $H^*(X)$ be the cohomology ring of a space X with coefficients $\mathbb{Z}_\ell$. Consider the elementary abelian ℓ-subgroups of G and make them the objects of a category $\underline{C}$ where a morphism from A to A' is by definition a component of $\{x \in G \mid xAx^{-1} \subset A'\}$. Then the restriction homomorphism defines a map

$$H^*(BG) \longrightarrow \operatorname*{invlim}_{\underline{C}} H^*(BA)$$

The main theorem asserts that this map is an F-isomorphism, i.e. any element of the kernel and cokernel is killed by a sufficiently high power of Frobenius: $z \mapsto z^\ell$. When ℓ is odd this theorem says something non-trivial only about the subrings of even degree elements. Some corollaries of the theorem are that the dimension of $H^*(BG)$, i.e. the order of the pole of the Poincare series at $t=1$, is equal to the maximal rank of an elementary abelian ℓ-subgroup and that the minimal primes of $H^{ev}(BG)$ are in natural 1-1 corespondence with the conjugacy classes of maximal elementary abelian ℓ-subgroups of G.

2. Cohomology of groups of rational points. Let G be a connected algebraic group defined over k, a finite field with q elements. Assume that the cohomology of G with coefficients in $\mathbb{Z}_\ell$ for the etale topology, denoted $H^*(G)$, has a simple system of transgressive generators for the spectral sequence

$$E_2^{pq} = H^p(BG) \otimes H^q(G) \Longrightarrow H^{p+q}(pt)),$$

where BG denotes the classifying topos of G in the sense of Grothendieck.

The structure theorem of Borel for this spectral sequence shows that $H^*(BG)$ is isomorphic to the symmetric algebra of the generating subspace P of $H^*(G)$. Since $G/G(k)$ is isomorphic to G acting on itself by $x,y \quad xy(Fx)^{-1}$ there is a spectral sequence

$$E_2^{pq} = H^p(BG) \otimes H^q(G) \Longrightarrow H^{p+q}(BG(k))$$

Theorem: Under the above hypotheses the subspace P of $H^*(G)$ is transgressive and an element x transgresses to $x - F^* x$, where F^* denotes the action of the Frobenius on $H^*(BG)$. Consequently $H^*(BG(k))$ is additively isomorphic to a symmetric algebra on P_F (coinvariants for the action of Frobenius) tensore with the exterior algebra on P^F (invariants under Frobenius). If l is odd, then this is a multiplicative isomorphism.

When G is reductive, the hypothesis on $H^*(G)$ may be verified by checking it for the compact form of G. Examples;

3. Algebraic K-theory: Let R be a ring, which need not be commutative. By an R-vector bundle over a topological space X, I mean a sheaf of R-modules locally isomorphic to $X \times P$ where P is a projective R-module of finite type. When X is connected and locally-connected and endowed with a basepoint such a thing is the same as a representation of the fundamental group of X as automorphisms of P.. Define the crude K-theory associated to R, denoted $k(X,R)$ to be the Grothendieck group of R-vector bundles over X. Problem: Prove the existence of a universal map $k(X,R) \longrightarrow K(X,R)$, where $K(X,R)$ is a representable functor on the homotppy category. If such a thing exists, then we may define $K_i(R) = K(S^i,R)$.

If H is an algebraic subgroup of G, then I need a Serre spectral sequence

$$E_2^{rs} = H^r(BG, H^s(G/H)) \Longrightarrow H^{r+s}(BH)$$

together with the fact that if G is connected, then

$$H^*(BG, H^*(G/H)) = H^*(BG) \otimes H^s(G/H) .$$

(the two situations where I wish to apply this are $H = G(k)$ ~~xxdx~~ in G and G as a diagonal subgroup in $G \times G$.) I need all of the convenience of the Leray spectral sequence used by Borel, e.g. multiplicative structures, Kunneth theorem, etc. Will return to this.

By the Kunneth formula, $H^*(G)$ is a finite dimensional connected Hopf ~~algebra~~ commutative algebra. We shall suppose that it admits a simple system of transgressive generators in the spectral sequence

$$E_2 = H^*(BG) \otimes H^*(G) \Longrightarrow H^*(pt)$$

which implies by Borel [ref] that $H^*(BG)$ is a polynomial ring with indecomposable space isomorphis to the subspace of generators for $H^*(G)$. If ℓ is odd, then what we have supposed is ~~that~~ equivalent to $H^*(G)$ being an exterior algebra with generators of even degree, or that $H^*(BG)$ is a polynomial ring with even dimensional generators, or that G has no ℓ-torsion.

<u>Theorem 1</u>: Suppose that $H^*(G)$ ADmits a simple system of transgressive generators, ~~so that $Q(H^*(BG)) = P(H^*(G))$ by the transgression. Then~~ so that $H^*(BG) = S(M)$ where M is the subspace of transgressive elements ~~by~~ of $H^*(G)$. Let M^F and M_F be the sub- and quotient space of invariant and ~~xxx~~ coinvariant elements of M under F. Then there is an isomorphism

$$H(BG(k)) = M \otimes S[M]$$

Cohomology of finite groups of rational points.

List of theorems

1. (announced at Nice). G connected alg. group / alg. cl. field k, l prime no. $\neq$ char. k, σ endo. of $G \ni$ G^σ finite ($\Longleftrightarrow$ $(1-\sigma)G = G$ by Steinberg). Assume $H^*(BG) \cong SV$ and V stable under σ. Then ($\Rightarrow H^*(G) = \Lambda V[-1]$

$$H^*(G^\sigma) \simeq S(V_\sigma) \otimes \Lambda(V^\sigma[-1])$$

You should be able to prove a complete topos version of this theorem with mentioning alg. geometry.

$\boxed{2.}$ What happens for $GL_n(\mathbb{F}_q)$, $O_n(\mathbb{F}_q)$, $Sp_n(\mathbb{F}_q)$, and unitary groups. Explicit computations + possibly a result computing the restriction of the basic classes.

Most of this is window-dressing and designed to keep others from computing $O_n(\mathbb{F}_q)$ by your Adams conj. method. You might want to know explicitly about ~~the restriction of~~ the class $c_i'' \in H^{2i-1}(GL_n \mathbb{F}_q, \mu_{q^i-1}^{\otimes i})$.

G group in a topos $\mathcal{T}$

$\sigma : G \longrightarrow G$ endomorphism.

G^σ fixed subgroup.

assume (1) $G/G^\sigma \xrightarrow{\sim} G$

$$xG^\sigma \longmapsto x(\sigma x^{-1})$$

(2) $\begin{cases} H^*(G \times X) \xleftarrow{\sim} H^*(G) \otimes_\Lambda H^*(X) \\ H^*(G) \text{ finite-dimensional} \end{cases}$ mod ℓ cohomology $A = \mathbb{Z}/\ell\mathbb{Z}$

---

(2) $\Longrightarrow$ $H^*(G)$ <u>Hopf algebra finite-dimensional</u>

$X = G/G^\sigma$

$G \times X \longrightarrow X$

$H^*(G \times X) \longleftarrow H^*(X)$

$G \times G \longrightarrow G$

$g \quad g' \qquad gg'(\sigma g)^{-1}$

$$\begin{array}{ccccccc} G\times G & \longrightarrow & G \times G \times G & \longrightarrow & G \times G \times G & \longrightarrow & G \\ g, g' & & g\ \sigma g\ g' & \longrightarrow & g\ \sigma g^{-1}\ g' & \longmapsto & g g' \sigma g^{-1} \end{array}$$

$H^*(G \times G)$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $H^*(G)$

$p\otimes 1\otimes 1 - 1\otimes p\otimes 1 + 1\otimes 1\otimes p \qquad p\otimes 1\otimes 1 + 1\otimes p\otimes 1 + 1\otimes 1\otimes p \qquad p$

$p\otimes 1 - \sigma p\otimes 1 + 1\otimes p.$

~~[illegible]~~ Choose $W \subset H_*(G)$ so that W ~~[illegible]~~ is ~~an orthogonal~~ complement for ~~f~~ the orth. to $PH^*(G)$. One knows

$$(+) \qquad \Lambda W \xrightarrow{\sim} H_*(G)$$

by Hopf alg. theory.

<u>Claim:</u> ~~[illegible]~~ The $H_*^*(G)$-module structure on $H^*(G)$ defined by $\nu : G \times G \longrightarrow G$ is given by

$$\omega \otimes p \longmapsto \omega \otimes ((p - \sigma^* p) \otimes 1 + 1 \otimes p)$$

$$\Big\downarrow \omega \otimes \mathrm{id}$$

$$\langle \omega - \sigma\omega, p \rangle$$

and ω acts as a derivation.

difference between p odd and $p = 2$: There is a squaring operation on $V = PH^*(G)$ and $H^*(G)$ is the universal ~~envelop~~ Inveloping algebra of V (restricted abel. Lie alg.)

$$\Lambda W \xrightarrow{\sim} H_*(G) \quad \text{still holds.}$$

$$\begin{array}{ccc} H^*(X) & \longrightarrow & H^*(G\times X) = H^*(G)\otimes H^*(X) \\ p & & (p-\sigma^*p)\otimes 1 + 1\otimes p \end{array}$$

Claim: ~~Let G^t denote the object G with G action $g(g') = gg'(\sigma g)^{-1}$. Then~~ Let

$$\begin{array}{rccc} \nu : & G\times G & \longrightarrow & G \\ & g \quad g' & \longmapsto & gg'(\sigma g)^{-1} \end{array}$$

then

$$\nu^* : H^*(G) \longrightarrow H^*(G\times G)$$

given by

$$\nu^*(p) = pr_1^*(p-\sigma^* p) + pr_2^*(p).$$

for $p \in PH^*(G)$.

Let G^t denote G considered as a G-object via ν. Then $H^*(G^t)$ is an $H_*(G)$-module. (Recall in general given $G\times X \longrightarrow X$ one makes $H^*(X)$ into an $H_*(G)$-module as follows:

$$\begin{tikzcd} H^*(X) \arrow[r] \arrow[rr, bend right] & H^*(G\times X) & H^*(G)\otimes H^*(X) \arrow[l, "\sim"'] \end{tikzcd}.$$

$$\begin{tikzcd} H_*^q(G)\otimes H^*(X) \arrow[r] \arrow[dr, bend right] & H_*(G)\otimes H^*(G)\otimes H^*(X) \arrow[d, "ev\cdot 1"] \\ & H^*(X) \end{tikzcd}$$

)

Computation of $\mathrm{Ext}^{p}_{H_*(G)}(k, H^*(G))^q = E_2^{pq}$

Can use Kunneth formula maybe! Maybe better to establish homotopy equivalence

$$\mathrm{Ext}^{p}_{H_*(G)}(k, M) = \text{homology of } S(V) \otimes M$$

with ~~differential~~ differential determined by fact should be an $S(V)$- ~~dif~~ module and

$$M \longrightarrow V \otimes M \longrightarrow S^2 V \otimes M \longrightarrow \cdots$$

~~let~~ let $V = P H^*(G)$.

$$\begin{array}{ccc} H^*(X) & \longrightarrow & H^*(G) \otimes H^*(X) \\ & & \downarrow \\ & & V \otimes H^*(X) \end{array}$$

choose this

logical steps in the above argument

T topos, G group in T, X in T_G, $F \in (T_G)_{ab}$

Then cohomology is $\boxed{\text{defined}}$ $H^*_G(X, F)$

$G \times X \longrightarrow X$ covering

Any covering has a Čech spectral sequence

Prop 1: The spectral sequence in this case takes the form

$$E_2 = \check{H}^p(\nu \longmapsto H^q(G^\nu \times X, F))$$

$\boxed{\text{mess}}$

into this proposition must go

- ~~[illegible]~~ definition of $H^*(G^\nu \times X, F)$ when $F \in (T_G)_{ab}$
- formulas for the simplicial operations.
- induction formula: $H^*_G(G \times^K Y, F) = H^*_K(Y, F)$

It will be necessary ~~to~~ ~~[illegible]~~ work with the definition of cohomology using injective resolutions.

$G, X \in T_G$

nerve

$G \times G \times X \rightrightarrows G \times X \rightrightarrows X$

this is a simp. obj. in T_G

Cech thing of covering $G \times X \longrightarrow X$

Cech spectral sequence:

$$E_2^{pq} = \check{H}^p(\nu \mapsto H^q_G(G^{\nu+1} \times X, F)) \Rightarrow H^{p+q}_G(X, F)$$

$$F \in (T_G)_{ab}$$

$$H^*_G(G^{\nu+1} \times X, F) = H^*(G^\nu \times X, F)$$

assume F a ring A ~~and~~ constant that Keunneth holds

$$H^*(G \times Y, A) = H^*(G, A) \otimes_A H^*(Y, A)$$

as well as $H^*(G)$ fin. type proj. each dim.

$$\Rightarrow \quad H^*(G^\nu \times X) = H^*(G)^{\otimes \nu} \otimes H^*(X)$$

$$= \operatorname{Hom}(H_*(G)^{\otimes \nu}, H^*(X))$$

where $H_*(G) \overset{\text{defn}}{=}$ dual of $H^*(G)$.

so

$$E_2^{pq} = \operatorname{Ext}^{p}_{H_*(G)}(A, H^*(X))^q \Rightarrow H^{p+q}_G(X).$$

that they notified me where the W2 forms were going to be sent in a monthly computer-printed statement which I seldom open until the end of the year. Anyhow it should be straight now.

I received notification that I will be promoted to a tenure position at M.I.T. in July, although the rank won't be specified until after the budget is prepared.

Thanks again

Dan

~~my arguments prove the degeneracy of the spec. seq~~

$$
\begin{tikzcd}
G^t \arrow[d] & & G^s \arrow[d] & \\
PG \times^G G^t \arrow[r] \arrow[d] & & P(G\times G) \times^{G\times G} G^s & \simeq BG \\
BG \arrow[r] & & B(G\times G) \arrow[from=u]
\end{tikzcd}
$$

In the Serre spectral sequence coh. of fibre transgresses in known fashion. ditto for rest.

~~3rd section:~~

~~(1.) definition of $K_i A$~~

~~(2.) Computation of $K_i \mathbb{F}_q$.~~

~~(3.) symmetric groups~~

$$H^i X \longrightarrow \bigoplus_j H^{i-j} X \otimes \underbrace{\wp H^j G}_{V^{j+1}}$$

G^t $\qquad\qquad$ G^s

$$G \xrightarrow{\ \Gamma\ } G \times G$$

induced G spaces.

gives a map of spectral sequences.

no good: I want to generate $\underline{H^*(G)^F}$

$$E_2^{pq} = \mathrm{Ext}^p_{H.G}(k, H^{\cdot}X)^q \Longrightarrow H^{\cdot}_G(X)$$

H. G algèbra de Hopf

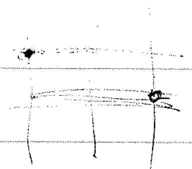
~~assume $H.G = \Lambda W$ W dual to $PH^{\cdot}(G)$.~~

~~Compute $E_2 =$~~

E_1 $H^{\cdot}(X) \otimes SV$ H^iX degree $0, i$

V^j degree $1, j-1$

and $V^j \simeq PH^{j-1}(G)$.



and differential is unique derivation $\ni$

$$H^iX \longrightarrow H^{i-j}X \otimes V^{j+1}$$

$$\begin{array}{ccccc} H^*X & \longrightarrow & H^*(X\times G) & = & H^*(X)\otimes H^*(G) \\ & & & & \downarrow \\ & & & & H^*(X)\otimes PH^*(G) \end{array}$$

$$H^iX \longrightarrow \bigoplus_j H^{i-j}X \otimes \underbrace{PH^jG}_{V^{j+1}}$$

$$
\begin{tikzcd}
\left(G^{t}\right)_{G} \arrow[r] \arrow[d] & \left(G^{s}\right)_{G\times G} \arrow[d] \\
BG \arrow[r] & B(G\times G)
\end{tikzcd}
$$

to such a square is associated <u>the operation Φ</u>!!!

$$
\begin{tikzcd}
X' \arrow[r, no head] \arrow[d, "f'"'] & X \arrow[d] \\
Y' \arrow[r] & Y
\end{tikzcd}
$$

$$
\begin{tikzcd}
H^{i}(X') & H^{i}(X) \arrow[l] \\
H^{i}(Y') \arrow[u] & H^{i}(Y) \arrow[l] \arrow[u] \\
\tilde{H}^{i}(\mathrm{Conef}') \arrow[u, "\delta"'] & \tilde{H}^{i}(\mathrm{Conef}) \arrow[l] \arrow[u, "\delta"'] \\
H^{i-1}(X') \arrow[u] & H^{i-1}(X) \arrow[l] \arrow[u] \\
H^{i-1}(Y') \arrow[u] & H^{i-1}(Y) \arrow[l] \arrow[u]
\end{tikzcd}
$$

$$
\begin{array}{ll}
\mathrm{Ker} & H^{i}(Y) \longrightarrow H^{i}(Y') \oplus H^{i}(X) \\
 & \quad \downarrow \Phi \\
\mathrm{Cokernel} & H^{i-1}(Y') \oplus H^{i-1}(X) \longrightarrow H^{i-1}(X')
\end{array}
$$

On the cohomology of groups of rational points: An outline

1) Abstract topos theorem about the cohomology of the group G^F where F is an endomorphism of G.

2) Then this must be applied in the special case ~~when~~ of an algebraic group with F satisfying the conditions of Steinberg.

3) Explicit examples. This will involve you in computing the cohomology for the classical groups. Conditions are satisfied when the flag manifold is generated by the cohomology in dimension 1.

I want to review all of the proofs, decide where the difficulties are, and break these down into small parts. ~~Summary~~

Typical problem: Let G be an algebraic group over an algebraically closed field k. ~~XXXXXXXXXXXXXXXXXXXXXXXXX~~ Then i) $H^*(G)$ cohomology mod ℓ is finite dimensional over $Z/\ell Z$, ii) $H^*(G\times X) = H^*(G)\otimes H^*(X)$ for any variety X. (map $G \to pt$ is cohomologically proper)

Corollary of this is that $H^*(G)$ is a Hopf algebra and hence satisfies the ~~conditions~~ Borel-Hopf theorem. Maybe the way to do things is to use the comp comparison theorems, using the Chevalley theorem that there is a nice family joining with characteristic zero and then the Artin comparison theorem.

In order to ~~dimm~~ decide what you need, you should decide in advance exactly what examples to be treated. classical groups. GL_n and unitary groups; here you start with the general linear group and~~xxxxxxx~~ an outer automorphism and then twist with Frobenius. Orthogonalgroups, and symplectic groups. In these examples the problem is with ~~understanding the~~ finding the cohomology of BG and deciding when it is a polynomial ring with generateos stable under F.

The Bockstein spectral sequence? Is this worth including?

Program

Paper breaks up into two parts: First an abstract topos-type theorem. Secondly the application of this abstract result in the specific case of an algebraic group over and algebraically closed field. The second part should not be written until the examples to be presented are decided upon. Examples should be the orthogonal and symplectic groups, the unitary groups, and possibly one of the exceptional forms-the Suzuki of Rhee groups. Perhaps an abelina variety form fun, and in any case the fixpoints of an enodo of th torus should be computed.

The hypothesis that V should be stable under F- it is really only necessary I think to know that there are enough invariants in SV.

(3) the spectral sequence degenerates if V can be chosen stable under F.

to show

we have $H^*(G)^F = E_2^{0*}$ is in E_∞

~~and we~~

we also have V_F in E_2^{1*}

this goes in (2)

the point is that these are infinite cycles because of the map

$$(G, X) \longrightarrow (G, pt)$$

$$\mathrm{Ext} \longleftarrow \mathrm{Ext}$$

land in the image. Thus $\exists$ canonical isom.

$$E_2 = E_2^{0*} \otimes S(V_F)$$

or there is a definite map $V_F^* \longrightarrow E_2^{1,*-1}$

So then the only thing to be proved is that

$$E_2^{0*} \cong H^*(G)^F$$

is in E_∞. The method is to use the operation Φ.

$$\Phi : S(V)^F \longrightarrow S(V_F) \otimes V^F$$

(2) Compute E_2 when

$X = G^t$, assume H_*G exterior algebra.

$$\cancel{H^*(G^p \times X) = H^*(G)^{\otimes p} \otimes H^*(X)}$$

$$H_*G = \Lambda W$$

$$H^*X = \Lambda W^*$$

and the action map

$$\alpha : H_*(G) \otimes H^*(X) \longrightarrow H^*(X)$$

$$\Lambda W \otimes \Lambda W^* \longrightarrow \Lambda W^*$$

is given by interior product

$$w \otimes z \longmapsto i(w - Fw)\, z$$

Answer

$$\mathrm{Ext}_{H_*G}(k, H^*X) = \underbrace{\Lambda(V^F[-1])}_{H^*(G)^F} \otimes S(V_F)$$

(1) spectral sequence

$$E_2 = \mathrm{Ext}_{H.G}(k, H^*X) \Rightarrow H_G^*(X).$$

Proof: Cech spectral sequence of covering $G\times X \to X$ is

$$E_2 = H^p(\nu \mapsto H_G^q(G^{\nu+1}\times X)) \Longrightarrow H_G^{p+q}(X)$$

$$H_G^q(G^{\nu+1}\times X) = H^q(G^\nu \times X)$$

$$E_1^{pq} = H_G^q(G^{p+1}\times X) = H^q(G^p \times X)$$

$$d_1 = \sum \delta_i$$

induction formula

Künneth formula $\Longrightarrow$

$$H^*(G^p\times X) \simeq \mathrm{Hom}((H_*G)^{\otimes p}, H^*X)$$

$$= \mathrm{Hom}_{H_*G}(H_*G^{\otimes p+1}, H^*X)$$

so $E_2 = \mathrm{Ext}_{H_*G}(k, H^*X)$

(2) properties of the spectral sequence

edge homo $H_G^*(X) \longrightarrow E_1^{0*} = H^*(X)$.

mult. structure

(seems better to work with E_1 term) except one needs the Ext to do the computation.

Income 1970
source:
Institute for Advanced Study
receipts Jan. 1 – April 30 $ 5250.

Now I recall from my earlier work that this operation is needed to get $H^*(G)^F = E_2^{0*}$ in E_∞!

$\boxed{\text{edge}}$ homom: $H^*_G(X) \longrightarrow H^*(X)^{\pi_0 G}$.

$\operatorname{Ker}\left\{ H^*(X) \rightrightarrows H^*(G \times X) \right\}$

$$\begin{tikzcd}
G \arrow[r] \arrow[d, no head] & (G^t)_G \arrow[d, no head] \\
pt \arrow[r, no head] & BG
\end{tikzcd}$$

March 26, 1970:

Problem: I have proposed to define algebraic K-theory by proving the existence of a universal map $k(X,R) \longrightarrow [X,B]$. Does such a B exist?

necessary conditions? Suppose given a fibration $Z' \to Z \to Z''$ of pointed spaces and a map $k \longrightarrow [\;,Z]$ which becomes zero in Z''. Then does it lift into Z'. So consider first the case of the symmetric group. Then I have that $m(X)$ acts freely on $m(X)\times m(X)$ with quotient $k(X)$, and I have a map

$$\begin{tikzcd}
m(X)\times m(X) \arrow[r] \arrow[d, equal] & {[X,Z]} \\
{[X, M\times M]} \arrow[ur] &
\end{tikzcd}$$

Thus I can lift: $M\times M \longrightarrow Z'$ i.e.

$$\begin{tikzcd}
m \arrow[d] & & {[\;,Z']} \arrow[d] \\
m\times m \arrow[d] \arrow[urr] \arrow[rr] & & {[\;,Z]} \arrow[d] \\
k \arrow[urr] & & {[\;,Z'']}
\end{tikzcd}$$

but there's no reason why this lift should be compatible with the m-action!!

On the other hand M is an associative monoid and we wish to form ΩBM. From earlier work it appears reasonable to ~~[illegible]~~ consider

$$EM \times_M (M \times M).$$

Question: Under what conditions might this be ~~[illegible]~~ homotopic to ΩBM?

The idea is that

$$M \xrightarrow{\Delta} M \times M \longrightarrow EM \times_M (M \times M) \longrightarrow BM \longrightarrow B(M \times M)$$

should be a fibration. NO Check this as follows: Let $N \subset M$ be an inclusion of monoids, then do we have a fibration in the homotopy category

(*) $$EN \times_N M \longrightarrow BN \longrightarrow BM$$

The answer is probably NO because the above rectangle gives a long exact sequence of homotopy groups NO

$$\pi_{g+1}(BM) \xrightarrow{\partial} \pi_g(M) \times \pi_g(M) \longrightarrow \pi_g(EM \times_M (M \times M)) \longrightarrow \pi_g(BM)$$

which suggests problems. Actually the ∂ map ought to be interesting since already for M discrete is it a map ~~[illegible]~~ from the group G generated by M to $M \times M$?

This is wrong because $EM \longrightarrow BM$ is not locally trivial since the transition functions lie in M and hence are not isomorphisms. Thus it might be true that (*)

is a fibration sequence. Consider the situation homologically

$$H_*(EN \times_N M) = H_*(\mathbb{Z}[EN] \otimes_{\mathbb{Z}[N]} \mathbb{Z}[M])$$

But

$$H_*(BN) = H_*(\mathbb{Z}[EN] \otimes_{\mathbb{Z}[N]} \mathbb{Z})$$

$$H_*(BM) = H_*(\mathbb{Z}[EM] \otimes_{\mathbb{Z}[M]} \mathbb{Z})$$

?

<u>Problem</u>: Let M be a monoid (or more generally a category). We have some idea how to handle BM as the topos of functors. How does one think of ΩBM?

Start with an M-object having on automorphism? BM is the classifying topos of M, ie. the category of left M sets. Given as map $f: T \longrightarrow BM$ where T is a topos (i.e. an f^* functor) we get $f^*(M)$ which is a right principal bundle for M. Thus one finds an object P of T with a right M action which is free?

There are **two** problems at the moment which prevent me from proving the existence of B.

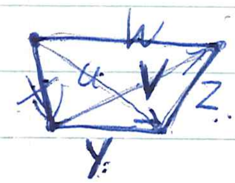
(i) how to put the inverses in.

(ii) how to split the exact sequences.

Start with the category of vector spaces and isomorphisms?

0-simplices	~~0 vector space~~ a fixed 0 vector space
1-simplices	V
2-simplices	$V' \to V \to V''$ exact

3-simplices = family of 4 2-simplices.



$$X \to U \to Y \quad ✓$$
$$Y \to V \to Z \quad ✓$$
$$U \to W \to Z$$
$$X \to W \to V$$

$\ni$
```latex
\begin{tikzcd}
X \arrow[r] \arrow[d, "id"] & U \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & V \arrow[d] \\
0 \arrow[r] & Z \arrow[r, "id"] & Z
\end{tikzcd}
```
commutes.

This defines a simplicial set which represents the classifying space of BGl_R.

So the conjecture is that this explicit simplicial set represents the functor you want it to. Perhaps you can settle the existence of a generalized coh. theory.

So a q-simplex is a family of vector spaces V_{ij} $0 \leq i \leq j \leq q$ together with maps

$$\varphi_{ij}^{i'j'} : V_{ij} \longrightarrow V_{i'j'}$$

if $i \leq i'$ and $j \leq j'$ such that

(i) categorical

(ii) $\forall$ $i < j < k$ want

$$0 \longrightarrow V_{ij} \longrightarrow V_{ik} \longrightarrow V_{jk} \longrightarrow 0$$

to be exact.

(Think of $V_{ij} = F_j / F_i$. This is very reminiscent of the immeuble.)

<u>Conjecture</u>: ① By a suitable use of Bass's graph of groups you ~~ought~~ might to be able to make this appear as a quotient in a suitable equivariant fashion of an immeuble.

② Can you get the higher classifying spaces ?

March 28, 1970

Derivation of the spectral sequence

$$E_2^{pq} = H^p(BG, H^q(G/H)) \Longrightarrow H^{p+q}(BH)$$

needed for your stuff on finite groups. Let G be group in a topos T and let H be a subgroup of G. Then there is an equivalence of categories

$$T_G/(G/H) \simeq T_H$$

$$G\times_H S \longleftarrow\!\!\!\mapsto S$$

~~Let $f: G/H \to pt$ be the canonical map in T_G~~ of the sort that

$$H_G^*(G/H) \simeq H_H^*(pt).$$

Let $f: G/H \longrightarrow pt$ be the unique such map in T_G and consider the Leray spectral sequence

$$E_2^{pq} = H_G^p(pt, R^qf_*(\mathbb{Z}_\ell)) \Longrightarrow H_G^{p+q}(G/H).$$

So the problem is to compute $R^qf_*(\mathbb{Z}_\ell)$. One knows (Ref.) that it is the object of T_G which represents the sheaf on T_G represented by the functor

$$S \longmapsto H_G^q(S\times(G/H), \mathbb{Z}_\ell).$$

The above should be generalized by replacing G/H by any object X of T_G.

Consider the sheaf on T associated to the presheaf $S \longmapsto H^q(S \times X)$ (coeffs. mod ℓ). and denote the object of T representing this sheaf by $\mathcal{H}^q(X)$. I claim that it is crucial to know that the base change theorem holds for

(*)
$$\begin{tikzcd} S \times X \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] & pt \end{tikzcd}$$

ie that $\mathcal{H}^q(X)$ is the constant sheaf $H^q(X)$. If so, then we can show that $R^q f_*(\mathbb{Z}_\ell)$ is the constant sheaf $H^q(X)$ on which G acts through $\pi_0 G$ in the following way.

~~Starting with the 2 maps~~

First suppose $g \in \Gamma(S, G)$; it induces an auto of $S' \times X \longrightarrow S'$ for any S' over S, hence gives an auto. of $\mathcal{H}^q(X)$ over S. Thus we get a G-action on $\mathcal{H}^q(X)$, ie a map $G \times \mathcal{H}^q(X) \longrightarrow \mathcal{H}^q(X)$ which if $\mathcal{H}^q(X)$ is a constant object of T factors through $\pi_0 G$.

Now define a map in T

$$\mathcal{H}^q(X) \longrightarrow R^q f_*(\mathbb{Z}_\ell)$$

to be map on sheaves given by the isomorphism

$$H^\delta(S \times X) \longrightarrow H^\delta_G(G \times S \times X).$$

Everythings clear.

It remains to check that the base change theorem (*) holds for $X = G/H$, in fact for G itself is all that we need. But this is immediate from the structure theorems for algebraic groups. Thus assume G connected so there is an extension

$$G_\ell \longrightarrow G \longrightarrow A$$

where G_ℓ is ~~maximal connected~~ the maximum connected linear subgroup of G. Let B be ~~the~~ a Borel subgroup. Of course you use <u>cohomologically proper</u> for maps. Thus enough to worry ~~that~~ about the map

$$\underbrace{G \longrightarrow G/N}_{\substack{\text{successive}\\ \text{extension of}\\ \mathbb{G}_a\text{-bundles}\\ \text{which you}\\ \text{can complete at } \infty}} \underbrace{\longrightarrow G/B}_{\substack{\text{successive}\\ \text{extension of}\\ \mathbb{G}_m\text{-bundles}\\ \checkmark}} \xrightarrow[\text{proper}\,\checkmark]{} pt$$

Proposition (Borel): $G \ni H^*(G)$ ~~[illegible]~~ primitively generated, K subgroup (conn) of $G \ni H^*(G) \to H^*(K)$ surjective. Then it should be true that $H^*(G)$ as an $H^*(G/K)$-module is isomorphic to

$$H^*(G/K) \otimes H^*(K)$$

and this will follow from the Leray spectral sequence

$$E_2^{pq} = H^p(G/K) \otimes H^q(K) \Longrightarrow H^{p+q}(G)$$

which should be valid as K has the Künneth property.

So let e_i $i \in I'$ be a basis for $\operatorname{Ker}\{PH^*(G) \longrightarrow PH^*(K)\}$ extend it to a basis e_i $i \in I$ of $PH^*(G)$ and choose generators c_i $i \in I - I'$ for $H^*(BK)$, lift them to $H(BG)$ and extend this system of generators to $H^*(BG)$ so that $c_i \,|\, BK = 0$ $i \in I'$.

Then e_i $i \in I'$ form a simple system of generators for $H^*(G/K)$ and in the spectral sequence

$$H^*(BG) \otimes H^*(G/K) \Longrightarrow H^*(BK)$$

they are transgressive with c_i representing τe_i $i \in I'$

Proof. If $c \in H^q(BG)$ lifts to zero in $H^q(BK)$, hence the image of c in E_q^{q0}, denoted $\bar{c}$ is of the form $d_q e$ with $e \in E_q^{0q-1} \subset E_2^{0q-1} = H^*(G/K)$. Now apply the map φ of spectral sequences and you find $\varphi(e) \in PH^*(G)$, in fact we have $\varphi(e) \in \operatorname{Ker} PH^*(G) \longrightarrow PH^*(K)$ because a non-zero element of $PH^*(K)$ transgresses to a non-zero indecomposable element of $H^*(BK)$. Thus we are constructing a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & PH^*(G/K) \arrow[r] \arrow[d] & PH^*(G) \arrow[r] \arrow[d, "\tau"', "\cong"] & PH^*(K) \arrow[r] \arrow[d, "\tau"', "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Ker} \arrow[r] & QH^*(BG) \arrow[r] & QH^*(BK) &
\end{tikzcd}$$

where $PH^*(G/K)$ is the image of

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\{H^*(G) \to PH^*(K)\} \arrow[r] \arrow[d] & P(H^*(G)) \arrow[d] & \\
0 \arrow[r] & H^*(G/K) \arrow[r] & H^*(G) \arrow[r] & H^*(G) \otimes H^*(K)
\end{tikzcd}$$

Thus we find that as c_i' runs over $\mathrm{Ker}\{QH^*(BG) \to QH^*(BK)\}$ we get a basis for $PH^*(G/K)$ showing all these elements transgress. qed.

§3. Theorem I

Let α be an endomorphism of a connected algebraic group G such that $(1-\alpha)\,G = G$, where $1-\alpha$ is the map $x \mapsto x(\alpha x)^{-1}$. Then the subgroup G^α of fixpoints under α is finite and the map

$$(3.1) \qquad \begin{array}{ccc} G/G^\alpha & \longrightarrow & G \\ xG^\alpha & \longmapsto & x.\alpha x^{-1} \end{array}$$

is bijective on k-valued points, hence it is a radidial surjective morphism (ref?), and consequently it induces an isomorphism

$$(3.2) \qquad H^*(G) \simeq H^*(G/G^\alpha).$$

Therefore the spectral sequence of the"fibration" $(G/G^\alpha, BG^\alpha, BG)$ takes the form

$$E_2^{pq} = H_G^p \otimes H^q(G) \implies H_G^{p+q}(G/G^\alpha) = H_{G^\alpha}^{p+q}.$$

We propose now to compute this spectral sequence when $H^*(G)$ satisfies the condition of Borel (2.).

Let X denote the homogeneous space for $G\times G$ given by the variety G with action $(g',g'').x = g'x(g'')^{-1}$. Consider X as a G-variety by means of the homomorphism $(id,\alpha): G \to G\times G$, we get G acting on itself via $g.x = gx(\alpha x)^{-1}$, which is the left side of 3.1. The restriction homomorphism gives rise to a map of spectral sequences

$$(3.3) \qquad \begin{array}{ccccc} E_2 & = & H^*_{G\times G} \otimes H^*(G) & \implies & H^*_G \\ 'E_2 & = & H^*_G \otimes H^*(G) & \implies & H^*_{G^\alpha} \end{array}$$

in which the map on the fibres is the identity, and on the base is the map $(id,\alpha)^*$, and which on the total space is the restriction homomorphism from G to G^α. By 2. If e is an element of the subspace P of $H^*(G)$ and if the transgression of e is represented by the element c of H^*_G, then in the upper spectral sequence of 3.3 one knows that e is transgressive and its ~~transgression~~ image under the transgression is represented by $c\otimes 1 - 1\otimes c$. Consequently by the morphism 3.3 we deduce

Proposition 3.4: Assume that $H^*(G)$ satisfies the condition of

§3. The main theorem.

Let o be an endomorphism of a connected algebraic group G such that the map $1 - o$ from G to itself given by $g \quad g(og)^{-1}$ is surjective. Then the subgroup G^o of fixpoints for o is finite [ ], and the map

(3.1) $$G/G^o \qquad G$$

which sends gG^o to $g(og)^{-1}$ is bijective on k-valued points, hence it is a radicial surjective morphism of schemes and so induces an isomorphism ([ ], Th)

(3.2) $$H^*(G) \qquad H^*(G/G^o) .$$

Therefore the spectral sequence of the "fibration" $(G/G^o, BG^o, BG)$ takes the form

(3.3) $$E_2 = H^*(BG) \otimes H^*(G) \implies H^*(BG^o).$$

We propose now to compute this spectral sequence under suitable hypotheses.

~~Suppose that $H^*(G)$ satisfies the condition of Borel, sot~~

Let us denote by $H^*(G)^o$ the largest sub Hopf algebra of $H^*(G)$ which is fixed elementwise by o. If $H^*(G) = U(P)$ is primitively generated, then $H^*(G)^o = U(P^o)$, where P^o denotes the subspace of o-invariants of P. Let I be the ideal in $H^*(BG)$ generated by all elements of the form $x - o^*x$, and set $H^*(BG)_o = H^*(BG)/I$; it is the largest o-invariant quotient ring of $H^*(BG)$.

Main theorem 3.4: Assume that $H^*(G)$ satisfies Borels condition and that the ideal I in $H^*(BG)$ is regular (i.e. generated by a regular sequence). Then in the spectral sequence 3.3 we have $E^{pq}_\infty = H^p(BG)_o \otimes H^q(G)^o$.

Corollary 3.5: Suppose that ℓ is odd and that the conditions of 3.5 hold. Then $H^*(BG^o)$ is isomorphic as a ring to $H^*(BG)_o \otimes H^*(G)^o$.

In this case $E^{o*}_\infty = H^*(G)^o = \Lambda(P^o)$ is an exterior algebra with odd degree generators, so lifting these generators to $H^*(BG^o)$ they generate an exterior algebra but anti-commutativity, so one gets an algebra isomorphism $E_\infty \cong H^*(BG^o)$.

In all the examples known to me, it is possible to choose a o-stable subspace Q such that $S(Q) = H^*(BG)$. In this case I is the ideal generated

Borel. Then in the spectral sequence 3.2, the elements of P are transgressive, and if c represents e in the universal spectral sequence, then e transgresses to $c - \alpha^* c$ in 3.2.

Proof: View G^α as a subgroup of G and G as the diagonal subgroup of $G \times G$ and consider the morphism of spectral sequences associated to the fibration $(G/G^\alpha, BG^\alpha, BG) \longrightarrow (G, BG, B(G\times G))$ which is furnished by the map $(id,\alpha): G \quad G\times G$. The induced map on cohomolgoy of the fibres is the isomorphism 3.2; as P is transgressive in the spectral sequence 2.

We consider the homomorphism (id,α) from G to $G\times G$. It carries the subgroup G^α into the diagonal subgroup and induces the map 3.1 on the coset spaces. Hence it gives rise to a map of spectral sequences

I

According to 2. the subspace P of $H^*(G)$ is transgressive in the fir st spectral sequence and an element x of P transgresses to $c \otimes 1 - 1 \otimes c$ in $H^*(BG\times BG)$ if x transgresses to c in the universal spectralseqenece 2. . Consequently P is also transgressive in the second spectral sequence and x transgresses to $(id,\alpha)^*(pr_1{}^* c - pr_2{}^* c) = c - \alpha^* c$, proving the proposition.

The analysis of the spectral sequence (3.?) is now standard. Indeed choose a basis $e_1 .. e_n$ for P such that $e_1, .., e_m$ form a basis for P^α, and choose a system $x_1 .. x_n$ of polynomial generators for $H^*(BG)$ such that in $QH^*(BG)$ which is naturally isomorphic to P by means of the transgression, x_i corresponds to

Thus we may assume that the fibre admits a simple system of transgressive generators, e_i the bases is a polynomial ring with generatoes x_i and that e_i transgresses to 0 for i less than or = m and to x_i for i greater than m. Then computation of the spectral sequence is then clear. E_r is the tensor product of E_r^{0*}, which admits a simple system of generators consisting of the e_i for $i \quad m$ and the e_i of degree $\quad r$ with $i \quad m$, and E_r^{*0}, which is the quotient of the ply ring with genertors x_i

Correct statement of the main theorem: Suppose that $H^*(G)$ satisfies Borel's condition and let P be the space of primitive elements. Let I be the ideal of $H^*(BG)$ generated by all elements of the form $x - o^*x$, and suppose that I is regular ,i.e. generated by a regular sequence. Let $c_1, \ldots, c_n$ be a system of polynomial generators for $H^*(BG)$, so chosen such that $c_i - o^*c_i$ for $i = 1, \ldots, m$ is a minimal system of generatoss for I. Thus we are looking at the map $m/m^2 = P - I/I^2 \otimes k$, and are

---

<u>Proposition 3.4</u>: Suppose that $H^*(G)$ satisfies Borel's condition. Then in the spectral sequence 3.3 the subspace P is transgressive, and if e P transgresses to c $H^*(BG)$ in the universal spectral sequence, then e transgresses to $c - o^*c$ in 3.3.

Consider the homomorphism (id, o) from G to $G \times G$. It carries the subgroup G^o into the diagonal subgroup and induces the map 3.1 on the coset spaces. Hence it gives rise to a map of spectral sequences

$$E_2 = H^*(BG) \; o \; H^*(G) \qquad H^*(BG^o)$$

$$E_2 = H^*(B(G \times G)) \; o \; H^*(G) \qquad H^*(BG)$$

According to 2.? then subspace P is transgressive in the second spectral sequence and the element of P transgresses to $pr_1^*c - pr_2^*c$. Hence in the first spectral sequence e transgresses to $c - o^*c$.

According to this proposition the ideal I of elements of $H^*(BG)$ which are in the image of the transgression of 3.3 restricted to P is the ideal generated by $x - o^*x$.

So the problem is that you must prove that the generators of the E_2 term of the Eilenberg-Moore spectral sequence survive. For this you want to use your map x in $H(BG)^o \xrightarrow{q}$ goes to Coker$(H(BG) \quad H(BG^o))$.

1. Equivariant cohomology for the etale topology

In [ ] Grothendieck has defined quite generally the classifying topos of a group scheme G. This replaces the classifying space BG of the topologists. In this section we review Grothendieck's construction in the special case (geometric rather than arithmetic) of an algebraic group (for the most part reductive) defined over an algebraically closed field k. More generally we shall consider the equivariant cohomology of an algebraic group acting on a variety X, and we shall derive the familiar spectral sequence(s) used in its computation.

1. So we let k be an algebraically closed field and let C be the category of schemes of finite type over k. Let T be the category of sheaves (of sets) on C for the etale topology, that is, contravariant functors F on C with values in the category of sets such that

(i) $F(S \quad S') = F(S) \times F(S') \qquad F(\emptyset) = e$

(ii) If $S' \longrightarrow S$ is etale and surjective, then

$$F(S) \longrightarrow F(S') \rightrightarrows F(S'') \qquad S'' = S' \times_S S'$$

is exact.

~~Any object X of C gives a sheaf~~ Associating to a scheme in C the contravariant functor Hom(?,X) gives a functor from C to T permitting us to view C as a full subcategory of T.

If F is a sheaf of abelian groups on C, ie.e. an abelain group object of T, then cohomology groups of an object are defined, and denoted $H^q(X,F)$. Recall that these are defined by taking injective rsolution of F, but in view of a theorem of Verdier [ ], they also have a Cech-like definition (which appeals to me because of earlier experience with semi-simplicial things.) Recall the definition of hypercovering It is a semi-simplicial object of schemes etale over X

$$U_1 \qquad U_o \qquad X$$

such that if we take the set of points with values in k, then we get a resolution (contactible Kan complex).

Given such a hypercovering $\underline{U}_\bullet$ we let $C^*(U_\bullet,F)$ be the cosimplicial abelian group with $C^q(U_\bullet,F) = \Gamma(U_q,F)$ and $H^q(U_\bullet,F)$ the cohomology of this cosimplical abelian group. Verdier shows that the hypercoverings of X form up to simplicial homotppy a filtered category and that the etale cohomolgoy can be computed quite simply as

$$H^q(X,F) = \text{dir.lim.}\ H^q(U_\bullet,F)$$

The point of this excursion is that in the case we are interested in F will be the constant sheaf $\mathbb{Z}_\ell$, whence $C^*(U_\bullet,F)$ ~~mkxx~~ will be the cochain complex on the simplicial set $\pi_0(U_\bullet)$. Hence cohomology mod ℓ will have cup product, Steenrod operations, all the ~~xxxxxxxxxxxx~~ usual structure of algebraic topology.

2. Let G be an ~~xxxxxxxxxxxxxxxxx~~ algebraic group, i.e. a nonsingular group object in C. Then G gives rise to a ~~xxxxxxxxxxxxxxx~~ group object, which we shall again denoted by G. Introduce the category T_G of objects F endowed with a left action $G\times F \to F$, the so-called classifying topos of G. If X is a sceme on which G acts, more generally an object of T_G, then its cohomology shall be denoted

$$H^*_G(X, F)$$

where F is an abelian group object of T_G. (Computation of this a la Verdier?)

We can now take up the derivation of the spectral sequences that we shall need. Let $f:X \to Y$ be a map of schemes on which G acts. Then the map $f: X \to Y$ gives rise to a map of topi $f_* : T_G/X \to T_G/Y$, where

$$\Gamma(U, f_*(F)) = \Gamma(f^{-1}U, F)$$

(More precisely f_* is the adjoint of the map ~~XX~~ $f^*(U) = X\times_Y U$.) This map gives rise to a Leray spectral sequence

$$E_2^{pq} = H^p_G(Y, R^q f_* F) \Longrightarrow H^q_G(X,F)$$

(Use the Grothendieck notation perhaps. $H^q(X,G;\ F)$?) Here $R^q f_*(F)$ is the presheaf associated to the sheaf $U \to H^q(f^*U,G;F)$, as U runs over T_G/Y. So it seems desirable to know that T_G/Y is the topos associated to the site consisting of free ~~xxxxxxxxxx~~ G-schemes over Y, with the etale topology.

To find a site for T_G: Situation is G is a topological group and we want to understand the gross topos T and the corresponding classifying topos.

Claim 1; Consider the category of principal G-bundles P — X with group G. Then the topos is the same as the category of contravariant functors on this category which are sheaves for the etale topology, i.e. If P_i --- P is a covering family then usual nonsense holds.

How does one compute the etale cohomology semi-simplicially?

If G is discrete there is no problem because one then works entirely in the situation where sheaves correspond to etale spaces. e.g. the map GxX—X is etale. In the general xase one puts oneself in this position by some trick such as working entirely in the topos. But it seems difficult to extract what is being used in the argument from the beginning. However it seems clear that if we start with defining hypercoverings as simplicial objects P. in the category of free G-spaces with augmentation to X suchthat on passing to sheaves we get a hypercovering of X. It's pretty clear that this is the correct thing but not exactly clear what has to be proved in order to be sure. In other words I know that free S-sapces generate the category of sheaves in T_G, and in fact that I get generators of the form GxU where U is a scheme. An alternative method is to take a hypercovering U. of X and then

The only point of this concretization is to make you feel more secure but it should not make it any easier to prove things.

C category of schemes of finite type over Spec k, k field which I suppose to be alg. closed.

$\mathbf{T}$ topos of sheaves for the etale topology

G group in C

X scheme on which G acts i.e. have

$$G \times X \longrightarrow X$$

~~$G \times X \rightrightarrows X$~~

$T_{\tilde{G}} \ni \tilde{X}$ $\quad F \in (T_{\tilde{G}}/\tilde{X})_{ab}$ has basic coh. groups.

~~T G-scheme~~

T_G topos of sheaves associated the category of principal G-bundles $P \to X$ for the etale topology. Consequently given a G-scheme X, we consider hypercoverings

$$\Rrightarrow P_1 \rightrightarrows P_0 \longrightarrow X$$

problem with this if G is not finite because then

$P_i \longrightarrow P_j$ <u>aren't etale.</u>

outline of paper

Part I: The first theorem

§1. derivation of the spectral sequences

2. Borel's theorem + analysis of spectral sequence $G, G\times G$ in the good case.

3. Proof of first theorem

4. Discussion of hypotheses guaranteeing the good case + 5. examples.

Part II: The second theorem on restriction to the torus

1. localization theorem
2. proof of 2nd theorem
3. Symmetric invariants theorem examples ℓ odd.

4. further analysis for $\ell = 2$.

$\left\{ \begin{array}{l} ? \\ ? \end{array} \right.$

Part III: General results on the homotopy type of $BG(\mathbb{F}_q)$ after completion.

A

<u>first theorem</u> §1.

k alg. closed field, Sch schemes of finite type over k with etale topology, T associated topos. ~~G alg. group over k, [illegible] T_G its classifying topos~~ If X is a scheme let $H^*(X)$ be its cohomology with coeffs in $\mathbb{Z}/\ell$ where ℓ is a prime no. $\neq$ char of k.

Suppose ~~G acts on X~~ G is an alg. group over k, which we identify with a group in T and let T_G be classifying topos. If X is an object of T_G let

$$H^*_G(X)$$

be the cohomology of X with coeffs. in $\mathbb{Z}_\ell$.

Thus if G acts on a scheme X we have its equivariant cohomology

$$H^*_G(X)$$

We put $pt = \operatorname{Spec} k$ and write H^*_G for $H^*_G(pt)$

~~Propos Defn A scheme~~

~~Defn: A map of schemes $f: X \to Y$ is called coh. proper if the base change thm. holds for arb. $Y' \to Y$.~~

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

$g^* f_* =$

Remark: coh. proper $\xRightarrow{?}$ Kunneth formula.

$$H^*(S\times X) = H^*(S)\otimes H^*(X)$$

~~Remarks: $X \xrightarrow{f} pt$ coh. proper (modulo resolutions of singularities). In this it is equivalent to X satisfying Künneth theorems~~

$$\cancel{H^*(S\times X) = H^*(S)\otimes H^*(X)}$$

Proposition 1: Let X be a scheme on which G acts and suppose that $X \to pt$ is cohomologically proper. Then $\exists$ spectral sequence

$$E_2^{pq} = H^p_G(pt, H^q(X)) \Longrightarrow H^{p+q}_G(X)$$

where G acts on $H^q(X)$ through $\pi_0(X)$.

Corollary: G connected + $X \to pt$ coh. proper $\Rightarrow \exists$ spec. seq.

$$E_2^{pq} = H^p_G \otimes H^q(X) \Longrightarrow H^{p+q}_G(X).$$

Proposition 2: $G \longrightarrow pt$ is coh. proper.

Correct (operating) hypothesis is that X satisfies Kunneth

$H^*(X)$ finite diml

& $H^*(S\times X) \xleftarrow{\sim} H^*(S)\otimes H^*(X)$

§2. Borel's theorem

Consider the spectral sequence

$$E_2^{pq} = H_G^p \otimes H^q(G) \Longrightarrow H^{p+q}(pt)$$

It is multiplicative. ~~Recall the transgression in dim is a map from a subgr is the map given from the subgroup~~ ~~$E_r^{0,r-1}$ of $E_2^{0,r-1} H^q(G)$ to the quotient group $E_r^{q,0}$ of $E_2^{q,0} \cong H^q(BG)$~~

Recall that the transgression in dimension r is the map from the subgroup $E_r^{0,r-1}$ of $E_2^{0,r-1} \cong H^{r-1}(G)$ to to quotient group $E_r^{r,0}$ of $E_2^{r,0} \cong H^r(BG)$ given by d_r. One knows that a transgressive element of $H^*(G)$ is primitive (Borel, 20.1) and that decomposable elements of $H^*(BG)$ go to zero in $E_*^{*,0}$ (easy consequence of multiplicative structure on the spectral sequence) so that the transgression can be viewed as a map from a subgroup of $PH^*(G)$ to a quotient of $QH^*(BG)$ given by

$$PH^{r-1}(G) \hookleftarrow E_r^{0,r-1} \xrightarrow{d_r} E_r^{r,0} \twoheadleftarrow QH^r(BG)$$

Easy ~~Borel's~~ Borel thm then says

Thm: If $H^*(G)$ has a simple system of transg. gen.

D

for the spectral sequence, then

1) $H^*(BG)$ is a polynomial ring

2) The transgression is isomorphism

$$\tau : PH^{q}(G) \xrightarrow{\cong} QH^{q}(BG).$$

(In this situation $H^*(G)$ is a primitively generated Hopf algebra with ~~odd~~ $P_{ev}H^{\ell}(G) = 0$ if ℓ is odd and $H_*(G)$ is an exterior algebra..)

Hard Borel thm. says that

$$H^*(G) \underset{alg}{\cong} \Lambda P \quad P = P_{odd} \Longrightarrow \text{above holds.}$$

Another Borel thm. says that $H^*(BG)$ poly ring

$\Longrightarrow$ rest.)

(Nature of the multiplicative structure of the spec. seq arises from Kunneth isom

$$H^*_{G\times G'}(X\times X') \xleftarrow{\sim} H^*_G(X) \otimes H^*_{G'}(X)$$

Must check this gives rise to a pairing of spectral sequences.

Conclude that)

~~Nat~~ Next situation: $K \subset G$ ~~both are~~ $\ni$

[(i) $H^*(G)$ has a simple system of transg. generators:
(ii) $H^*(G) \longrightarrow H^*(K)$ surjective

then $H^*(K)$ is primitively generated since $H^*(G)$ is. Moreover since the (G, EG, BG) spectral sequence maps to the one for (K, EK, BK) it follows that $PH^*(K)$ ~~is primitive. The is primitive~~ consists of transgressive elements. By structure theory ~~applied to $H_*(K)$~~ of restricted Lie algebras one knows that $PH^*(K)$ has a simple system of trans. gens. ~~Also if we~~ Define $PH^*(G/K)$ by requiring

$$0 \longrightarrow PH^*(G/K) \longrightarrow PH^*(G) \longrightarrow PH^*(K) \longrightarrow 0$$

~~Then $H^*(G/K)$ has~~ to be exact. Now ~~by the~~ assumption (ii) implies that the fiber is thick in the spectral sequence

$$H^*(G/K) \otimes H^*(K) \Longrightarrow H^*(G)$$

so ~~actually~~ at least as $H^*(G/K)$-modules

$$H^*(G) \cong H^*(G/K) \otimes H^*(K)$$

~~so that~~

$$\cancel{0 \to H^*(G/K) \to H^*(G) \to H^*(G) \otimes H^*(K)}$$

~~Should be exact.~~ Consequently if c_i $i \in I$ is a base for $PH^*(G)$ such that ~~c_i~~ $c_i \longmapsto 0$ $i \in I'$ and c_i / K form a base for $PH^*(K)$ $i \in I''$ (where $I = I' \cup I''$), then the c_i $i \in I'$ form a ~~sy~~ simple system of gens. for $H^*(G/K)$.

F

Now consider the three spectral sequences

$$\begin{array}{llcl}
1) & & H^p(BG)\otimes H^q(G/K) & \Longrightarrow H^{p+q}(BK) \\
& j^* & \downarrow & \qquad\downarrow \\
2) & & H^p(BG)\otimes H^q(G) & \Longrightarrow H^{p+q}(pt) \\
& i^* & \downarrow & \qquad\downarrow \\
3) & & H^p(BK)\otimes H^q(K) & \Longrightarrow H^{p+q}(pt)
\end{array}$$

Notation: e_i $i\in I$ basis for $PH^*(G)$

e_i'' $i\in I''$ " " $PH^*(K)$

view $I''\subset I$ and then

$$i^* e_i = \begin{cases} 0 & i\in I' = I-I'' \\ e_i'' & i\in I''. \end{cases}$$

e_i' base for $PH^*(G/K)$ such that

$$K \xrightarrow{i} G \xrightarrow{j} G/K$$

$j^* e_i' = e_i$ $\quad i\in I'$.

c_i'' representing $\tau e_i''$ $\quad i\in I''$

c_i representing τe_i $\quad i\in I$

$$i^* c_i = \begin{cases} 0 & i\in I' \\ c_i'' & i\in I'' \end{cases}$$

Then the argument runs as follows. Since $i^*(c_i)=0$ $\quad i\in I'$ we have $\quad (r=\deg c_i)$

$K_r c_i = d_r z$ in E_r^{r0} $\quad$ with $z\in E_r^{0,r-1} \subset H^{r-1}(G/K)$

in spectral sequence 1). Now apply j^* and look at this equation in spec. sequence 2)

$$K_r c_i = d_r(j^* z)$$

But in spectral sequence 2), the transgression is an isomorphism of $PH^*(G) \xrightarrow{\sim} QH^*(BG)$, hence

$$j^* z = e_i$$

and so as $j^* : H^*(G/K) \longrightarrow H^*(G)$ is injective

$$z = e_i'$$

Consequently in spectral sequence 1) $PH(G/K)$ is transgressive and $\tau e_i' = c_i$ for all $i \in I'$. Thus we have proved

<u>Prop:</u> Suppose $K \subset G$ satisfies (i) and (ii) on page E. Then in spectral sequence

$$E_2 = H^*(BG) \otimes H^*(G/K) \Longrightarrow H^*(BK)$$

the fiber has a simple system of transgressive generators and in fact the transgression is an isomorphism

$$PH^*(G/K) \xrightarrow{\sim} \operatorname{Ker}\{QH^*(BG) \xrightarrow{i^*} QH^*(BK)\}$$

H

Now the situation I want to apply this to ~~is where~~ is ~~where the subgroup~~ the diagonal subgroup $\Delta: G \longrightarrow G \times G$, where G is a comm. alg. ~~group~~ and where $H^*(G)$ has a simple system of universally transgressive generators. In this case the hypotheses (i) and (ii) apply since the diagonal

$$\Delta^*: H^*(G \times G) \longrightarrow H^*(G)$$

is surjective. ~~[illegible]~~

In this case ~~[illegible]~~ the homogeneous space is G^s ~~[illegible]~~ where $G \times G$ ~~acting~~ acts by $(g_1, g_2)g = g_1 g g_2^{-1}$.

Proposition: If G satisfies Borel's condition, then in the spectral sequence

$$E_2 = H^*(B(G \times G)) \otimes H^*(G^s) \Longrightarrow H^*(BG)$$

the cohomology of the fibre admits a simple system of transgressive generators $e_i \in PH^*(G)$ such that $\tau(e_i)$ is represented by $c_i \otimes 1 - 1 \otimes c_i$ where $\tau e_i = c_i$ in the spectral sequence

Outline of paper for Nice: Cohomology rings of groups.

Goal should be an announcement of your work and should consist of an outline of a theory together with indications for further research. The theory itself might be organized ~~xmxm~~ according to your papers:

1) Spectrum

2) Groups of rational points

3) Families of groups Gn

Axiomatization of a system of groups - one G_n for each integer n together with Whitney sum homomorphisms $G_n \times G_m$ G_{n+m} and wreath product maps of the sort that the disjoint unions of BG_n gives rise to a cohomology theory. ~~GmxemxExampkesxof~~ Such systems should give rise to generalized cohomology theoryes whose cohomology, cobordism theory, etc. i.e. maps into other generalized cohomology theoryes ~~shmm~~ should be calculable before forming the K-theory. Therefore it should be possible to define the cohomology of the true classifying space for the general linear groups over the finite fields.

There exists a basic duality here: You have a very good idea of what ~~gmesxmmxfmx~~ are the fibrant objects: the trace theoryes with inverses. This is not quite correct because ~~K~~ k(X,R) gave the wrong results. The correct apprach is to ~~wxfmxmxthexgenmraixxed~~ form the appropriate motive categories form the universal functors and then check that they satisfy the exactness axiom.

Key problem; Let ~~Rxbmxx~~ k be a representable trace theory and let K be the universal functor on the motive category endowed with a map from k to K compatible with traces. Then i) K should be graded and ii) $k = K^0$.

Conjecture

Summary of results on cohomology rings of finite groups of rational points.

Let k be a finite field with q elements and let G be a connected algebraic group defined over k. View G as a variety over an algebraic closure K of ~~K~~ k, together with a geometric Frobenius endomorphism giving its structure as variety defined over k. This Frobenius will be denoted F; it probably suffices to have any radicial surjective endomorphism. (The test is whether $G/H \quad G$, where H is the finite group of points of $G(K)$ fixed by F. Thus I want the map $x \quad x(Fx)^{-1}$ to be tangent to the identity, hence that $dF = 0$; it is not clear that this holds for the Steinberg generalizations.)

~~Followingxfromxthexnext~~ In the following we shall let $H^*(X)$ denote the cohomology of the scheme X for the etale topology with coefficients in $\mathbb{Z}_\ell$, where ℓ is a prime number distint from the characteristic of K. This requires us to ~~xxxxxxxxxxxxxxxxxxxxxxxx~~ embed the category of ~~xxxxxxxx~~ schemes of finite type over K into the category ~~S~~ T of sheaves on this category for the etale topology. In particular G becomes a group object in T. Following Grothendieck we define the classifying topos of G to be the category T_G of objects of T endowed with an action of G. This topos ~~xxxxxxxxx~~ is ~~xxxxxxxxx~~ endowed with a canonical morphism $T_G \xrightarrow{f} T$ as well as a torsor(principal homogeneous space) for f^*G namely G itself with the ~~right~~ right translation, the left action making it an object of T_G. Moreover the pair f and the torsor have the universal property in the category of topi over T that one expects from the classifying space.

We will use the notation BG ~~xxxxxx~~ for the classifying topos of G. The cohomolgoy groups with coefficients mod ℓ will be dentoed $H^*(BG)$. More generally if M is ~~xxxxxxxxxxx~~ an object of T endowed with an action of G, then we have sheaves $R^q f_*(M)$ in T.

~~xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx~~

(started March 23)

~~On the cohomology groups of~~ Cohomology of finite groups of rational points.

Borel's work established a general framework by whcih we can understand the cohomolgoy ~~of~~ rings modulo ℓ of BG, where G is a compact connected Lie group. in most cases, leaving open certain exceptions which then have to be examined on an individual basis. The ~~prim~~ purpose of the present paper is to extend Borel's theory to encompass finite groups ~~of~~ which occur as the group of rational points of an algebraic group defined over a finite field of characteristic p different from ℓ. Our tool here is to use the etale cohomology of the algebraic group ~~[illegible]~~ which ~~[illegible]~~ is the same as that of the compact ~~form~~ form of the reductive semisimple factor group. The following examples indicate ~~the complexity~~ rather well the general situation.

~~[illegible]~~ Take GL_n : Then $H^*(BGL_n)$ is an exterior algebra with generators. of degrees 1,3, 2n-1 e_i of degree 2i-1 for i = 1,..,n. Frobenius acts by $Fe_i = q^i e_i$. Let d be the least positive integer ~~such~~ such that q^d-1 is divisible by ℓ. Then e_{jd} ~~[illegible]~~ j= 1,..,[n/d] are invariant under Frobenius

Statement of the theorem: Hypothesis: G has no ℓ-torsion. Let P be the subspace of $H^*(G)$ of primitive elements, and let P_F and P^F be respectively the ~~spaces of~~ spaces of ~~[illegible]~~ co-invariant and invariant elements for the action of Frobenius. Then $H^*(BG_F)$ is isomorphic to $\Lambda(P^F) \otimes S(P_F)^*$

Example GL_n ..

Example of a torus. ~~[illegible]~~

The mod 2 situation is expeptional and will be discussed separately. Don't know GL_n O_n or Sp_{2n} with cohomology mod 2 yet, nor