February 4, 1960.

I want to test conjecture 3 of yesterday in the case $p=2$. It says that to give a multiplicative operation

$$(*) \qquad \gamma : H^*(X) \longrightarrow R \otimes H^*(X)$$

is the same as giving a sequence of elements $r_\nu \in R_{2^\nu - 1}$ $\nu \geq 0$ and that this sequence is determined from γ by

$$\gamma x = \sum_{\nu \geq 0} r_\nu x^{2^\nu} \qquad x \in H^1(X).$$

Using Kunneth we can prove the existence of a universal such operation as follows. Suppose γ given as above. Then

$$\gamma_q : H^q(X) \longrightarrow [R \otimes H^*(X)]^q$$

is an additive operation. As $H^q(X) = [X, K_q]$, where $K_q = K(\mathbb{Z}_2, q)$, γ_q is the same as an element of degree q of

$$P(R \otimes H^*(K_q)) = \operatorname{Hom}_{\mathbb{Z}_2\ v.s.}(Q H_*(K_q) \sigma^q, R) \qquad (Q = I/I^2)$$

where σ^q is a place-keeping element of homological degree $-q$. The fact that the $\{\gamma_q\}$ form a multiplicative operation means that

$$(**) \qquad \bigoplus_{q \geq 0} Q H_*(K_q) \sigma^q \longrightarrow R$$

is a ring homomorphism. So the conjecture is the following

<u>Theorem:</u> Let $\xi_\nu \in H_{2\nu}(\mathbb{R}P^\infty)\sigma$ be a generator. Then

$$\mathbb{Z}_2[\xi_\nu]_{\nu\geq 0} \longrightarrow \bigoplus_{q\geq 0} 2H_*(K_q)\sigma^q$$

is an isomorphism.

<u>Proof:</u> First note that from ~~[illegible]~~ Serre's calculations of the cohomology of E-M spaces one knows that the suspension isom. induces a surjection

$$\mathcal{P}H^*(K_n) \longrightarrow \mathcal{P}H^{*-1}(K_{n-1}).$$

Put in different terms this means that multiplication by ξ_0 in the universal ring (left side of (**) page 1 which ~~[illegible]~~ we will now denote by R_{univ}) is injective. Thus R_{univ} is a free $\mathbb{Z}_2[\xi_0]$-module. Consider $R_{univ}/(\xi_0-1)$ which represents ~~[illegible]~~ stable multiplicative natural transformations. By Milnor $R_{univ}/(\xi_0-1)$ is $\cong\mathbb{Z}_2[\xi_\nu]_{\nu>0}$, so one sees (using homogeneity of ξ_0) that R_{univ} is the polynomial ring generated by the ξ_ν.

~~Everything in this argument ~~[illegible]~~ should go through for p odd.~~

~~Hence~~

$$\cancel{\mathbb{Z}_p[\xi_\nu, \tau_\nu]_{\nu\geq 0} \xrightarrow{\cong} \bigoplus_{q\geq 0} 2H_*(K_q)\sigma^q} \qquad \cancel{K_q = K(\mathbb{Z}_p, q)}$$

~~and Ring operations $H^*(X) \to R\otimes H^*(X)$ are the same as~~

~~[illegible] where the ξ_ν, τ_ν are defined in terms of the universal multiplicative operations ~~[illegible]~~ by~~

$$\gamma(\beta x) = \sum_{\nu\geq 0}' \xi_\nu (\beta x)^{p^\nu}$$

$$\gamma(x) = \left[\sum_{\nu\geq 0}' \tau_\nu (\beta x)^{p^\nu} + \xi_0 x\right].$$

This needs some checking especially to understand what exactly happens to the signs.

So we start with a multiplicative operation

$$\gamma : H^*(X) \longrightarrow R \otimes H^*(X)$$

Again we get elements

$$\gamma_q \in P[R \otimes H^*(K_q)]^q \cong \underset{\mathbb{Z}_p\text{-v.s. graded}}{\mathrm{Hom}} (QH_*(K_q)\sigma^q R)$$

to describe the additive operation in each degree. ~~These maps work as follows. Given $\lambda \in \mathrm{Hom}_{gr/\mathbb{Z}_p}(H_*(K_q)\sigma^q, R)$ and $x \in H^q(X) = [X, K_q]$ we consider the composition~~

~~$$\lambda \circ x_* : H_*(X)\sigma^q \longrightarrow R$$~~

~~Then $\gamma(x) \in [R \otimes H^*(X)]^q$ is the unique element such that~~

~~$$(\lambda \circ x_*)(y\sigma^q) = \langle x, y \rangle \quad \text{for all } y \in H_*(X)$$~~

Again the γ_q should fit together to form a ring homomorphism. We can obviously construct the universal multiplicative operation

$$H^*(X) \longrightarrow R_{univ} \otimes H^*(X)$$

For the prime 2 claim $\exists$ an operation

$$H(X) \xrightarrow{\gamma} \mathbb{Z}_2[\xi_\nu]_{\nu \geq 0} \otimes H(X)$$

$$\ni \quad \gamma x = \sum{}' \xi_\nu x^{2^\nu} \qquad\qquad x \in H^1$$

moreover γ is a universal multiplicative operation.

<u>Proof of existence:</u> Start with

$$P^{(r)} : H(X) \longrightarrow \mathbb{Z}_2[w_{2^r-2^{r-1}}, \dots, w_{2^r-2^0}] \otimes H(X)$$

such that
$$P^{(r)} x = \sum_{i=0}^{r} w_{2^r-2^i}\, x^{2^i}$$

Setting $\xi_\nu = w_{2^r-2^\nu}$ $\quad \nu = 0, \dots, r-1$ we get an operation

$$P^{(r)} : H(X) \longrightarrow \mathbb{Z}_2[\xi_0, \dots, \xi_{r-1}] \otimes H(X) \quad \ni$$

$$\gamma x = \sum_{\nu=0}^{r-1} \xi_\nu x^{2^\nu} + x^{2^r}$$

<u>Question:</u> Does

$$\begin{tikzcd} H(X) \arrow[r] \arrow[dr] & \mathbb{Z}_2[\xi_0, \dots, \xi_{r-1}] \otimes H(X) \\ & \mathbb{Z}_2[\xi_0, \dots, \xi_r] \otimes H(X) \arrow[u] \end{tikzcd}$$

commute for $r \gg \dim X$? If so then we get the operation we want. To prove commutes we can invert ξ_0 and what is γ in terms of the usual Steenrod operations?

$$H(X) \xrightarrow{\gamma} \mathbb{Z}_2[\xi_\nu]_{\nu \geq 0} \otimes H(X)$$

$$Ru = \xi_0^{-\deg u} \gamma u \qquad\qquad Rx = \sum_{\nu \geq 0} \frac{\xi_\nu}{\xi_0} x^{2^\nu}$$

Then

$$Ru = \sum_\alpha \left(\frac{\xi}{\xi_0}\right)^\alpha Sq_\alpha u \qquad\qquad \alpha = (\alpha_1, \alpha_2, \cdots), \quad \alpha_i > 0.$$

$$\gamma u = \xi_0^{\deg u} \sum_\alpha \left(\frac{\xi}{\xi_0}\right)^\alpha Sq_\alpha u$$

$$\boxed{\gamma u = \sum_\alpha \xi_0^{\deg u - \ell(\alpha)} \xi^\alpha Sq_\alpha u}$$

can you see why ~~Sq~~ $Sq_\alpha u = 0$ if $\ell(\alpha) > \deg u$?

$$\ell(\alpha) = \sum_{i=1}^{\infty} \alpha_i$$

by cobordism (deg u) $\quad u = f_* 1 \quad f: Z \to X \quad$ ~~deg~~ codim $f =$

$$Sq_\alpha u = f_* c_\alpha(\nu_f)$$

not much help.

example: $\alpha = (i, 0, 0, \cdots)$ $\quad Sq_\alpha = Sq^i$ and one knows that $Sq^i u = 0$ if $i > \deg u$!

Proof that $Sq_\alpha u = 0$ if $\ell(\alpha) > \deg u$.

take $\quad \alpha = (\alpha_1, \dots, \alpha_n, 0, \dots\dots)$

Then look at $P^{(n)}u \qquad \xi_\nu \longmapsto w_{2^n-2^\nu} \quad 0 \le \nu < n, \quad \xi_n \mapsto 1.$

$$\gamma u \longmapsto \sum_{\alpha_1, \dots, \alpha_n} w_{2^n-1}^{\deg u - \sum_{i=0}^{n} \alpha_i} (w_{2^n-2^1})^{\alpha_1} \cdots (\underset{1}{\overset{\shortparallel}{w_{2^n-2^n}}})^{\alpha_n} Sq_\alpha u.$$

~~[illegible]~~

$$\deg \underline{\underline{Sq_\alpha}} u = \deg u + \sum_{i=1}^{n} \alpha_i (2^i - 1)$$

take n so large that if $\alpha_n > 0$ then $Sq_\alpha u = 0$ for diml. reason $2^n - 1 > \dim X$. Then

$$\gamma u \longmapsto \sum_{\substack{\alpha_0, \alpha_1, \dots, \alpha_{n-1} \\ \sum \alpha_i = \deg u}} \overset{\alpha_0}{w_{2^n-1}^{\deg u - \sum_{i=0}^{n-1} \alpha_i}} \underbrace{(w_{2^n-2^1})^{\alpha_1} \cdots (w_{2^n-2^{n-1}})^{\alpha_{n-1}}}_{\text{all of these are distinct}} Sq_\alpha u$$

$$\overset{\shortparallel}{P^{(n)}u}$$

from which we conclude that that since $P^{(n)}u$ has no denominators

$$Sq_\alpha u \neq 0 \implies \sum_{i=0}^{\infty} \alpha_i \le \deg u$$

restrict from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^{n-1}$ and then $w_{2^n-2^i} \longmapsto w_{2^{n-1}-2^{i-1}}^2$

$$(P^{(n-1)}u)^2 = \sum_{\alpha_1 + \dots + \alpha_{n-1} = q} \left(w_{2^{n-1}-1}^{\alpha_1} \cdots w_{2^{n-1}-2^{n-2}}^{\alpha_{n-1}} \right)^2 Sq_\alpha u$$

$\shortparallel$

$$= \sum_{\beta_1 + \dots + \beta_{n-2}} w_{2^{n-1}-1}^{q - \sum \beta_j} (w_{2^{n-1}-1})^{2\beta_1} \cdots (w_{2^{n-1}-2^{n-2}})^{2\beta_{n-2}} (Sq_\beta u)^2$$

$$\underset{\sum \alpha_i = q}{Sq_{\alpha_1 \dots \alpha_{n-1}} u} = \cancel{(Sq_{\alpha_2 \dots \alpha_{n-1}})^2} \; (Sq_{\alpha_2 \dots \alpha_{n-1}} u)^2$$

$$Sq_{(q,0,\ldots)}\, u = (Sq_0 u)^2 = u^2.$$

check: $n=1$. $\gamma u = \sum_{\alpha_0+\alpha_1=q} \xi_0^{\alpha_0}\xi_1^{\alpha_1} Sq^{\alpha_1} u$

degree $\xi_1 = - (2-1) = -1$

The assertion is that no neg. powers of ξ_0 occur because

$$Sq^i u = 0 \quad \text{if } i > q.$$

and $\gamma x = \xi_0 u + \xi_1 u^2$.

This tests my conjecture for $p=2$ that $\exists$ an operation

$$\gamma : H(X) \longrightarrow \mathbb{Z}_2[\xi_\nu]_{\nu \geq 0} \otimes H(X)$$

and shows that

$$\gamma u = \sum_{\sum_{i\geq 0} \alpha_i = \deg u} \xi^\alpha Sq_{\alpha_1,\ldots}\, u = \sum_{(\alpha_1,\ldots)} \xi_0^{q-\ell(\alpha)} \xi^\alpha Sq_\alpha u.$$

Next I want to knows that γ is a universal multiplicative operation. Thus I ~~want to~~ suppose given

$$\Gamma : H(X) \longrightarrow R \otimes H(X)$$

and I let $\Gamma x = \sum_{\nu \geq 0} r_\nu x^{2^\nu}$. So I want to show that

$$\begin{tikzcd}
H(X) \arrow[r, "\gamma"] \arrow[dr, "\Gamma"'] & \mathbb{Z}_2[\xi_\nu] \otimes H(X) \arrow[d] \\
& R \otimes H(X)
\end{tikzcd}$$

commutes, or equivalently that

Obsolete

February 6, 1970.

Let ℓ be a prime numbers such that the Chow ring of G/B is generated by its elements of degree 1 mod ℓ. Then according to Borel it should be possible by spectral sequence arguments to show that $H^*(BG)$ is a polynomial ring and $H^*(G)$ is an exterior algebra and the suspension homomorphism induces an isomorphism

$$\mathcal{Q}^q H(BG) \xrightarrow{\sim} P^{q-1} H(G).$$

Assume all of this. Suppose $\ell \mid q-1$ and that G is defined over $\mathbb{F}_q$ such that the torus T of B is split, i.e. isomorphic to $\mathbb{G}_m^{\text{rank}}$ over $\mathbb{F}_q$. Then ${}_\ell T \cong \mu_\ell^r$ is a finite ~~abelian~~ group without Galois twisting, i.e. ~~[illegible]~~ $F = \text{identity}$ on ${}_\ell T$. I claim that this implies that the spectral sequence

1) $$E_2^{pq} = H^p(BG) \otimes H^q(G) \Longrightarrow H^{p+q}(BG(k)) \qquad k = \mathbb{F}_q$$

is degenerate. Indeed this is the spectral sequence for $H^*_G(G/G(k))$. Now since $H^*(BT) \longrightarrow H^*(G/T) = H^*(G/B)$ is surjective by assumption and $H^*(B_\ell T) \longrightarrow H^*(T/{}_\ell T)$ is surjective by direct calculation, we have descent for the map $B_\ell T \longrightarrow BG$. ~~and hence Thus pulling back the fibration~~ Thus the ~~[illegible]~~ spectral sequence

2) $$E_2^{pq} = H^p(B_\ell T) \otimes H^q(G) \Longrightarrow H^{p+q}_{{}_\ell T}(G/G(k))$$

is a direct sum of copies of ~~[illegible]~~ 1) and so it suffices to prove that ~~[illegible]~~ 2) degenerates. More precisely we have the following triangle already encountered in Spin

$$
\begin{tikzcd}
 & H^*_G(G/G(k)) \arrow[dl] \arrow[d] & \\
H^*(G/G(k)) & H^*_{{}_eT}(G/G(k)) \arrow[l] & \text{gen. by } H^*_G(G/G(k)) \text{ and } H^*_{{}_eT}(pt) \arrow[l]
\end{tikzcd}
$$

Since $\tilde{H}^*_{{}_eT}(pt)$ goes to zero one sees that one arrow is surj. iff the other arrow is.

But now if we identify $G/G(k)$ with G via the map $gG(k) \longmapsto g(Fg)^{-1}$ one sees that the left multiplication by ${}_eT$ corresponds to ($tg \longmapsto tg\,Fg^{-1}Ft^{-1} = t(gF^{-1}g)t^{-1}$) the conjugation action of ${}_eT$ on G. Thus it suffices to show that

$$H^*_{{}_eT}(G) \longrightarrow H^*(G)$$

is surjective where ${}_eT$ acts by conjugation. I claim that in fact

3) $$H^*_G(G) \longrightarrow H^*(G)$$

is surjective. To see this I use that $H^*(G)$ is generated as an algebra by the image of the suspension homomorphism. Geometrically the suspension is defined by the principal bundle $G * G$ over $G \wedge \Sigma^1$

$$
\begin{tikzcd}
tg_0 + (1-t)g_1 \arrow[d, mapsto] & G * G \arrow[r] \arrow[d] & EG \arrow[d] \\
t\, g_0 g_1^{-1} & G \wedge \Sigma^1 \arrow[r] & BG
\end{tikzcd}
$$

We note that G acts on the left of $G*G$ and this corresponds to the conjugation action on $G \wedge \Sigma^1$. Thus the suspension homomorphism ~~[illegible]~~ factors

$$H^*(BG) \longrightarrow H^*_G(G \wedge \Sigma) \longrightarrow H^*(G \wedge \Sigma)$$

proving that the image of 3) contains the image of the suspension and hence is surjective.

<u>Conclusion</u>: Let G be a (connected) reductive algebraic group defined over $\mathbb{F}_q$. Let l be a prime number dividing $q-1$ which is good for G, i.e. Chow(G/B) generated by its elements of dimension 1. (mod l) * Then the spectral sequence

$$H^*(BG) \otimes H^*(G/G(\mathbb{F}_q)) \Longrightarrow H^*(BG(\mathbb{F}_q))$$

degenerates. ~~Thus~~ If $l \neq 2$, $H^*(BG(\mathbb{F}_q))$ is isomorphic to the tensor product of $H^*(BG)$ and $H^*(G)$.

(* and suppose G has a split torus defined over $\mathbb{F}_q$)

The philosophy behind the above argument is that we can descend to a subgroup of $G(k)$ and that then ~~then~~ the twisted conjugation is equivalent to the conjugation action which one knows has a totally non-homologous to zero fibres. When $l \mid q-1$ ~~[illegible]~~ and there is a split torus T one can descend to ${}_lT \subset G(k)$ (Chevalley groups have split tori!)

February 8, 1970:

Let G be a connected algebraic group defined over $\mathbb{F}_q = k$. ~~[illegible]~~ Let F be the Frobenius endomorphism of G and let G_t be G as a variety regarded as a G-variety with action ~~[illegible]~~ map

$$G \times G_t \longrightarrow G_t$$

$$g, x \longmapsto gx\,Fg^{-1}$$

Then there is a map of G-varieties

$$G/G(k) \xrightarrow{\ \simeq\ } G_t$$

$$gG(k) \longmapsto g\,Fg^{-1}$$

and this map is an isomorphism since G is connected (thm. of Lang). Thus G_t is a homogeneous space of G, and consequently ~~[illegible]~~

$$H^*_G(G_t) \cong H^*(BG(k))$$

equivariant ~~[illegible]~~ étale cohomology with coefficients $\mathbb{Z}_\ell$.

We want now to compute $H^*(BG(k))$ by using the spectral sequence

$$(*) \qquad E_2^{pq} = H^p(BG) \otimes H^q(G_t) \Longrightarrow H^{p+q}_G(G_t)$$

The key result is the following

Theorem 1: Suppose that ~~[illegible]~~ ℓ is a good prime for G in the sense of Borel. (This means that the spectral sequence for (G, EG, BG) has the good form: $H^*(G)$ has a simple system of transgressive generators e_i, whence if $c_i \in H^*(BG)$ represents τe_i, then $H^*(BG)$ is a polynomial ring with generators c_i.) Then in $(*)$ the elements e_i are transgressive and τe_i is represented by $c_i - F^* c_i$.

Comments: 1) As $H^*(G)$ is a Hopf algebra, if l is odd, then all the e_i are of odd dimension and $H^*(G)$ is an exterior algebra.

2) From Borel's theory on knows that the suspension $QH(BG) \xrightarrow{\sim} PH(G)$ has the transgression for its inverse. It follows that $H^*(G)$ is generated by its primitives hence $H_*(G)$ is commutative and killed by Frobenius.

3) If $H_*(G)$ is an exterior algebra with odd degree generators then Eilenberg-Moore $\Longrightarrow$ $H^*(BG)$ is a poly. ring

Corollary: Under the conditions of the theorem suppose p is odd. Let P_0 be the kernel of $1 - F^*$ on $PH^*(G)$ and let Q_0 be the kernel of $1 - F^*$ on $QH^*(BG) = PH^*(G)$. Then

$$H^*(BG(k)) \simeq \Lambda(P_0) \otimes S(Q_0)$$

When $p = 2$, then $\quad gr\, H^*(BG(k)) \cong \Lambda(P_0) \otimes S(Q_0)$.

~~Write P as a direct sum $P = P_0 \oplus P_1$ respecting the grading. And lift the transgression $PH^*(G) \xrightarrow{\sim} QH^*(BG)$ to a map $PH^*(G) \longrightarrow QH^*(BG)$~~

Proof: Consider the spectral sequence $(*)$. The cohomology of the fibre $H^*(G^t) \cong H^*(G)$ is transgressively generated by $PH^*(G)$. Choose a splitting $P^* = P_0^* + P_1^*$ where $P_0 = \mathrm{Ker}(id - F^*)$ and let e_{i0}, e_{i1} be homogeneous bases for P_0^* and P_1^*. Then we can choose representatives for the transgression ∋

$$\tau(e_{i0}) = 0$$

$\tau(e_{i1})$ forms part of a generating system for $H^*(BG)$.

Then I can map a perfect spectral sequence into (*)

$$\mathbb{Z}_\ell[e_{i0}] \otimes \mathbb{Z}_\ell[e_{i1}, \tau(e_{i1})] \otimes \mathbb{Z}[c_{j1}] \longrightarrow H^*(BG) \otimes H^*(G^t)$$

and I find that

$$E_\infty = \mathbb{Z}_\ell[e_{i0}] \otimes \mathbb{Z}_\ell[c_{j1}] \simeq \Lambda P_0 \otimes S(Q_0).$$

Thus if $\ell \neq 2$ E_∞ is a free graded anti-commutative algebra, hence $H^*(BG(k))$ must ~~also~~ be isomorphic to E_∞.

The theorem will follow from a more general assertion.

Theorem: Assume $H^*(G)$ has a simple system $\{e_i\}$ of transgressive generators for the spectral sequence of (G, EG, BG) with transgression $\{\tau e_i\}$. Consider G as a homogeneous space for $G \times G$ by $(x, y) \cdot z = xyz^{-1}$. Thus G is the homogeneous space for the diagonal subgroup $\Delta : G \to G \times G$. Then given the simple system $\{e_i\}$ of generators is also transgressive in the fibration $(G \times G/\Delta G, B(\Delta G), B(G \times G))$ and the transgression of e_i in this last spec. sequence is represented by $\tau' e_i = \tau e_i \otimes 1 - 1 \otimes \tau e_i$.

To prove the theorem on page 1 we note that G acting on itself via the twisted conjugation is the **pull-back** of $G \times G$ acting on G by left + right multiplication by means of the map $(id, F): G \longrightarrow G \times G$. Thus the is a map of "fibrations" $(G, BG(k), BG) \longrightarrow (G, BG, B(G\times G))$ and as the e_i transgress in the latter to $\tau e_i \otimes 1 - 1 \otimes \tau e_i$, they must transgress to $(\tau e_i) - F^*(\tau e_i)$ in the former spectral sequence. q.e.d.

Proof of theorem on page 3: Consider the diagram

$$\begin{tikzcd}
 & H^{q-1}(G\times G) \arrow[r, "\delta", "\cong"'] & H^q(E(G\times G), G\times G) & \\
0 \arrow[r] & H^{q-1}(G) \arrow[u, "\varphi^*"'] \arrow[r, "\delta"] & H^q(BG, G) \arrow[u, "\psi^*"'] \arrow[r, "j"] & H^q(BG) \\
 & & \tilde{H}^q(BG\times BG) \arrow[u, "\pi"'] \arrow[ur, "\Delta^*"'] &
\end{tikzcd}$$

where $\varphi: G\times G \longrightarrow G$ $\varphi(x,y) = xy^{-1}$, and ψ comes from the homotopy equivalence $E(G\times G) \times_{G\times G} G \sim BG$. (Note that the map $G \longrightarrow BG$ coming from the inclusion of a fibre is homotopic to zero since $\varphi: G\times G \longrightarrow G$ admits a section.) The middle row is exact. Now given $e \in PH(G)$ we know that

$$\varphi^*(e) = e\otimes 1 - 1 \otimes e$$

and that

$$\delta\, \varphi^*(e) = \psi^*(\delta e) \in \operatorname{Im} \psi^* \pi^*$$

since $\varphi^*(e)$ is transgressive in $(G\times G, E(G\times G), BG\times BG)$.

Consider the element $\tau e \otimes 1 - 1 \otimes \tau e \in \tilde{H}^{8}(BG \times BG)$. Then

$$\pi^*(\tau e \otimes 1 - 1 \otimes \tau e) = \delta u$$

for a unique u in $H^{8-1}(G)$. Then as $\delta^{-1}\varphi^*\pi^*$ is the suspension we have

$$\begin{aligned} \delta \varphi^*(u) &= \varphi^*\pi^*(\tau e \otimes 1 - 1 \otimes \tau e) \\ &= \delta(e \otimes 1 - 1 \otimes e) \end{aligned}$$

and so $\varphi^*(u) = e \otimes 1 - 1 \otimes e$. But φ^* is injective so $u = e$. Thus

$$\delta e = \pi^*(e \otimes 1 - 1 \otimes e)$$

completing the proof of the theorem.

February 9, 1970

Question: Is $H^*(BG(k)) \longrightarrow H^*(BT(k))$ injective?
A related question is whether

1) $$H^*_G(G^c) \longrightarrow H^*_T(T^c)$$ ($G^c = G$ with conj. action)

is injective. I conjecture that this latter is the map

2) $$H^*(BG) \otimes H^*(G) \longrightarrow H^*(BT) \otimes H^*(T)$$
$$c_i \; , \; c_i \longmapsto c_i \; , \; dc_i$$

which in many cases is an isomorphism with the W-invariants.
Note that 2) is not free, since on killing $\bar{H}(BG)$ it is ~~the~~ the map

$$H^*(G) \longrightarrow H^*(G/T) \otimes H^*(T)$$

$$G^c \longleftarrow G/T \times T$$
$$gtg^{-1} \longleftarrow (gT, t)$$

which isn't free for $G = SU(2)$. Thus injectivity of 2) is due to other reasons.

Start with the map

$$f: G \times_N T \longrightarrow G^c$$

which can be rewritten $G/T \times_W T \longrightarrow G^c$. The map f is an isomorphism on G_{reg}, hence $f_* 1 = 1$ in cohomology. I claim that in fact $f_* 1 = 1$ in equivariant cohomology since $1 \in H^0_G(G \times_N T)$ has nowhere to go but into 1. Thus quite generally

3) $$f^*: H^*_G(G^c) \longrightarrow H^*_N(T)$$

is injective. By Hockschild-Serre there is a spectral sequence

$$H^*(N/T, H^*_T(T)) \Longrightarrow H^*_N(T)$$

which degenerates at primes not dividing the order of W giving an isomorphism

$$H^*_N(T) = \{H^*(BT) \otimes H^*(T)\}^W$$

Question: Any relation between $\varinjlim_k Gl_n(\mathbb{F}_{p^k})$ and $Gl_{n,\overline{\mathbb{F}}_p}$. The conjecture is that

$$\varinjlim_k B\,Gl_n(\mathbb{F}_{p^k}) \longrightarrow (B\,Gl_{n,\overline{\mathbb{F}}_p})_{et}$$

is the "off-p" completion of the former. (Recall that if X is a space, it has a completion with respect to any "class" of groups. In this case we take all finite groups of order prime to p. On the right we take the etale homotopy type of the classifying topos of $Gl_{n,\overline{\mathbb{F}}_p}$.)

~~Note that the homotopy groups of the simplicial set on the left are finite. Here is an example to fortify our nerves:~~

~~Consider the ℓ-completion à la Artin-Mazur of the group $\hat{\mathbb{Q}}_\ell/\hat{\mathbb{Z}}_\ell$. What this means is that we must pro-represent the functor~~

$$\cancel{X \mapsto [K(\hat{\mathbb{Q}}_\ell/\hat{\mathbb{Z}}_\ell, 1), X] = [\varprojlim S^1 \xleftarrow{\ell} S^1, X]}$$

if X has finite homotopy groups of order powers of ℓ.

$$0 \to R^1 \varprojlim \{\pi_2 X \xleftarrow{\ell} \cdots\} \to [\varinjlim S^1 \xrightarrow{\ell} S^1, X] \to \varprojlim \{\pi_1 X \xleftarrow{\ell} \pi_1 X\} \to 0$$

$$R^1 \varprojlim \{\pi_2 X \xleftarrow{\ell} \cdots\} = 0, \qquad \varprojlim \{\pi_1 X \xleftarrow{\ell} \pi_1 X\} = 0$$

Thus $\hat{Q}_\ell / \hat{\mathbb{Z}}_\ell$ completed is zero.

Recall that it is easy to compute the cohomology groups of a completion

$$H^q(\hat{X}, \mathbb{Z}_{\ell^\nu}) = [\hat{X}, K(\mathbb{Z}_{\ell^\nu}, q)]$$

$$= [X, K(\mathbb{Z}_{\ell^\nu}, q)]$$

$$= H^q(X, \mathbb{Z}_{\ell^\nu})$$

Let's use the cohomology criterion! Then

$$H^*(BGl_n(\overline{\mathbb{F}}_p), \hat{\mathbb{Z}}_\ell) = \hat{\mathbb{Z}}_\ell [c_1, \cdots, c_n]$$ (take inverse of spec. seq. over $\hat{\mathbb{Z}}_\ell$)

$$H^*(BGl_{n, \overline{\mathbb{F}}_p}, \hat{\mathbb{Z}}_\ell) = \text{"}$$

which shows that the conjecture is correct. (Note that $BGl_n(\overline{\mathbb{F}}_p)\hat{}$ is simply-connected since ~~[illegible]~~ $Gl_n(\overline{\mathbb{F}}_p)$ has no finite quotient groups, since ~~[illegible]~~ $Sl_n(\overline{\mathbb{F}}_p)$ would have to go to zero in such a thing by simplicity and since $\overline{\mathbb{F}}_p^{\,*}$ has no finite quotients, being divisible.)

February 10, 1970

Proposal for a definition of algebraic K-theory:

Recall that if G is a group it has a "classifying space" simplicial set $\overline{W}(G)$. The functor $\overline{W}$ commutes with direct products.

Now given a ring R let

$$X(R) = \coprod_{n \geq 0} \overline{W}(Gl_n(R)) \qquad Gl_0(R) = \{1\}$$

Then $X(R)$ is a simplicial set which depends functorially on R and commutes with direct products (in fact arbitrary inverse limits). Define a simplicial monoid structure on $X(R)$ by defining

$$X_p(R) \times X_q(R) \longrightarrow X_{p+q}(R)$$

to be the map induced by the natural inclusion

$$Gl_p \times Gl_q \longrightarrow Gl_{p+q}.$$

Note that this is associative since the diagram of groups

$$\begin{tikzcd}
Gl_p \times Gl_q \times Gl_r \arrow[r] \arrow[d] & Gl_p \times Gl_{q+r} \arrow[d] \\
Gl_{p+q} \times Gl_r \arrow[r] & Gl_{p+q+r}
\end{tikzcd}$$

is commutative.

Since $X(R)$ is a simplicial monoid, it has a "classifying space" $\overline{W}(X(R))$. We propose the following definition

$$K_i(R) = \pi_{i+1}\,\overline{W}(X(R)) \qquad i \geq 0. \qquad \left(\begin{array}{l} i=0 \text{ get } \mathbb{Z} \\ \text{so wrong} \end{array}\right)$$

This is reasonable because $\Omega\overline{W}(X(R))$ is the invertible H-space generated by the simplicial monoid $X(R)$.

The problem with this definition is that $K_0(R)$ seems to be $\mathbb{Z}$. Thus $\overline{W}(X(R))_1 = X(R)_0 = \coprod_{n\geq 0} c$. It seems that

$$[\overline{W}(X(R)), \overline{W}(\pi)] = [X(R), \pi] = \pi$$

for any group π. Thus as $\overline{W}(\pi)$ is a Kan complex one sees that $\pi_1 \overline{W}(X(R)) = \mathbb{Z}$.

This shows that we have failed to get the correct K_0. Somehow we have failed to understand putting in direct summands of free modules. Might there be any way of of obtaining the category of projective modules from the category of free modules and automorphisms? Unfortunately idempotents are not automorphisms.

assume your calculations are correct so that

M monoid

$$\begin{array}{c} EM \\ \downarrow \\ BM \end{array}$$

---

semi-simp.

$$\begin{array}{ccc} \bar{W}(M) \times_\tau M & & \bar{W}(M) \times_\tau G(\bar{W}(M)) \\ \downarrow & & \downarrow \\ \bar{W}(M) & & \bar{W}(M) \end{array}$$

$\bar{W}(M) \longrightarrow \bar{W}(M)$ if M gp $\Longrightarrow$ $G\bar{W}(M) \to M$.

but you do get

$\bar{W}($

$$\begin{array}{c} \bar{W}(M) \times_\tau M \\ \downarrow \\ \bar{W}(M) \end{array}$$ contractible ~~so is not clear.~~

question: M

$$\begin{tikzcd} M \arrow[r] \arrow[d] & G(\bar{W}(M)) \arrow[d] \\ W(M) \arrow[r, "\mathrm{we}"] \arrow[d] & E\bar{W}(M) \arrow[d, "\text{fibration}"] \\ \bar{W}(M) \arrow[r, "\mathrm{id}"] & \bar{W}(M) \end{tikzcd}$$

and the question is whether this map from M to $G(\bar{W}(M))$ is a homology isomorphism or if

February 12, 1970. norms + non-ab. class field theory

Conjecture: Let R be a henselian local ring with finite residue field k and let l be a prime number different from char(k). Then the map

$$Gl_n(R) \longrightarrow Gl_n(k)$$

induces an isomorphism on group cohomology modulo l. More generally this should hold for any algebraic group defined over R.

Examples: 1) $n=1$. Then we have an exact sequence

$$0 \longrightarrow 1+m \longrightarrow R^* \longrightarrow k^* \longrightarrow 0.$$

because the equation $aX = 1$ $a \in R$, $a \not\equiv 0 \pmod{m}$ has a root by Hensel's lemma. Since $1+m$ is uniquely l-divisible it has no cohomology modulo l.

2) Suppose G is a nilpotent algebraic group over R. Then $G(R)$ is a nilpotent group which is uniquely l-divisible. Here one sees $G(R)$ has no cohomology mod l by means of a composition series for G which reduces one to the abelian case.

The situation probably ~~must~~ should be generalized: Let G be a group scheme over S and consider the simplicial object

$$\overline{W}(G): \qquad G\times G \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G \;\substack{\longrightarrow\\ \longrightarrow}\; 1$$

as a simplicial object in the category of ~~étale~~ sheaves for the étale topology on Sch/S. According to Deligne(?) there is a spectral sequence

$$E_2^{pq} = \check{H}^p(\nu \mapsto H^q(U, G^{\nu+1})) \Longrightarrow \text{something}$$

where U is an object over S. I think the "something" is

$$H^*(BG_U)$$

which is the cohomology of the classifying topos of the group $G_U = U \times_S G$. ~~If this is all correct, then suppose that the higher cohomology of $G^{\nu+1}$ are all zero.~~ For example suppose U is the spec. of an algebraically closed field. Then

$$H^*(U, G^{\nu+1}) = H^*(U, G)^{\nu+1}$$

and $H^*(U, G)$ is an exterior algebra in good cases. This unfortunately shows that you are not getting what you want.

Problem: Is there any way to get a spectral sequence involving the cohomology of the group $G(S)$?

The corresponding topological problem is this: Suppose S is a fixed space and G is a sheaf of groups over S. Then can one relate the cohomology of G and the cohomology of $G(S)$?

February 16, 1970:

Take the spectrum of the family of symmetric groups:

$$X = \coprod_{n\geq 0} \operatorname{Spec} H^*(B\Sigma_n) \qquad \text{cohomology mod } p.$$

Since $G \longmapsto \operatorname{Spec} H(BG)$ is covariant and commutes with products, X is a commutative monoid scheme over $\operatorname{Spec} \mathbb{Z}_p$ (provided we work with perfect schemes.) For each integer $a \geq 0$ there is a map

$$\begin{array}{ccc} \operatorname{Spec}\left\{H^*(B\mathbb{Z}_p^a)^{Gl_a(\mathbb{Z}_p)}\right\} & \xhookrightarrow{\text{closed immersion}} & X_{p^a} \\ \wr\| & & \\ A^a = \operatorname{Spec} \mathbb{Z}_p[c_{p^a-p^{a-1}}, \ldots, c_{p^a-1}] & & \end{array}$$

and the conjecture is that X is a free commutative monoid with these generators possibly with some relations. Example of a relation: The diagram of groups

$$\begin{tikzcd} \mathbb{Z}_p^a \arrow[r] \arrow[d] & \Sigma_{p^a} \arrow[r, "\Delta"] \arrow[d, "\mathrm{in}_1"] & \Sigma_{p^a}^p \arrow[d, "\text{via wreath product}"] \\ \mathbb{Z}_p^{a+1} \arrow[r] & \Sigma_{p^a} \times \mathbb{Z}_p \arrow[r] & \Sigma_{p^{a+1}} \end{tikzcd}$$

gives rise to a diagram of rings

$$\begin{tikzcd} (c_{p^a-p^{i-1}})^p \quad 0 & \mathbb{Z}_p[c_{p^a-p^{a-1}}, \ldots, c_{p^a-1}] & H^*(B\Sigma_{p^a}) \arrow[l] & H(B\Sigma_{p^a})^{\otimes p} \arrow[l] \\ c_{p^{a+1}-p^i} \arrow[u, mapsto] \quad c_{p^{a+1}-1} \arrow[u, mapsto] & \mathbb{Z}_p[c_{p^{a+1}-p^a}, \ldots, c_{p^{a+1}-1}] \arrow[u] & H^*(B\Sigma_{p^{a+1}}) \arrow[l] \arrow[ur] & \end{tikzcd}$$

which shows that

$$\begin{tikzcd} (x_{a-1}, \cdots, x_0) \arrow[d, mapsto] & \mathbb{A}^a \arrow[r, hook] \arrow[d] & X_{p^a} \arrow[d, "\text{multiplication by } p"] \\ (x_{a-1}^p, \cdots, x_0^p, 0) & \mathbb{A}^{a+1} \arrow[r, hook] & X_{p^{a+1}} \end{tikzcd}$$

commutes.

Perhaps it is better to use that there is a map

$$\begin{tikzcd} \operatorname{Spec}\{H^*(B(\mathbb{Z}_p \wr \Sigma_n))\} \arrow[r] & \operatorname{Spec}\{H^*(B\Sigma_{pn})\} \\ \mathbb{A}^1 \times \operatorname{Spec} H^*(B\Sigma_n) \amalg (\operatorname{Spec} H^*(B\Sigma_n))^p . \arrow[u] & \end{tikzcd}$$

Thus there is a basic map

$$\begin{tikzcd} \mathbb{A}^1 \times X_n \arrow[r] & X_{pn} \\ 0 \times X_n \arrow[u] \arrow[ur, "\text{mult. by } p \text{ in } X"'] & \end{tikzcd}$$

such that

commutes. Moreover

$$\begin{tikzcd} (\mathbb{A}^1)^a \times X_1 \arrow[r] \arrow[d, "\text{divide by } Gl_a(\mathbb{Z}_p)"] & X_{p^a} \\ \mathbb{A}^a \arrow[ur, hook] & \end{tikzcd}$$

the map considered before.

Suppose you have a numbers $\lambda_1, \dots, \lambda_a$. Do you have any felling for Dicksons invariants

$$\sum_{i=0}^{a} t^{p^i} c_{p^a - p^i}(\lambda_1, \dots, \lambda_a) = \prod_{0 \le j_1, \dots, j_a < p} \left(t + \sum_{i=1}^{a} j_i \lambda_i\right)$$

which generate the ~~functions~~ functions $\varphi(\lambda_1, \dots, \lambda_a)$ invariant under $Gl_a(\mathbb{Z}_p)$. Thus if the operation

$$\mathbb{A}^1 \times X \longrightarrow X$$

is denoted Γ we know that

$$\Gamma(\lambda_1, \dots, \lambda_a) \cdot x$$

is a $Gl_a(\mathbb{Z}_p)$-invariant of $\lambda_1, \dots, \lambda_a$.

Simpler example:

$$\coprod_{n \ge 0} \operatorname{Spec} H^*(BGl_n) \cong \coprod_{n \ge 0} SP_n(\mathbb{G}_a)$$

which is the free ~~[illegible]~~ commutative monoid generated by $\mathbb{G}_a = \mathbb{A}^1$.

February 17, 1970.

Question: Can you compute the cohomology or homotopy groups of $\Omega B\{\coprod_n BGl_n(\mathbb{F}_q)\}$?

The idea suggested by conversation with Sullivan is to regard $BGl_n(\mathbb{F}_q)$ as the homotopy kernel of the pair $BGl_n \underset{id}{\overset{F}{\rightrightarrows}} BGl_n$ where BGl_n denotes the classifying topos of the group Gl_n over $\bar{\mathbb{F}}_q$, and then form the "exact sequence" of monoids

$$\coprod_n BGl_n(\mathbb{F}_q) \longrightarrow \coprod_n BGl_n \rightrightarrows \coprod_n BGl_n$$

which hopefully should be preserved by passage to classifying spaces. If everything works then we expect a fibration

$$\Omega B\Big(\coprod_n BGl_n(\mathbb{F}_q)\Big)^{(\frac{1}{p})} \longrightarrow (\mathbb{Z}\times BU)(\tfrac{1}{p}) \xrightarrow{\psi^q - 1} (\mathbb{Z}\times BU)(\tfrac{1}{p})$$

which on taking the long exact sequence of homotopy groups should yield the exact sequence

$$\begin{array}{ccccc}
\to K_1(\mathbb{F}_q)(\frac{1}{p}) & \longrightarrow & 0 & \longrightarrow & 0 \\
\to K_2(\mathbb{F}_q)(\frac{1}{p}) & \longrightarrow & \mathbb{Z}(\frac{1}{p}) & \xrightarrow{q-1} & \mathbb{Z}(\frac{1}{p}) \\
\to K_3(\mathbb{F}_q)(\frac{1}{p}) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

from which we obtain the formulas

$$K_{2j}(\mathbb{F}_q)_{(\frac{1}{p})} = 0 \qquad j>0$$

$$K_{2j-1}(\mathbb{F}_q)_{(\frac{1}{p})} = \mathbb{Z}/q^j-1 \ , \ j>0$$

Given a space X I form

$$\coprod_n E\Sigma_n \times_{\Sigma_n} X^n$$

which is a monoid. I conjecture that

$$\Omega B \coprod_n (E\Sigma_n \times_{\Sigma_n} X^n)$$

is the free homotopy ~~[illegible]~~ symmetric H-space generated by X and is ~~[illegible]~~ naturally isomorphic to

$$\varinjlim_n \Omega^n \Sigma^n X. = Q(X)$$

One method of testing this conjecture would be to ~~[illegible]~~ produce a map and then check that it induces ~~[illegible]~~ an isomorphism on cohomology. But a much better method would be to produce the ~~[illegible]~~ (generalized) cohomology theory involved.

So the idea might be to produce ~~[illegible]~~ the functor represented by

$$B \coprod_n (E\Sigma_n \times_{\Sigma_n} X^n).$$

First the functorial aspect: Given a space X, I can form the monoid

$$M(X) = \coprod_n E\Sigma_n \times_{\Sigma_n} X^n$$

and its classifying space $BM(X)$. ~~[illegible]~~ ~~[illegible]~~ Moreover M is a triple (co-?)

$$M(X) \longleftarrow M(M(X))$$

$$X \longrightarrow M(X)$$

where the first map comes from the wreath product maps

$$E\Sigma_j \times_{\Sigma_j} (E\Sigma_k \times_{\Sigma_k} X^k)^j = E(\Sigma_j \wr \Sigma_k) \times_{\Sigma_j \wr \Sigma_k} X^{jk}$$

For this it is necessary to check that

$$W(\Sigma_k \wr G) = W\Sigma_k \times (WG)^k$$

but in dimension q we have

$$W_q(\Sigma_k \wr G) = (\Sigma_k \times G^k)^q = \Sigma_k^q \times (G^q)^k = W_q(\Sigma_k) \times W_q(G)^k$$

so it's OKAY.

Now if X is a homotopy symmetric H-space, we are given maps

$$M(X) \longrightarrow X$$

which make X is into an algebra over the triple i.e.

$$X \longrightarrow M(X) \xrightarrow{\mu} X \qquad (\text{composite} = id)$$

$$M(M(X)) \underset{\text{can.}}{\overset{M(\mu)}{\rightrightarrows}} M(X) .$$

~~[illegible]~~ Now the next step is to understand inverting.

Suppose we admit Boardman's theorem that an invertible homotopy symmetric H-space extends uniquely to a connected generalized cohomology theory. Apply this to $\Omega BM(X) = J(X)$. I ought to be able to show that $J(X) \xrightarrow{\sim} Q(X)$.

First of all $Q(X)$ is a homotopy symmetric H-space so there is a map

$$M(X) \longrightarrow Q(X)$$

and since $Q(X)$ is invertible this extends to a map

$$J(X) \longrightarrow Q(X).$$

On the other hand by Boardman one gets a connected gen. coh. theory with spectrum

$$E_i = \Omega^{-i+n} B^n M(X) \qquad n \geq 1$$

and there is a canonical map $X \longrightarrow E_0$ whence a morphism of spectra

$$\varinjlim_n \Omega^{-i+n} S^n X \longrightarrow E_i$$

defined by

$$X \longrightarrow E_0 = \Omega^n E_n \Rightarrow S^n X \longrightarrow E_n \Longrightarrow \Omega^{-i+n} S^n X \longrightarrow \Omega^{n-i} E_n = E_i$$

which yields ~~this basic map~~ (take $i=0$) a map

$$Q(X) \longrightarrow J(X).$$

So there are maps in both directions which is a good indication.

To put this all on a good footing all that is needed is an argument which will allow you to extend ~~pass from~~ a functor with traces for coverings to a gen. coh. theory. ~~to it~~

February 17, 1970.

Some comments on your conjecture that $BG(R) \to BG(k)$ is a cohomology isomorphism mod ℓ, where R is a henselian d.v.r. with residue field k of char $p \neq \ell$.

(One technique would be to deduce it somehow from the ~~comparison~~ specialization theorems for etale cohomology. Thus take $k = \mathbb{F}_q$, whence we can realize $BG(k)$ as ~~the~~ a fibre "space" over BG_k, or better we get our hands on $G(k)$ through its action on the scheme G?)

The covering

$$1 \longrightarrow G(k) \longrightarrow G_k \xrightarrow{x \mapsto x(Fx)^{-1}} G_k \longrightarrow 1 \qquad \left(G_k = Sp(k) \times_{Sp(R)} G\right)$$

is a principal $G(k)$ bundle hence gives rise to a map

$$\pi_1(G_k) \twoheadrightarrow G(k)$$

which is surjective since G is connected. Now if G were proper over Spec R, then one would have an isomorphism

$$\pi_1(G) \xleftrightarrow{\cong} \pi_1(G_k)$$

by the theorem on specialization of the fundamental group, more precisely the categories of etale coverings of G and G_k are equivalent. Hence to the above covering of G_k by itself would correspond a Galois covering of G

$$G(k) \longrightarrow Z \longrightarrow G$$

with group $G(k)$. Now what?

General problem: Let G be a group in a topos $\mathcal{T}$ and let $f: \mathcal{T}_G \longrightarrow \mathcal{T}$ be the classifying topos of G. Is there anything that one can say about the cohomology of the group $G(S)$ where S is an object of $\mathcal{T}$?

One has a natural map of groups in $\mathcal{T}/S$

$$\begin{tikzcd} G(S)_S \arrow[rr] \arrow[dr] & & G_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $G(S)_S$ is the constant group $S \times G(S)$ over S. Consequently one has a homomorphism of abelian sheaves in $\mathcal{T}/S$

$$(*) \qquad R^q (f_S)_*(M) \longrightarrow H^q_{G(S)}(M)$$ (just take discrete group cohomology of M fibre by fibre)

for any abelian sheaf M of $\mathcal{T}_G/S$. If you take a point $s \in S$ then the stalk of the latter is just the cohomology of $G(S)$ acting on the the abelian group M_s

$$H^q_{G(S)}(M)_s = H^q_{G(S)}(M_s).$$

Example: Take $\mathcal{T}$ to be the category of π-sets so that G is a group on which π-acts and $\mathcal{T}_G = \text{Sets}_{\pi \times_s G}$. Then if M is a $\pi \times_s G$-module

$$R^q f_*(M) = H^q_G(M)$$

with its natural π-action. If $S = \pi/\rho$, then the map $*$ above is the restriction homomorphism

$$H^q_G(M) \longrightarrow H^q_{G^\natural}(M)$$

with the natural ρ-action.

~~Let G be an algebraic group defined over [illegible] a henselian d.v.r. with residue field $\mathbb{F}_q$.~~

Let G be a connected algebraic group defined over $\mathbb{F}_q$. Then the covering $G/G(\mathbb{F}_q) \longrightarrow G$ defines a surjection

$$\pi_1(G) \twoheadrightarrow G(\mathbb{F}_q)$$

If G is abelian we have the following diagram

$$\begin{tikzcd}
G(\mathbb{F}_{q^n}) \arrow[r] \arrow[d, "\mathrm{norm}"] & G \arrow[r, "x \mapsto x - F^n x"] \arrow[d, "x \mapsto \sum_{i=0}^{n-1} F^i x"] & G \arrow[d, "\mathrm{id}"] \\
G(\mathbb{F}_q) \arrow[r] & G \arrow[r, "x \mapsto x - Fx"'] & G
\end{tikzcd}$$

so that we end up with a homomorphism

$$\pi_1(G) \twoheadrightarrow \varprojlim_n G(\mathbb{F}_{q^n}) \qquad \text{(for the norm maps)}$$

For example

$$\pi_1(\mathbb{G}_m) \twoheadrightarrow \prod_{\ell \neq p} T_\ell(\mathbb{G}_m)$$

$$\pi_1(\mathbb{G}_a) \twoheadrightarrow \prod_{\infty} \mathbb{Z}/p\mathbb{Z}$$

In fact

$$\mathrm{Hom}(\pi_1 \mathbb{G}_m, \mathbb{Z}/\ell\mathbb{Z}) \cong H^1(\mathbb{G}_m, \mathbb{Z}/\ell\mathbb{Z}) \cong \frac{\bar{k}[T,T^{-1}]^*}{(\bar{k}[T,T^{-1}]^*)^\ell} \cong \mathbb{Z}/\ell\mathbb{Z}$$

because the ~~xx~~ units in $k[T,T^{-1}]$ are $\langle k^*, T^n \rangle$. Also by Artin-Schreier

$$\mathrm{Hom}(\pi_1 \mathbb{G}_a, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Coker}\left\{ k[T] \xrightarrow{T \mapsto T^p - T} k[T] \right\}.$$

If G is non-abelian, there seems to be no good way to connect the various ~~xx~~ homomorphisms

$$\pi_1(G) \twoheadrightarrow G(\mathbb{F}_q)$$

together. Thus for $G = PSl_2$ the groups $G(\mathbb{F}_q)$ are simple and can't be mapped to each other.

New idea for non-abelian class field theory.(?) According to the general yoga in Serre's book one obtains the abelian ~~x~~ coverings of a variety V by mapping $f: V \to A$ where A is an abelian algebraic group (Albanese variety?) and pulling back an isogeny $B \to A$. The next thing to do is to replace A by ~~a~~ homotopy symmetric animal.

<u>Serre's idea</u>: Suppose K is a local field and $K \to L$ is a finite extension. Then one gets an isogeny

$$\mathrm{Norm}: U_L \to U_K$$

of pro-algebraic groups whose kernel is canonically ~~x~~ isomorphic to $\mathrm{Gal}(K/L)$ (needs checking).

From course notes of Iwasawa:

$(k : \mathbb{Q}) < \infty$.

$\mathcal{O}$ = ring of integers of k.

S = some collection of non-archimedean prime divisors of k.

For $v \in S$, $\mathfrak{p}_v$ is the corresponding prime ideal of $\mathcal{O}$.

$\mathcal{O}' = \{x \mid x \in k$, denominator of (x) is a product of prime ideals of the form $\mathfrak{p}_v$, $v \in S$, for $x \neq 0\}$.

I = ideal group of $\mathcal{O}$.

I' = " of $\mathcal{O}'$ = group of finitely-generated non-zero $\mathcal{O}'$-submodules of k.

FACTS:

① $\varphi : I \longrightarrow I'$ given by $\mathfrak{a} \longmapsto \mathfrak{a}\mathcal{O}'$, $\mathfrak{a} \in I$, is a surjective group homomorphism.

② $H = \mathrm{Ker}(\varphi)$ = subgroup of I generated by all $\mathfrak{p}_v$, $v \in S$.

③ Let P be the subgroup of principal ideals of I

P' = subgroup of I' generated by the "principal S-ideals": $a \cdot \mathcal{O}'$, $a \in k^\times$.

$$C = I/P.$$

φ maps P onto P' and induces surjection

$$C \longrightarrow C' = I'/P'$$

with kernel HP/P. So we have the exact sequence

$$0 \longrightarrow HP/P \longrightarrow C \longrightarrow C' \longrightarrow 0.$$

In particular if $\mathfrak{p}_v$, $\forall v \in S$, is a principal ideal of $\mathcal{O}$,

$$C \xrightarrow{\sim} C'$$

which gives the answer to your query.

February 18, 1970:

Attempt to understand Boardman's theorem on the construction of conn. gen. coh. theories for homotopy symmetric H-spaces.

I want to start with a contravariant functor k on the category of ~~[illegible]~~ C^∞-manifolds to Ab which is endowed with ~~a~~ Gysin homomorphisms for finite covering maps with the following properties

(i) base change

(ii) composition

(iii) disjoint union

This to me is ~~that~~ how a homotopy symmetric H-space should be defined.

~~Boardman's theorem should say that gr~~

Example of how you get such a k: Let h^* be a gen. coh. theory, that is, a contravariant functor from the category of manifolds to the category of graded abelian groups which is endowed with Gysin homomorphisms for proper framed maps. Let k be h^0.

Boardman's theorem should say that the forgetful functor

$$h^* \longmapsto h^0$$

$$(\text{gen. coh. th.}) \longrightarrow (\text{trace theories})$$

has a faithful left adjoint which ~~[illegible]~~ allows one to identify trace theories with the full subcategory of connected ($h^+(pt)=0$) gen. coh. theories.

(In all the above add the word <u>representable</u>!)

Starting with the category of compact C^∞-manifolds ~~the~~ we form the category of motives; this should be the suspension category and its key property is that ~~additive~~ a functor to Ab with Gysin homomorphisms for proper framed degree 0 maps on manifolds extends to an additive functor from the suspension category to Ab.

we are given a ~~[illegible]~~ functor $h(X) = [X, \overset{K}{\cancel{B}}]$
on the category of finite complexes endowed with
(i) abelian grp. structure
(ii) traces for finite coverings. Axioms
coverings can be composed, unioned and base changed
I want to show how to extend h to a generalized coh. theory.
~~[illegible]~~
~~[illegible] part of this relies on being~~
If $h = h^0$ of a gen. coh. theory, then I can define f_* for any framed map of degree 0. $f: Z \longrightarrow X$

Conversely if you have succeeded in doing this much then maybe you have a way of extending to ~~[illegible]~~ a gen. coh. theory!

~~[illegible]~~ $h^{+q}(X) = ?$ $q > 0$
If I know h^{q-1} can I extend? So I take X. If $X = SY$ I know what to do so I look at

$$[X, B(K)] = [\Omega X, K]_{\text{grps.}}$$

so therefore $h^1(X)$ is something like $P h^0(\Omega X)$.
Similarly

$$h^q(X) = [X, B^q(K)] = [\Omega^q X, K]_{q\text{th order groups.}}$$

$$= \underset{\uparrow}{P}\, h^0(\Omega^q X)$$

hyper-primitive functor order q.

now the feeling I have is that the commut. structure of $\Omega^q X$ is difficult to understand because of instability. Thus the thing to do is first show

$$h^0(X) = [Q(X), K]_{\text{hom. sym. } H\text{-spaces.}}$$

$[?, Q(X)] = \pi^0(?;X)$. So suppose you can prove that

$$h^0(X) = \text{natural transf from } \pi^0(?;X) \text{ to } h^0(?) \text{ compatible with traces}$$

(The arrow $\longleftarrow$ is clear. For the arrow $\longrightarrow$ one needs only to construct Gysin hom. for framed maps!). Then I can define

$$h^q(X) = \text{natural transf. from } \pi^{-q}(?;X) \text{ to } h^0(?) \text{ compatible with traces.}$$

(get signs correct. $h^q(X) = h^0(S^{-q}X) = \mathrm{Hom}(\pi^0(?;S^{-q}X), h^0(?))$

$$\pi^0(?;S^{-q}X) \cong \{?, S^{-q}X\}$$

$$\pi^{-q}(?;X) = \pi^0(S^q ?;X) = \{S^q ?, X\} \quad)$$

Note that

$$h^q(pt) = \left[\frac{\text{natural transf. from } \pi^{-q}(?,pt) \text{ to } h^0(?)}{\text{trace compatible}}\right]$$

is intuitively $h^0(S^{-q}) = h^q(pt) = \pi_0(E_q) = 0$ since $q \geq 1$ hopefully $E_\cdot$ is a connected spectrum.

This aspect of the theory should be formal. Why a functor $X \longmapsto h^0(X)$ + Gysin for degree 0 extends.

Lemma: $h^0(X) = \mathrm{Hom}_{\mathcal{G}}(\pi^0(?;X), h^0(?))$ $\mathcal{G}$ contra functors on manifolds to ab ~~with~~ Gysin for proper degree 0 framed maps.

Suppose you have an element $\xi \in h^0(X)$ and an element α of $\pi^0(Y;X)$ i.e. a diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & X \\ Y & \end{tikzcd}$$

where f is proper framed of degree 0. Then can define $\alpha^*(\xi) = f_* g^* \xi \in h^0(Y)$. By the axioms this should be natural.

~~Next suppose that X is a finite~~

Point is that a functor with Gysin of degree is an (additive) functor on the suspension cat.

$$\begin{cases} \mathrm{Obj} = \text{f.c. } X \text{ with basepoint} \\ \mathrm{Hom}_{\mathrm{Susp}}(X,Y) = \{X,Y\}_. = \pi^0(X;Y) \end{cases}$$

so I have $\{h^0 : \mathrm{Susp} \longrightarrow ab\}$ $h^0(X) = \mathrm{Hom}(h_X, h^0)$

~~Next I try to make~~

February 18, 1970:

(don't believe argument)

Sullivan claims that $BU^{\otimes}$ not isomorphic as an h-space to ΩX where X is a space with finite k-invariants, and ~~p~~ in fact

$$BU^{\otimes} \underset{h}{\simeq} \Omega X \qquad X \text{ has finite k-invariants } X \text{ conn.}$$

$$\Downarrow$$

$$\exists\ \varphi : X \simeq SU \qquad \text{at each prime } p$$

$$\Downarrow$$

$$BU^{\otimes} \underset{h}{\simeq} \Omega X \underset{\Omega(\varphi)\ h\text{ isom}}{\simeq} \Omega SU \underset{h}{\simeq} BU^{\oplus} \qquad \text{at each } p$$

but this is false since $[X, BU^{\otimes}] = (1+\bar{K}(X))^{\times}$ is not isomorphic to $[X, BU^{\oplus}] = \bar{K}(X)$, take $X = B\mathbb{Z}_p$ former has an element of order p and the latter is torsion-free, in fact it is free of rank $p-1$ over $\hat{\mathbb{Z}}_p$.

Work with the prime 2 take $X = RP^4$ where

$$\bar{K}(RP^4) = \mathbb{Z}_4(H-1)$$

where H = Hopf bundle. Note that

$$(H-1)^2 = 1 - 2H + 1 = 2(1-H) \neq 0$$

since

$$c_2(2(1-H)) = (c_1(1-H))^2 = (-\omega)^2 = \omega^2 \neq 0.$$

Now so we start with $\Omega X \underset{h}{\cong} BU^{\otimes}$ and try to deduce a contradiction. Now the homotopy groups of X must be

$$\begin{array}{lccccc} \pi_q(X) & 0 & 0 & \mathbb{Z} & 0 & \mathbb{Z} & \cdots \\ q & 1 & 2 & 3 & 4 & 5 \end{array}$$

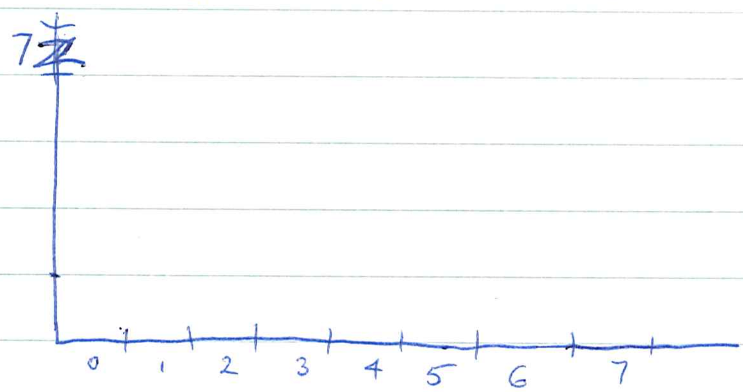
Let $X(0,5)$ be the part of the Postnikov system of X with the same homotopy groups $0 \le q \le 5$ as X.

$$\begin{array}{c} K(\mathbb{Z},5) \\ \downarrow \\ X(0,5) \\ \downarrow \\ K(\mathbb{Z},3) \longrightarrow K(\mathbb{Z},6) \end{array}$$

Now one knows that $H^6(K(\mathbb{Z},3),\mathbb{Z}) = \mathbb{Z}_2(\beta Sq^2)$ and the k-invariant of X must be different from zero, otherwise we have

$$X(7,\infty) \longrightarrow X \longrightarrow K(\mathbb{Z},3) \times K(\mathbb{Z},5)$$

so



$$H^q(K(\mathbb{Z},3)\times K(\mathbb{Z},5)) \xrightarrow{\approx} H^q(X) \qquad q \le 6$$

$$0 \longrightarrow H^7(K(\mathbb{Z},3)\times K(\mathbb{Z},5)) \longrightarrow H^7(X) \longrightarrow H^7(X(7,\infty)) \xrightarrow{d}$$

and therefore we would have $H^6(X) = \mathbb{Z}_2$. **Still no contradiction.**

Sullivan claims that if $BU^{\otimes} \underset{h}{\simeq} \Omega X$ as h-spaces, then

$$\pi_{2i+1}(X) = \mathbb{Z} \qquad i \geq 1$$
$$\pi_{2i}(X) = 0 \qquad i \geq 1$$

and that

$$H^{2i+2}(X(0,\cdots,2i-1), \mathbb{Z})$$

is cyclic generated by k_{2i+1}

$$\begin{array}{ccccc} & & K(\mathbb{Z}, 2i+1) & & \\ & & \downarrow & & \\ & & X(0,\cdots,2i+1) & & \\ & & \downarrow & & \\ K(\mathbb{Z},2i-1) & \longrightarrow & X(0,\cdots,2i-1) & \xrightarrow[k_{2i+1}]{} & K(\mathbb{Z}, 2i+2) \end{array}$$

and that

$$H^{2i+2}(K(\mathbb{Z}, 2i-1)) = \mathbb{Z}_2 \cdot \beta Sq^2 u_{2i-1}$$

is generated by the restriction of k_{2i+1}

He might get in trouble right at the bottom. Sullivan's argument consists in the following inductive set. He assumes constructed an isomorphism of $X(0,2i-1)$ and $SU(0,2i-1)$ at the prime 2. Then because SU has torsion-free homology one knows that the k invariant in

$$H^{2i+2}(SU(0,2i-1), \mathbb{Z})$$

generates ~~this~~ the cyclic torsion subgroup and restricts to generate $H^{2i+2}(K(\mathbb{Z}, 2i-1), \mathbb{Z}) = \mathbb{Z}_2(\beta Sq^2)$ (here must have $i \geq 2$).
The k invariant for $X(0, 2i+1)$ also generates the H^{2i+2} of the fibre because of the isom. $\Omega X \simeq BU$, so by Nakyama this k invariant must generate $H^{2i+2}(X(0, 2i-1), \mathbb{Z})$ (working at 2) (here must have $i \geq 3$ since the restriction of the k-invariant in ΩX is in $H^{2i+1}(K(\mathbb{Z}, 2i-2), \mathbb{Z})$ which is $\mathbb{Z}_2$ only if $i \geq 3$.) By means of a suitable 2-unit $\mathbb{Z} \to \mathbb{Z}$ we can carry one generator to another and so extend the isom. to $SU(0, 2i+1) \simeq X(0, 2i+1)$.

Sullivan's argument would seem* to imply that if $BU^{\otimes} \simeq \Omega X$ and X has finite k-invariants, then

$$X(5, \infty) \cong SU(5, \infty) \qquad \text{(at 2)}$$

so that as

$$BU^{\otimes} \cong BU(1) \times BSU^{\otimes}$$

as H-spaces, then perhaps

$$(BSU)^{\otimes} \simeq (BSU)^{\oplus} . \qquad \text{(at 2)}.$$

Maybe this can be proved by Segal's method. *This isn't correct because $SU(5, \infty)$ doesn't necessarily have torsion-free homology at 2.

February 19, 1970

To understand Brauer theory from the point of view of representation rings.

Start with a finite group G of order $p^a m$ where $(m, p) = 1$, and let A be a d.v.r. ~~of~~ with quotient field K of char 0 and residue field k of char. p. ~~[illegible]~~ There is a basic homomorphism (called decomposition) or specialization

$$d : R_K(G) \longrightarrow R_k(G)$$

which is a λ-ring homomorphism defined as follows. The maps

$$\operatorname{Spec} k \xrightarrow{i} \operatorname{Spec} A \xleftarrow{j} \operatorname{Spec} K$$

should give rise to an exact sequence

$$\begin{tikzcd} R_k(G) \arrow[r, "i_*"] \arrow[dr, bend right, ".\, i^* i_* 1"'] & R_A(G) \arrow[r, "j^*"] \arrow[d, "i^*"] & R_K(G) \arrow[r] & 0 \\ & R_k(G) & & \end{tikzcd}$$

And ~~[illegible]~~ $i_* 1$ is the $A[G]$-module k with trivial action and admits the resolution

$$0 \longrightarrow A \xrightarrow{\pi} A \to k \to 0$$

π = uniformizing parameter, hence

$$i^* i_* 1 = [A] - [A] = 0.$$

Thus the homomorphism $d = i^*(j^*)^{-1} : R_K(G) \to R_k(G)$ is well-defined and is obviously a λ-ring homomorphism.

In more concrete terms we take a $K[G]$-module V choose a lattice M inside V which is invariant and then take $V \otimes_A k$. This is obviously $[V] = j^*[M]$ and $d[V] = i^*[M]$.

The next point is to show that $\psi^q = id$ on $R_k(G)$ for $q =$ some power of p.

So for simplicity I assume that k is finite and sufficiently large so that $R_k(G) \xrightarrow{\cong} R_{\bar{k}}(G)$. One knows that $\psi^p =$ action of Frobenius. But $\psi^q = id$ on $R_k(G)$ if $k = \mathbb{F}_q$.

On the other hand one knows that if $q = p^\nu$ and

$$q \equiv 1 \mod m \qquad\qquad q \equiv 0 \mod p^a$$

recall $|G| = p^a m$.

then ψ^q is an idempotent operation on $R_K(G)$. Here you use the description of $R_K(G)$ as central functions on the group and the formula $\psi^q(\chi)(g) = \chi(g^q)$. The image of ψ^q consists of those characters χ such that

$$\chi(g) = \chi(g_r)$$

(since $\psi^q(\chi)(g) = \chi(g^q) = \chi(g_r^q g_s^q) = \chi(g_r)$, where g_r, g_s are the regular and singular components of g.)

Theorem:

~~[illegible]~~ If K has enough roots of unity, then

$$(*) \qquad \mathrm{Im}\{\psi^{g} : R_K(G) \longrightarrow R_K(G)\} \xrightarrow{\sim} R_k(G)$$

in other words a modular virtual representation lifts to a unique virtual representation with character satisfying

$$\chi(g) = \chi(g_r).$$

Proof: According to Serre's book p. III-13, d is surjective so we have

$$\begin{array}{ccc} R_K(G) & \xrightarrow{d} & R_k(G) \\ \downarrow \psi^{g} & & \downarrow \psi^{g} = id \\ R_K(G) & \longrightarrow & R_k(G) \end{array}$$

which proves that the map $(*)$ is surjective. But now count the ranks of these free abelian groups. One knows that $R_k(G)$ has rank equal to the number of p-reg conjugacy classes, and on the other hand $(\mathrm{Im}\,\psi^{g})\otimes K$ ~~[illegible]~~ $\hookrightarrow R(G)\otimes K \xrightarrow{\sim}$ Class functions on G, with $(\mathrm{Im}\,\psi^{g})\otimes K$ corresp. to those class functions $\varphi \ni \varphi(g) = \varphi(g_{reg})$. Thus $\mathrm{rank}(\mathrm{Im}\,\psi^{g}) \leq$ no. of p-reg conjugacy classes. With surjectivity this means the map $(*)$ is an isomorphism.

In this way we construct a ring homomorphism

$$s : R_k(G) \longrightarrow R_K(G).$$

which is a section of d

In fact this is a λ-ring homomorphism. To see this note that

$$\psi^k(s\,d\,x) = \psi^k\psi^q x = \psi^q\psi^k x = s\,d\,\psi^k x = s(\psi^k dx)$$

so s commutes with ψ^k. To check that $s\lambda_t(x) = \lambda_t(sx)$ its enough to embed $R_K(G)$ into $R_K(G)\otimes\mathbb{Q}$ whence we have the Newton formula:

$$\lambda_{-t}(x)^{-1} = \exp \sum_{m\geq 1} \frac{t^m}{m}\psi^m(x)$$

expressing the λ^i in terms of the ψ^k.

Next I would like to understand the Brauer lifting of the standard representation of $Gl_n(\mathbb{F}_q)$. This perhaps should be entirely algebraic and contained in Serre's paper. Thus if A is a d.v.r. and if G_A is a (linalg.) group scheme over Spec A we should have a decomposition homomorphism

$$R_K(G) \xrightarrow{d} R_k(G)$$

defined as above. More precisely denote by C the affine coordinate ring of G so that C is a bigebra

$$C \xrightarrow{\Delta} C\otimes C$$
$$C \longrightarrow A$$

over A. I assume that C is flat (possibly free as an A-module if necessary.) Then G_K, G_k have coordinate rings C_K, C_k, resp.

$R_K(G)$ and $R_k(G)$ are formed from the categories of C_K, C_k comodules which are finite dimensional over K, k. For $R_A(G)$ we can use C-comodules which are either free f.g. over A or just f.g. over A, getting two Grothendieck groups

$$R_A(G)^{\circ} \longrightarrow R_A(G).$$

I work with the former. It is necessary to check the existence of lattices, so suppose V is a K-module and a C-comodule

$$V \xrightarrow{\Delta} C \otimes_A V$$

By Serre Gèbres page 12, V is a union of sub-comodules of finite type over A, i.e. invariant lattices. (Details: given M an A-submodule of V consider

$$M^{\circ} = \{v \mid \Delta v \in C \otimes M\} \subset M$$

To show that M° is a sub-comodule i.e. $\Delta(M^{\circ}) \subset C \otimes M^{\circ}$.
But $M^{\circ} = \mathrm{Ker}\{V \longrightarrow C\otimes V \longrightarrow C \otimes (V/M)\}$ so

$$\begin{tikzcd}
C\otimes M^{\circ} \arrow[r, "id\otimes inc."] & C\otimes V \arrow[r, "id\otimes(id\otimes p\circ\Delta)"] & C\otimes C\otimes V/M \\
M^{\circ} \arrow[u, dashed, "\exists"] \arrow[ur, "\Delta"] \arrow[r, "\Delta"'] \arrow[urr, dashed, bend right, "0"] & C\otimes V \arrow[ur, "\Delta\otimes p"'] &
\end{tikzcd}$$

so it's clear. If $N \subset V$ f.t. over A, then $\Delta(N) \subset C \otimes_A M$ for some f.t. A-module M, whence M° is a f.t. sub comodule of V containing N.)

Thus lattices exist and so as ~~[scribbled]~~ in Serre's book one can show that the decomposition map

$$d: R_K(G_K) \longrightarrow R_k(G_k)$$

is defined. Serre probably proves that d is surjective (if K is large), which for $G = Gl_n$ is no surprise although perhaps non-trivial. Now I have the following situation:

$$\begin{array}{ccc} R_K(G(\mathbb{F}_q)) & \xrightarrow{d} & R_k(G(\mathbb{F}_q)) \\ & & \uparrow \\ R_K(G_K) & \xrightarrow{d} & R_k(G_k) \end{array}$$

which doesn't seem to yield anything. <u>Conjecture</u>:

$$R_k(G_k) \overset{id}{\underset{\psi^q}{\rightrightarrows}} R_k(G_k) \longrightarrow R_k(G(\mathbb{F}_q))$$

is exact in the category of λ-rings.

Let's go back to the case of $Gl_n(\mathbb{F}_q)$. The problem is to construct the virtual representation which lifts the standard representation. Its character χ is characterized by the fact that $\chi(g) = \chi(g_r)$ and $\chi(g_r) =$ Brauer character of the standard representation. Things should be easier stably.

<u>Idea</u>: Start with standard representation of $Gl_n(\mathbb{F}_q)$, lift

it in the obvious way to $Gl_n(\mathbb{Z}[\zeta_{q-1}])$ getting a representation E of the latter in $\mathbb{Q}[\zeta_{q-1}]^n$. Form the virtual representation $\psi^q E$, and the hope is that this comes from $Gl_n(\mathbb{F}_q)$. The hope is therefore that if E is the standard representation of $Gl_n(\mathbb{Z}[\zeta_{q-1}])$, then

$$\psi^q[E] \in \operatorname{Im}\left\{R(Gl_n(\mathbb{F}_q)) \longrightarrow R(Gl_n(\mathbb{Z}[\zeta_{q-1}]))\right\}.$$

This is a pretty concrete assertion which should be easy to settle.

For $n=1$ we have $(\mathbb{Z}[\zeta_{q-1}])^* \subset \mathbb{C}^*$. We are asking that the qth power of this character dies on the units $\equiv 1 \bmod p$. This is absurd when the group of units is infinite. NO good.

February 19, 1970

$\boxed{\text{Brauer theory}}$ for modular "real" and "symplectic" representations:

Let G be a finite group. Denote by $R(G)$ ~~the complex~~, $RO(G)$, and $RSp(G)$ the Grothendieck groups of complex, real, and quaternionic representations of G, respectively. Then there is an exact sequence

$$(*)\quad R(G) \xrightarrow{\psi^{-1}-id} R(G) \longrightarrow RO(G) \oplus RSp(G) \longrightarrow R(G) \xrightarrow{\psi^{-1}-id} R(G)$$

$$E \longmapsto (E, -\mathbb{H}\otimes_{\mathbb{C}} E)$$

$$E \oplus F \longmapsto (\mathbb{C}\otimes_{\mathbb{R}} E) + F$$

$$x \longmapsto \psi^{-1}x - x$$

which I shall want to check is compatible with ψ^q at least for q odd.

The good way to think of $RO(G)$ (resp. $RSp(G)$) is as the Grothendieck group of orthogonal (resp. symplectic) representations of G. (An orthogonal (symplectic) representation over a field k is a representation on a k-vector space V endowed with a non-degenerate symmetric (resp. skew-symmetric) quadratic form.) The reason is that if E is a complex representation with a ~~symplectic~~ symmetric form B and if $(\,,\,)$ is an invariant hermitian form, then we get an operator $A: E \rightarrow E$ which is conjugate linear defined by

$$B(x,y) = (x, Ay).$$

(In general A^2 has several eigenspaces ~~and~~ necessarily orthogonal for $(\,,\,)$ and you change $(\,,\,)$ on each eigenspace.)

Moreover B symmetric $\Rightarrow$ $(x, Ay) = (y, Ax) \Rightarrow (x, A^2x) = \|Ax\|^2 > 0$. Thus A^2 is a complex linear $\overbrace{\text{positive}}$ operator commuting with G so $A^2 = c \cdot id$ with $c > 0$ and so changing $(\;)$ by $c^{-1/2}$ we get that $B(x,y) = (x, Ay)$ and $A^2 = id$ so A is a conjugation and $E \simeq \mathbb{C} \otimes_{\mathbb{R}} (E^A)$. Similarly if B is anti-symmetric, then

(need to check A^2 & J^2 are hermitian.)

$$B(x,y) = (x, Jy) \qquad J \text{ conj. linear}$$

implies
$$(x, Jy) = -(y, Jx)$$

$$\Rightarrow \quad (x, J^2x) = -\|Jx\|^2$$

so by changing $(\;)$ by a scalar $\overbrace{\text{on each of the eigenspaces of } J^2}$ we can assume that $J^2 = -1$, whence J defines a quaternionic vector space structure on E.

In these terms the map

$$R(G) \longrightarrow RO(G)$$

is
$$E \longmapsto E \oplus E^* \quad \text{with} \quad Q(x+\lambda) = \langle x, \lambda \rangle.$$

so if

$$A(x+y^*) = y + x^*$$

where x^* is the linear functional given in terms of a hermitian product $(\,,\,)$ on E by

$$(z, x) = \langle z, x^* \rangle$$

this means we have put

$$(x+y^*, x'+y'^*) = B(x+y^*, y'+x'^*)$$

$$= Q(x+y^* + y' + x'^*) - Q(x+y^*) - Q(y'+x'^*)$$

$$= \langle x+y', y^*+x'^* \rangle - \langle x, y^* \rangle - \langle y', x'^* \rangle$$

$$= (x+y', y+x') - (x,y) - (y',x')$$

$$= (x,x') + (y',y)$$

hence $\|x+y^*\|^2 = \|x\|^2 + \|y\|^2$ which is what we expect.

The map

$$R(G) \longrightarrow RSp(G)$$

is given by

$$E \longmapsto E \oplus E^* \quad \text{with}$$

$$B(x+\lambda, x'+\lambda') = \langle x, \lambda' \rangle - \langle x', \lambda \rangle$$

Thus if

$$J(x+y^*) = y - x^*$$

we have

$$(x+y^*, x'+y'^*) = B(x+y^*, -y'+x'^*)$$

$$= +\langle x, x'^* \rangle + \langle y', y^* \rangle$$

$$= (x,x') + (y',y)$$

which agrees with the expected hermitian product.

The two maps

$$RO(G) \longrightarrow R(G) \quad \text{and} \quad RSp(G) \longrightarrow R(G)$$

in these terms are the maps which forget the bilinear forms.

The exactness of the sequence can now be checked using the basis for $R(G)$ consisting of the irreducible representations. Note that $RO(G) \longrightarrow R(G)$ and $RSp(G) \longrightarrow R(G)$ are injective since on composing them with ~~multiplication by 2 gives~~ the natural maps the other way multiplies by 2. Consequently we have inclusions

$$(**) \qquad \begin{tikzcd}[column sep=small] & RO(G) \arrow[dr, phantom, "\subset" sloped] & \\ \mathrm{Norm}\{R(G)\} \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & R(G)^{\mathbb{Z}_2} \\ & RSp(G) \arrow[ur, phantom, "\subset" sloped] & \end{tikzcd}$$

where Norm denotes the image of $\psi^{-1} + id$ from $R(G)$ to $R(G)$. Now I claim that

$$R(G) \xrightarrow{\psi^{-1} - id} R(G) \xrightarrow{\psi^{-1} + id} R(G)$$

is exact. This is because $R(G)$ is the free abelian group generated by the irreducible complex representations which are permuted by ψ^{-1}. Thus the exactness follows from

$$H^1(\mathbb{Z}_2, \mathbb{Z}) = 0 \qquad H^1(\mathbb{Z}_2, \mathrm{ind}_{1 \to \mathbb{Z}_2} \mathbb{Z}) = 0$$

Therefore the exactness of (*) is equivalent to (**) being bicartesian. But now we can compute. Suppose x is an invariant element of $R(G)$; to show its the sum of something in RO and RSp. We have that x is a integral linear combination of terms of the form $[E]$, where E is irreducible and isomorphic to E^*, and $[F]+[F^*]$ where F is irreducible and $F \not\cong F^*$. The latter is already a norm so can be forgotten; so we can assume $x = [E]$. Then $(E^* \otimes E^*)^G = \mathbb{C}$ by Shur's lemma so either $\Lambda^2 E^*$ or $S^2 E^*$ has an invariant; the resulting form is necessarily non-degenerate by irreducibility of E. Thus x comes from either RO or RSp. Note that no such $[E]$ comes from both RO and RSp so

$$RO(G) \cap RSp(G) = (\psi^{-1} + id)\, R(G)$$

finishing the proof that (*) is exact.

Remark: This argument generalizes to representations over "sufficiently large" fields of odd characteristic. To make things clearer you might put

$$\hat{H}^0(\mathbb{Z}_2, R(G)) = \mathbb{Z}_2\text{-vector space with basis } [E] \ni E \cong E^*.$$

which is the sum of the image of RO and RSp. In order to see these are disjoint we need an argument allowing us to write elements of RO and RSp as sums of irreducibles whose behavior under $RO \to R$, $RSp \to R$ is clear. Thus I want to know that an irreducible RO(G) or RSp(G) element becomes in R(G) either a norm or an irreducible self-conjugate $[E]$.

February 20, 1970:

Let k be a field of odd characteristic (eventually an arbitrary commutative ring). ~~...~~ If G is a finite group, then I can form the Grothendieck groups $RO_k(G)$ and ~~...~~ $RSp_k(G)$ of orthogonal and symplectic representations of G over k. (It should ultimately be possible to make this construction in an arbitrary ringed topos.) For the moment we think of $RO_k(G)$ as the free abelian group generated by the free abelian monoid of positive elements, and a positive element is represented by an orthogonal ~~...~~ representation E under an equivalence relation to be made precise later. I want now to lay the framework for putting ~~...~~ a λ-ring structure on $RO_k(G) \oplus RSp_k(G)$.

If E and E' are repns. with bilinear forms so is $E \otimes E'$ with the form

$$(B \otimes B')(x \otimes x', y \otimes y') = \cancel{} \; B(x,y)\, B'(x',y').$$

If B and B' ~~...~~ are skew-symmetric then $B \otimes B'$ is symmetric etc. This should put a ring structure on $RO_k(G) \oplus RSp_k(G)$.

If E is endowed with a form B, then $\Lambda^q E$ ~~is~~ inherits the form

$$(\Lambda^q B)(x_1 \cdots x_q, y_1 \cdots y_q) = \det B(x_i, y_j)$$

If B is skew-symmetric then changing x's and y's the matrix $B(x_i, y_j)$ changes to its negative transpose so the determinant changes by $(-1)^q$. Thus $\Lambda^q B$ is symmetric for q even and skew-symmetric for q odd. This should put a λ-ring structure on

$RO_k(G) \oplus RSp_k(G)$ in such a way that the map into $R_k(G)$ is a λ-ring homomorphism.

How to form $RO_k(G)$: Think of an orthogonal representation as a map $BG \longrightarrow BO_k$. The analogue of an exact sequence should be a lifting to a parabolic subgroup of O_k. Not quite. There are two basic ways of decomposing a quadratic space: (i) If $E' \subset E$ is a subspace and $B|E'$ is non-degenerate, then $E \cong E' \oplus (E')^{\perp}$. (ii) If $E' \subset E$ is an isotropic subspace, then we want to have an equivalence

$$[E] \sim [(E' \oplus E/(E')^{\perp})] + [(E')^{\perp}/E'].$$

Only case (ii) fits the parabolic subgroup description.

Definition: $E \longmapsto [E] \in RO_k(G)$ is the universal function such that

(i) $E = E_1 \oplus E_2$ o.d.s $\Longrightarrow [E] = [E_1] \oplus [E_2]$

(ii) $0 \to E' \to E \to E'' \to 0$ exact and $B(E', E') \equiv 0 \Longrightarrow$

$$[E] = [E' \oplus E/(E')^{\perp}] + [(E')^{\perp}/E]$$

The above definition should work for a commutative ringed topos. It's necessary to check that we get a λ-ring structure on $RO_k(G) \oplus RSp_k(G)$ and that over a field of odd characteristic one gets a free ~~[illegible]~~ abelian group with irreducibles for basis.
~~The last point is clear if granted a Jordan-Hölder~~

The last point may be proved as follows. Divide up the simple $k[G]$-modules into three groups I_1, I_0, I_{-1} according to whether they support a non-degenerate symmetric form, no form, or an anti-symmetric form and then split up the 0 group $I_0 = I_0' + I_0''$ into a ψ^{-1} system of representatives. Let E be an orthogonal representation. Then by induction on the length of E as a $k[G]$-modules one sees that

$$[E] = \sum_{i\in I_0'}{}' \, n_i [E_i + E_i^*] + \sum_{i \in I_1}{}' \, m_i [E_i] \qquad \text{in } RO_k(G)$$

if the same formula holds in $R_k(G)$. The inductive step consists of choosing a minimal G-submodule M of E whence either M is non-isotropic and $[E] = [M] + [M^\perp]$ or M is isotropic and $[E] = [M \oplus M^*] + [M^\perp / M]$. This shows that $RO_k(G) \longrightarrow R_k(G)$ is injective so $RO_k(G)$ is a free group with the desired basis.

<u>**Conclusion:**</u> k field not of charac. 2 such that all $k[G]$-simple modules are absolutely irreducible. Then

$$R_k(G) \xrightarrow{\psi^{-1} - id} R_k(G) \longrightarrow RO_k(G) \oplus RSp_k(G) \longrightarrow R_k(G) \xrightarrow{\psi^{-1} - id} R_k(G)$$

is exact.

Moreover $RO_k(G) \longrightarrow R_k(G)$, $RSp_k(G) \longrightarrow R_k(G)$ are injective with given basis.

February 20, 1970

Sullivan suggests constructing the basic (virtual) representation of $Gl_n(\mathbb{F}_q)$ furnished by the Brauer theory by inducing up the standard lifting of the modular representation of the normalizer $\Sigma_n \int \mathbb{F}_q^*$ on $\mathbb{C}^n$ furnished by the basic identification $\mathbb{F}_q^* \simeq \mu_q \subset \mathbb{C}^*$ used in the definition of the Brauer character. ~~[illegible]~~ By Brauer theory the (generalized) character χ we are after is characterized by the conditions

$$\begin{cases} \psi^q(\chi) = \chi \\ \chi\begin{bmatrix} \zeta_1 & & \\ & \ddots & \\ & & \zeta_n \end{bmatrix} = \sum_{i=1}^n \zeta_i \qquad \zeta_i \in \mu_q \subset \mathbb{C}^* \\ \chi = \text{Brauer character of the (standard) modular repn. of } Gl_n(\mathbb{F}_q) \end{cases}$$

As the standard lifting S of $\Sigma_n \int \mathbb{F}_q^* = N$ is fixed by ψ^q, it follows that (the character of) $\operatorname{ind}_{N\to G}(S)$ will be invariant.

Let's compute ~~the~~ $\operatorname{res}^G_T \operatorname{ind}_{N\to G}(S)$. By the double coset formula this will be a sum over the orbits of T on G/N. A point of G/N is the same as a family of n-independent lines $\{L_i\}$ in $V = \mathbb{F}_q^n$ and the stabilizer of this is of special form. Indeed if $(a_1, \ldots, a_n)$ is a generator of a line L, then the stabilizer is the subgroup of diagonal matrices $(d_1, \ldots, d_n)$ such that $d_i = d_j$ if $a_i \neq a_j$. Thus the stabilizers are all of the form T_I where $\{1, \ldots, n\} = \coprod_j I_j$ and $T_I = \{(d_i) \mid d_i = d_j$ if i, j belong to the same $I_j\}$. The families of n-independent lines ("axes") stabilized by the subgroup T_I break up into families ~~[illegible]~~ contained in the eigenspaces. Therefore we have a terrible sum

$$\operatorname{res}^G_T \operatorname{ind}_{N\to G}(S) = \sum_{\substack{I \text{ partition} \\ \text{of } \{1,\dots,n\}}} \ \sum_{\substack{TxN \in G/N \\ \operatorname{stab}(xN) = T_I}} \operatorname{ind}_{T_I \to T} \rho_x(S)$$

where $\rho_x(S)$ is defined as follows. $x \cdot N$ is the same as a set of axes stabilized by T_I, hence ~~[illegible]~~ using x we get a map of these axes with the standard set and so can transform the representation to the new axes via x, whence we get a new representation of T_I. The important thing at the moment is to note that unless I is the finest partition ~~[illegible]~~ then $T_I \oplus A \simeq T$ where $A \cong \mathbb{Z}^r_{\ell-1}$ and ~~so~~ since

$$\operatorname{ind}_{T_I \to T} \rho_x(S) = \rho_x(S) \otimes \operatorname{reg}(A)$$

we have that all the $\overbrace{\text{Chern}}^{\text{non-zero}}$ $\overbrace{\text{classes}}^{\text{mod } \ell}$ of this are of degrees $\geq \ell^r - \ell^{(r-1)}$.

Various details of the Adams conjecture in the real case.

First stage is computation of $H^*(BO_n(\bar{\mathbb{F}}_p), \mathbb{Z}_2)$, p an odd prime. The claim is that

$$H^*(BO_n(\bar{\mathbb{F}}_p), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_n]$$

where w_j is the jth Stiefel-Whitney class, or more precisely the image of the universal Stiefel-Whitney class under the map $BO_n(\mathbb{F}_q) \longrightarrow BO_n$.

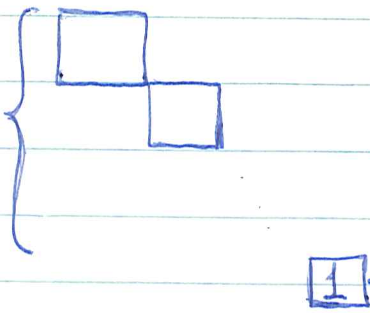
By O_n we mean the algebraic group which is the subgroup of Gl_n leaving the quadratic form $\sum_{i=1}^n x_i^2$ invariant. Since we are off 2 we have the canonical isomorphism $\mathbb{Z}_2 \xrightarrow{\sim} \mu_2$ and hence the corresponding 2-torus $\mathbb{Z}_2^n \hookrightarrow O_n$ of diagonal matrices. A more precise claim is that

$$H^*(BO_n(\bar{\mathbb{F}}_p), \mathbb{Z}_2) \xrightarrow{\simeq} H^*(B\mathbb{Z}_2^n, \mathbb{Z}_2)^{\Sigma_n} = \mathbb{Z}_2[x_1, \ldots, x_n]^{\Sigma_n}$$

$$w_j \longmapsto \text{jth elementary symmetric function of the } x_i.$$

We break up the proof of this isomorphism in 2 parts, the first being to show that the restriction homomorphism to this 2-torus is injective.

By Chevalley if $4 | q-1$, then the (group of points in $\mathbb{F}_q$ of the) normalizer of the torus in O_n contains ~~the~~ a Sylow 2-subgroup of $O_n(\mathbb{F}_q)$. More precisely let T_n be the maximal torus of O_n. It appears so:



$m = \left[\frac{n}{2}\right]$

$\square$ = torus in $O_2(\cancel{\ })$.

$\boxed{1}$ ← depending on whether $n = 2m$ or $2m+1$.

(~~For simplicity suppose that $n = 2m$~~) We have to describe the 2-torus in $O_2(\cancel{\ })$ which is $SO_2(\cancel{\ })$ and consists of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \ni \quad A = (A^t)^{-1}, \quad \det A = 1$$

that is

$$SO_2(\cancel{\ }) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a^2 + b^2 = 1 \right\}.$$

(Note that ~~since~~ once a fourth root of 1, i, becomes available then this torus becomes isomorphic to $\mathbb{G}_m$ via the formulas

$$\begin{aligned} a + ib &= z \qquad & a &= \tfrac{1}{2}\left(z + \tfrac{1}{z}\right) \\ a - ib &= \tfrac{1}{z} \qquad & b &= \tfrac{1}{2i}\left(z - \tfrac{1}{z}\right) \end{aligned}$$

).

The normalizer N_n of the torus T_n is clearly

$$\left(\Sigma_m \mathbin{S} O_2(\cancel{\ })\right) \times \mathbb{Z}_2^{\varepsilon} \qquad n = 2m + \varepsilon \qquad \varepsilon = 0, 1.$$

~~[illegible]~~ and its group of points over $\mathbb{F}_q$ has order

$$|N_n(\mathbb{F}_q)| = m!\,(2\cdot(q-1))^m \cdot 2^{\varepsilon}.$$

~~I wish now to compute the order of $O_n(\mathbb{F}_q)$. Suppose i is around so that we can assume the quadratic form is hyperbolic $V = W + W^*$ (case $n = 2m$) with $Q(w + \lambda) = \langle w, \lambda \rangle$.~~ ~~Then I wish to compute the number of isotropic subspaces~~ ~~Then the number of choices for~~

I wish now to compute the order of $O_n(\mathbb{F}_q)$. ~~suppose~~ suppose for simplicity that $i \in \mathbb{F}_q$ so I can write $x^2+y^2=(x+iy)(x-iy)$. ~~[illegible]~~ This implies that a form in 2-variables $\alpha x^2+\beta y^2$ with ~~[illegible]~~ discriminant 1 in $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2$ is isotropic. One knows that any form with ≥ 3 variables represents zero by ~~[illegible]~~ Warning so my hypothesis implies that the quadratic function $\sum_{i=1}^{2m} x_i^2$ with discriminant 1 is hyperbolic. Without assuming $i \in \mathbb{F}_q$ one has to be slightly more careful.

I begin by counting the number of isotropic flags in the case $n=2m$. These are flags $0 \subset W_1 \subset \cdots \subset W_m$ where W_m is totally isotropic. ~~[illegible]~~ I write $V = W + W^*$ $\quad Q(w+\lambda) = \langle w, \lambda \rangle$. Then the number of non-zero isotropic vectors in V is

$$\operatorname{card}\{\cancel{}\, w+\lambda \mid \langle w,\lambda\rangle = 0\} = \underset{w\neq 0 \quad \lambda\perp w}{(q^m-1)\,q^{m-1}} + \underset{w=0,\ \lambda\neq 0}{q^m-1} = (q^m-1)(q^{m-1}+1)$$

Thus

$$\text{no. of choices for } W_1 = \frac{q^m-1}{q-1}(q^{m-1}+1).$$

Then W_2 is an isotropic line in $W_1^\perp/W_1$ which is ~~[illegible]~~ hyperbolic since it has determinant 1. Thus can induct and we find

$$\text{no. of isotropic flags} = \prod_{j=1}^{m} \frac{(q^j-1)(q^{j-1}+1)}{(q-1)} = 2\,\frac{q^m-1}{(q-1)} \prod_{j=1}^{m-1} \frac{q^{2j}-1}{q-1}$$

If W is a maximal isotropic subspace, then it has an isotropic complement Z (Choose a complement Z; then $Q(w+z) = B(w,z) + Q(z)$ where $B: W \times Z \to \mathbb{F}_q$ is a non-degenerate pairing, so $Q(z) = B(Tz, z)$ for a unique transf. $T: Z \to W$, whence $\{-Tz+z\}$ will be an isotropic complement to W.)

Moreover the other isotropic complements are given by skew-symmetric maps $T: W \to Z$, hence are in number $q^{\frac{m(m-1)}{2}}$. Thus

$$|O_{2m}(\mathbb{F}_q)| = \underbrace{2\,\frac{q^m-1}{q-1}\prod_{j=1}^{m-1}\frac{q^{2j}-1}{q-1}}_{\text{no of isot. flags.}} \cdot \underbrace{(q-1)^m q^{\frac{m(m-1)}{2}}}_{\text{no. of bases splitting the flag}} \cdot \underbrace{q^{\frac{m(m-1)}{2}}}_{\text{no. of complements to an isotropic subspace}}$$

$$= \underset{\substack{\uparrow\\ \text{2 components}\\ \text{of } O(2m)}}{2\cdot q^{m(m-1)}}\ \underset{\substack{\uparrow\\ \text{Euler class}\\ e}}{(q^m-1)}\ \prod_{j=1}^{m-1}\underset{\substack{\uparrow\\ \text{Pont. classes}\\ p_1\cdots p_{m-1}}}{(q^{2j}-1)}$$

For $n = 2m+1$ we have $V = W + W^* + \mathbb{F}_q$ $\quad Q(w+\lambda+z) = \langle w,\lambda\rangle + z^2$ and the number of choices for the first vector of an orthonormal frame is

$$\operatorname{card}\{(w+\lambda+z) \mid \langle w,\lambda\rangle + z^2 = 1\}.$$

$$= \underset{z\neq\pm1}{(q-2)}\Big\{\underset{w\neq0\ \ \langle w,\lambda\rangle=1-z^2}{(q^m-1)\,q^{m-1}}\Big\} + \underset{z=\pm1}{2}\Big\{\underset{w\neq0\ \ \langle w,\lambda\rangle=0}{(q^m-1)\,q^{m-1}} + \underset{w=0,\ \lambda\ \text{arb}}{1\cdot q^m}\Big\}.$$

$$= q^{m-1}\left[q^{m+1} - q - 2q^m + 2 + 2q^m - 2 + 2q\right\}$$

$$= q^m(q^m+1)$$

Once this vector is chosen the rest is in the complement which has dimension 2m so

$$|O_{2m+1}(\mathbb{F}_q)| = \underset{\substack{\uparrow\\ \text{2components}}}{2}\cdot q^{m^2}\cdot\prod_{j=1}^{m}\underset{\substack{\uparrow\\ p_1,\dots,p_m}}{(q^{2j}-1)}$$

So therefore

$$\frac{|O_{2m}(\mathbb{F}_q)|}{|N_{2m}(\mathbb{F}_q)|} = \frac{2}{2}\cdot q^{m(m-1)} \cdot \frac{q^m-1}{m(q-1)} \prod_{j=1}^{m-1} \frac{q^{2j}-1}{2j(q-1)}$$

$$\frac{|O_{2m+1}(\mathbb{F}_q)|}{|N_{2m+1}(\mathbb{F}_q)|} = \frac{2}{2^1}\cdot q^{m^2} \cdot \prod_{j=1}^{m} \frac{q^{2j}-1}{2j(q-1)}$$

and this is prime to 2 provided $4 \mid q-1$.

(The above formulas hold even if $4 \nmid q-1$ but $2 \mid q-1$. In effect your counting argument works for a hyperbolic form and you can arrange this by selecting the first vector if $n=2m+1$ or the first two vectors if $n=2m$ so as to render the remainder hyperbolic.)

<u>Conclusion:</u> If $4 \mid q-1$, then $N_n(\mathbb{F}_q)$ contains a Sylow 2-subgroup of $O_n(\mathbb{F}_q)$. (This result occurs in Chevalley's Tohoku paper.)

The next point is to consider the structure of the Sylow 2-subgroup of $N_n(\mathbb{F}_q)$.

<u>Assertion:</u> The mod 2 cohomology of $N_n(\mathbb{F}_q)$ is detected by the family of elementary abelian 2-subgroups.

We know that

$$N_n(\mathbb{F}_q) = \left(\Sigma_m \int O_2(\mathbb{F}_q)\right) \times \mathbb{Z}_2^{\varepsilon} \qquad n = 2m+\varepsilon$$

By our lemma it suffices to prove the result for $O_2 = N_2$.

~~Lemma: Suppose that H_i is a family of subgroups of G with detect mod p cohomology. Then one can construct a detecting family for $\Sigma_m \wr G$ as follows. For each dyadic partition~~

In more detail we know that the Sylow 2-subgroup of $N_n(\mathbb{F}_q)$ (say $n = 2m$) is of the form

$$\prod_{i=0}^{n} \left\{ \Sigma^{(2)}_{2^i} \wr O_2(\mathbb{F}_q) \right\}^{a_i}$$

where $m = \sum_{i=0}^{n} a_i 2^i$ is the dyadic expansion of m, and where $\Sigma^{(2)}_{2^i}$ is an iterated wreath product, so

$$\Sigma^{(2)}_{2^i} \wr O_2(\mathbb{F}_q) = \underbrace{\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr}_{i \text{ times}} \cdots (\mathbb{Z}_2 \wr O_2(\mathbb{F}_q))\cdots)$$

If I know that $\Sigma^{(2)}_{2^{i-1}} \wr O_2(\mathbb{F}_q)$ is detected by a family A_j, $j \in J$, of elementary abelian 2-groups, the by the known facts on wreath products, one knows that the family

$$A_j \times A_{j'} \qquad \text{and} \qquad \mathbb{Z}_2 \times A_j$$

detects for $\Sigma^{(2)}_{2^i} \wr O_2(\mathbb{F}_q)$. By induction these will be elementary abelian 2-groups.

So for $O_2(\mathbb{F}_q) = \mathbb{Z}_{q-1} \times_\sigma \mathbb{Z}_2$ a dihedral group it is necessary to check that its mod 2 cohomology is detected by elementary abelian 2-groups. Can restrict to the Sylow subgroup $\mathbb{Z}_{2^\nu} \times_\sigma \mathbb{Z}_2$. ~~[illegible]~~ Take generators x, y with relations

$x^{2^\nu}=1$, $y^2=1$, $yxy^{-1}=x^{-1}$. To compute the cohomology mod 2 I use the ~~Serre~~ Hochschild- spectral sequence of the exact sequence

$$0 \longrightarrow \mathbb{Z}_{2^{\nu-1}}\cdot x^2 \longrightarrow \mathbb{Z}_{2^\nu}\times_\sigma \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2\times\mathbb{Z}_2 \longrightarrow 0$$

Now note that the cohomology of the fibre

$$H^*(B\mathbb{Z}_{2^{\nu-1}}) = \begin{cases} \mathbb{Z}_2[u,y]/(u^2+y) & \text{if } \nu=2 \\ \mathbb{Z}_2[u,y]/(u^2) & \text{if } \nu>2. \end{cases}$$

$u\in H^1(B\mathbb{Z}_{2^{\nu-1}})$ is represented by the map $\mathbb{Z}_{2^{\nu-1}} \twoheadrightarrow \mathbb{Z}_2$ and transgresses to the quadratic function $t_1^2+t_1t_2$ where $t_i:\mathbb{Z}_2\times\mathbb{Z}_2\to\mathbb{Z}_2$ are the projection maps viewed as elements of $H^1(B(\mathbb{Z}_2\times\mathbb{Z}_2))$. Now y is an infinite cycle in the spectral sequence, since the natural representation E of $\mathbb{Z}_{2^\nu}\times_\sigma\mathbb{Z}_2$ on R^2 has an Euler class which restricts to y. Thus only d_2 is non-zero in the spectral sequence and so

$$H^*(B(\mathbb{Z}_{2^\nu}\times_\sigma\mathbb{Z}_2)) = \mathbb{Z}_2[t_1,t_2,e]/(t_1^2+t_1t_2)$$

where the t_i are the elements of degree 1 represented by the maps $(x,y)\longmapsto(1,0)$ and $(0,1)$ respectively and where e is the Euler class of the natural representation E.

Consider the two elementary abelian subgroups $\{x^{2^{\nu-1}},y\}$ and $\{x^{2^{\nu-1}},xy\}$. Then for the first the restriction map

$$\begin{array}{ccc} H^*(B(\mathbb{Z}_{2^\nu}\times_\sigma\mathbb{Z}_2)) & \longrightarrow & H^*(B(\mathbb{Z}_2\times\mathbb{Z}_2)) = \mathbb{Z}_2[v_1,v_2] \\ t_1 & \longmapsto & 0 \\ t_2 & \longmapsto & v_2 \\ e & \longmapsto & v_1\cdot(v_1+v_2) \end{array}$$

char. $y\mapsto 1$, $x^{2^{\nu-1}}\mapsto -1$ char $y\mapsto -1$, $x^{2^{\nu-1}}\mapsto -1$

and for the second the restriction map sends

$$t_1 \longmapsto v_2$$

$$t_2 \longmapsto v_2$$

$$e \longmapsto v_1 \cdot (v_1 + v_2)$$

$x^{2^{\nu-1}}$ acts as -1

xy acts as a reflection

and we see that each restriction map picks up each of the minimal primes. (The first kills t_1, the second t_1+t_2). Thus the assertion on page 5 is proved. ~~Now any elementary~~ abelian 2-subgroup of $O_n(\mathbb{F}_q)$ is conjugate to a subgroup of the diagonal matrices so we have proved: false

Theorem: If $4 | q-1$, then

$$H^*(BO_n(\mathbb{F}_q)) \longrightarrow H^*(B\mathbb{Z}_2^n)$$

is injective on mod 2 cohomology.

Now the image is obviously contained in the image of the invariants under Σ_n and so we have the inclusion

$$H^*(BO_n(\mathbb{F}_q)) \hookrightarrow \mathbb{Z}_2[w_1, \dots, w_n].$$

Actually this map is an isomorphism since the universal etale Stiefel-Whitney classes give one elements in $H^*(BO_n(\mathbb{F}_q))$ with the correct restrictions to the 2-torus

Thus we have proved

Proposition: If $4 / q-1$, then

$$H^*(BO_n(\mathbb{F}_q)) \longrightarrow \prod_A H^*(BA)$$

is injective where A runs over representatives for the conjugacy classes of maximal elementary abelian 2-subgroups.

Given such an A ~~[illegible] decompose the quadratic~~ in $O_n(\mathbb{F}_q)$ we can decompose $V = \mathbb{F}_q^{\,n}$ into eigenspaces under A. If v, w belong to different eigenspaces then $\exists\, a \in A$ such that $av = v$, $aw = -w$ or possibly this holds with v, w interchanged, hence ~~[illegible]~~ $B(v,w) = B(av, aw) = -B(v,w)$ so $B(v,w) = 0$ as q is odd. Thus the eigenspaces are perpendicular and so if A is a maximal elementary abelian 2-group, then the eigenspaces give a decomposition of $V = L_1 + \cdots + L_n$ as an orthogonal sum of 1-dimensional quadratic spaces. A 1-dimensional quadratic space L is classified by the image of $Q: L - 0 \longrightarrow \mathbb{F}_q^*$, which is a coset of $(\mathbb{F}_q^*)^2$. Thus to A we can associate the number of L_i which belong to the 0-coset of $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2 = \mathbb{Z}_2$ and this gives an invariant of the conjugacy class of A. It's clear that we get any number ~~[illegible]~~ $j = 0, 2, 4, \ldots, [\frac{n}{2}]$ from some A. In effect the discriminant (mod squares) is the only invariant of a quadratic space over $\mathbb{F}_q$ and hence ~~the number~~ if $\varepsilon(L_i) \in \mathbb{F}_q^*/\mathbb{F}_q^{*2}$ is this invariant we must have $j = \sum_{i=1}^{n} \varepsilon(L_i) \equiv 0 \;(2)$. But taking $n=2$, we know that the form x^2+y^2 is equivalent to $\alpha x^2 + \alpha y^2$, hence we can get in a 2-dim'l space two orthogonal lines with non-zero

ε-invariant. Thus

<u>Proposition:</u> There are $\left[\frac{n}{2}\right]+1$ conjugacy classes of maximal elementary abelian 2-subgroups of $O_n(\mathbb{F}_q)$, $(4 \mid q-1)$.

The normalizer $N(A_j)$ of the A_j corresponding to $V = L_1 + \cdots + L_n$ where $\varepsilon(L_i) = 1$, $1 \le i \le 2j$ and $\varepsilon(L_i) = 0$ for $i > 2j$ is clearly A_j semi-direct product $\Sigma_{2j} \times \Sigma_{n-2j}$. Thus

$$H^*(BA_j)^{N(A_j)} \cong H^*(BO_{2j}) \otimes H^*(BO_{n-2j})$$

is a free $H^*(BO_n)$ module of rank $\binom{n}{2j}$. Indeed $[H^*(BO_1^{\,n}) : H^*(BO_n)] = n!$ and $[H^*(BO_1^{\,n}) : H^*(BO_{2j} \times BO_{n-2j})] = (2j)!\,(n-2j)!$ Thus we have a diagram

$$(*) \qquad \boxed{H^*(BO_n) \hookrightarrow H^*(BO_n(\mathbb{F}_q)) \xhookrightarrow{\ \theta\ } \prod_{j=0}^{\left[\frac{n}{2}\right]} H^*(BO_{2j}) \otimes H^*(BO_{n-2j})}$$

where the **composite** is the product of the natural maps $H^*(BO_n) \to H^*(BO_{2j} \times BO_{n-2j})$. I want to see how far the second map is from being an isomorphism. From the spectral sequence of the fibration

$$\begin{array}{c} O_n/O_n(\mathbb{F}_q) \longrightarrow BO_n(\mathbb{F}_q) \longrightarrow BO_n \\ \wr\! \mid \\ SO_n \end{array}$$

in etale homotopy theory I ~~should ought~~ should be able to show that $H^*(BO_n(\mathbb{F}_q)) \cong H^*(BO_n) \otimes H^*(SO_n)$ as $H^*(BO_n)$-modules, hence $H^*(BO_n(\mathbb{F}_q))$ should be of rank 2^{n-1} over $H^*(BO_n)$. Note that

$$\sum_{j=0}^{[\frac{n}{2}]} \binom{n}{2j} = \sum_{j=0}^{[\frac{n}{2}]} \binom{n-1}{2j-1} + \binom{n-1}{2j} = \sum_{i=0}^{n-1} \binom{n-1}{i} = (1+1)^{n-1} = 2^{n-1}.$$

Thus we see that the map Ⓗ of $(*)$ on page 10 should be an isomorphism after localizing at the zero ideal of $H^*(BO_n)$. (Well actually this has to be true for spectral reasons, so that our feeling that the etale spectral sequence degenerates is correct. In fact this can be used to prove the spectral sequence degenerates, since the first non-zero differential would have to change the rank over $H^*(BO_n)$ after localizing at 0-ideal.)

<u>Conclusion</u>: $H^*(BO_n(\mathbb{F}_q)) = \mathbb{Z}_2[w_1, \dots, w_n, e_1, \dots, e_{n-1}] / J$

where J is some ideal of relations which after a suitable filtration become like $e_i^2 = 0$. This ought to imply that $J = (r_1, r_2, \dots, r_{n-1})$ where degree $(r_i) = 2i$ and the sequence $\{r_i\}$ is regular.

If we form the unique quadratic extension $\mathbb{F}_q \subset \mathbb{F}_{q^2}$, then all of the A_j in $O_n(\mathbb{F}_q)$ become conjugate to A_0 in $O_n(\mathbb{F}_{q^2})$. This is because the quadratic space $(\mathbb{F}_q, \alpha x^2, \alpha \notin (\mathbb{F}_q^*)^2)$ becomes standard since $\sqrt{\alpha}$ exists in $\mathbb{F}_{q^2}$. Therefore we obtain a commutative diagram

$$\begin{tikzcd}
H^*(BO_n(\mathbb{F}_q)) \arrow[r, hook] & \prod_{j=0}^{[\frac{n}{2}]} H^*(BA_j(\mathbb{F}_q))^{N_j} \arrow[r, "pr_0"] & H^*(BA_0)^{N_0} \arrow[d, equal] \\
H^*(BO_n(\mathbb{F}_{q^2})) \arrow[r, hook] \arrow[u] & \prod_{j=0}^{[\frac{n}{2}]} H^*(BA_j(\mathbb{F}_{q^2}))^{N_j} \arrow[r, "pr_0"] & H^*(BA_0)^{N_0} \arrow[ul, "\{\rho_j^*\}"']
\end{tikzcd}$$

where $\rho_j : A_j(\mathbb{F}_q) \xrightarrow{\cong} A_0$ is defined by conjugation with a matrix in $O_n(\mathbb{F}_{q^2})$ and where A_0 denotes the diagonal matrices. This diagram shows that

$$\operatorname{Im}\left\{H^*(BO_n(\mathbb{F}_{q^2})) \longrightarrow H^*(BO_n(\mathbb{F}_q))\right\} \hookrightarrow H^*(BA_0)^{N_0}$$

and so we have proved

Proposition: $\quad H^*(BO_n(\mathbb{F}_{p^{2^\infty}})) \hookrightarrow H^*(BA_0)^{N_0} \cong \mathbb{Z}_2[w_1, \ldots, w_n]$ for any odd prime p.

Actually this map is an isomorphism, as we see from the étale Stiefel-Whitney classes. To produce these ~~[illegible]~~ independently of étale cohomology I need a Brauer theorem for quadratic representations.

Analogues for the symplectic groups: $Sp_{2n} = Sp(n)$ is the algebraic group of $n \times n$ matrices preserving the form $\sum_{i=1}^{n} x_i \wedge y_i$, athough it's probably easier to work ~~[illegible]~~ invariantly since the isomorphism problem is clear (i.e. no Witt group).

$$|Sp_{2n}(\mathbb{F}_q)| = \underbrace{\underbrace{\prod_{j=1}^{n} \frac{q^{2j}-1}{q-1}}_{\text{isot. flags}} \cdot \underbrace{(q-1)^n q^{\frac{n(n-1)}{2}}}_{\text{bases per flag}}}_{\text{isot subspace + bases}} \underbrace{q^{\frac{n(n+1)}{2}}}_{\text{complements}}.$$

$$= q^{n^2} \prod_{j=1}^{n} (q^{2j}-1)$$

$cq_1, \dots, cq_n$ quaternionic Chern classes.

For example $Sp_2 = Sl_2$ has $q(q^2-1)$ elements. Note that the Weyl group of Sp_{2n} is the same as that for O_{2n} namely $\Sigma_n \wr \mathbb{Z}_2$. Let N_{2n} be the normalizer of the torus. Then

$$|N_{2n}(\mathbb{F}_q)| = (q-1)^n \cdot 2^n \cdot n!$$

which agains checks Chevalley's theorem.

Note that the normalizer N_2 has a different structure here than for O_2. Here

$$N_2 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ semi-direct product + amalg. with } \mathbb{Z}_4 \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Thus

$$N_2(\mathbb{F}_q) \cong \mathbb{Z}_{q-1} \,\&\, \mathbb{Z}_4. \qquad (\& \text{ fancy symbol})$$

Restricting to the Sylow subgroup we get generalized quaternion gp.

$$\mathbb{Z}_{2^\nu} \,\&\, \mathbb{Z}_4 \qquad \text{generators } x, y \qquad x^{2^\nu} = 1,\ y^4 = 1,\ y^2 = x^{2^{\nu-1}}, \quad yxy^{-1} = x^{-1}$$

This is the subgroup of $\mathbb{H}$ generated by $x = \exp 2\pi i/2^\nu$ and $y = j$.

Before computing the cohomology of this we do first that of $N_2 = S^1 \wr \mathbb{Z}_4$ by means of the spectral sequence for the extension

$$0 \longrightarrow S^1 \longrightarrow S^1 \wr \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Note that $\mathbb{Z}_2$ acts trivially on $H^*(BS^1) = \mathbb{Z}_2[y]$, that y^2 comes from the Euler class e of the representation of $S^1 \wr \mathbb{Z}_4$ on $\mathbb{H}$, and that the group acts freely on S^3 hence has periodic cohomology. This forces $\tau(y) = t^3$ where t generates $H^1(B\mathbb{Z}_2)$ so

$$H^*(B(S^1 \wr \mathbb{Z}_4)) = \mathbb{Z}_2[t, e]/(t^3).$$

Now for the group $\mathbb{Z}_{2^\nu} \wr \mathbb{Z}_4$ we use the extension

$$0 \longrightarrow \mathbb{Z}_{2^{\nu-1}} \cdot x^2 \longrightarrow \mathbb{Z}_{2^\nu} \wr \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 0$$

Now $H^*(B\mathbb{Z}_{2^{\nu-1}}) = \mathbb{Z}_2[u, y]/(u^2 = 2^{\nu-2}y)$ and by killed x^4 we get a map of $\mathbb{Z}_{2^\nu} \wr \mathbb{Z}_4$ to a familiar extension, showing that

$$\tau(u) = t_1^2 + t_1 t_2 + 2^{\nu-2} t_2^2$$

where $t_i \in H^1(B(\mathbb{Z}_2 \times \mathbb{Z}_2))$ are characterized by

$$t_1(x) = 1 , \quad t_1(y) = 0$$
$$t_2(x) = 0 , \quad t_2(y) = 1$$

By use of the map $\mathbb{Z}_{2^\nu} \wr \mathbb{Z}_4 \longrightarrow S^1 \wr \mathbb{Z}_4$ which takes x to t_2 we see that $t_2^3 = 0$ in $H^3(B(\mathbb{Z}_{2^{\nu-1}} \wr \mathbb{Z}_4))$. As t_2^3 is relatively prime to $t_1^2 + t_1 t_2 + 2^{\nu-2} t_2^2$ we see that this is what $\tau(y)$ must be and also that the spectral sequence can be calculated to show that

$$H^*(B(\mathbb{Z}_{2^\nu} \wr \mathbb{Z}_4)) = \mathbb{Z}_2[t_1, t_2, e]/(t_1^2 + t_1 t_2 + 2^{\nu-2} t_2^2, t_2^3)$$

I want to determine if there are proper subgroups of $\mathbb{Z}_{2^\nu} \wr \mathbb{Z}_4$ which detect the non-zero element of H^3 which one computes is represented by $t_1^2 t_2$. Suppose H is such a subgroup; if $H \subset \mathbb{Z}_{2^\nu}$, as $\nu \geq 2$ the generator $z \in H^1(B\mathbb{Z}_{2^\nu})$ has square zero, hence as $t_i \mapsto \lambda_i z$, $t_1^2 t_2$ goes to zero. Thus $H \not\subset \mathbb{Z}_{2^\nu}$ so $H = \langle x^{2^j}, x^i y \rangle$ with $j \geq 1$; as $x^a (x^i y) x^{-a} = x^{i+2a} y$ we may, by replacing H by a conjugate, assume that $i = 0$ or 1. In the former case $t_1 | H = 0$, hence $t_1^2 t_2$ goes to zero. In the latter $(t_1 + t_2) | H = 0$ hence $t_1^2 t_2$ and $t_2^3 = 0$ have the same restriction to H, so $t_1^2 t_2$ goes to zero.

<u>Conclusion:</u> If $4 | q-1$, then no proper subgroup of the Sylow 2-subgroup of $N_2(\mathbb{F}_q) = \mathbb{Z}_{q-1} \wr \mathbb{Z}_4$ detects cohomology in dimension 3.

Since $N_{2n} = \Sigma_n \wr N_2$ we find that there is a detecting family of subgroups for $N_{2n}(\mathbb{F}_q)$ which are of form $(Q_{2^{\nu+1}})^i \times \mathbb{Z}_2^j$ where $q-1 = 2^\nu \cdot (\text{odd})$ and $Q_{2^{\nu+1}} =$ the generalized quaternion group $\mathbb{Z}_{2^\nu} \wr \mathbb{Z}_4$. So we conclude

<u>Proposition:</u> If $4 | q-1$, then

$$H^*(BSp_{2n}(\mathbb{F}_q)) \hookrightarrow \prod_K H^*(BK)$$

where K runs over representatives for the conjugacy classes of maximal subgroups of $Sp_{2n}(\mathbb{F}_q)$ of the form $(Q_{2^{\nu+1}})^i \times \mathbb{Z}_2^j$.

Note that $\mathbb{Z}_2^n \subset Sp_{2n}$ is the unique (up to conjugation) maximal elementary abelian 2-subgroup, where this inclusion comes from putting $\mathbb{Z}_2 \hookrightarrow Sp_2$ as $\pm I$ and taking the direct sum n-times.

The proposition on the preceding page should be more precisely formulated as follows.

Proposition: Let $Q_{2^{\nu+1}}^n \hookrightarrow Sp_{2n}(\mathbb{F}_q)$ be the subgroup obtained by taking the direct sum n-times of the inclusion $Q_{2^{\nu+1}} \hookrightarrow N_2(\mathbb{F}_q)$ considered above. Then

$$H^*(BSp_{2n}(\mathbb{F}_q)) \hookrightarrow H^*(BQ^n_{2^{\nu+1}}).$$

Proof: By induction on n. If n is not a power of 2 then we can write $n = i+j$ with $i,j < n$ such that the subgroup $Sp_{2i}(\mathbb{F}_q) \times Sp_{2j}(\mathbb{F}_q) \longrightarrow Sp_{2n}(\mathbb{F}_q)$ contains a Sylow 2-subgroup. If $n=1$ its OKAY. If $n = 2m$, then the subgroup $\mathbb{Z}_2 \wr Sp_{2m}(\mathbb{F}_q) \hookrightarrow Sp_{2n}(\mathbb{F}_q)$ also contains a Sylow subgroup. Then by basic wreath theory we know that $\mathbb{Z}_2 \times \Delta Q^m_{2^{\nu+1}} \longrightarrow Sp_{2n}(\mathbb{F}_q)$ and $Q^m_{2^{\nu+1}} \times Q^m_{2^{\nu+1}} \to Sp_{2n}(\mathbb{F}_q)$ detect cohomology. I have to produce an element of $Sp_{2n}(\mathbb{F}_q)$ which conjugates $\mathbb{Z}_2 \times \Delta Q^m_{2^{\nu+1}}$ into $Q^{2m}_{2^{\nu+1}}$. Look at it from the point of representations; we are given a group H acting on W and we consider $\mathbb{Z}_2 \times \Delta H$ and $H \times H$ acting on $W \times W$. Now we can decompose $W \times W$ under $\mathbb{Z}_2$, since the char. is odd, say

$$W \times W = \Delta W \oplus \{(\omega, -\omega)\}$$

and using the isomorphism

$$(*) \qquad \omega_1, \omega_2 \longmapsto \left(\omega_1 + \omega_2, \frac{-\omega_1 + \omega_2}{2}\right)$$

which is symplectic, we can conjugate and suppose that $\mathbb{Z}_2 \times H$ acts on $W \times W$ where $\mathbb{Z}_2$ acts non-trivially on the first factor and trivially by the second. Now here H has an element ε in its center acting as -1 on W, so we see that the representation is the restriction of that of $H \times H$ on $W \times W$ by means of the homomorphism $\theta : \mathbb{Z}_2 \times H \longrightarrow H \times H$ such that $\theta(-1, h) = (\varepsilon h, h)$. This gives the desired conjugation of $\mathbb{Z}_2 \times Q^{m}_{2^{\nu+1}}$ into $Q^{2m}_{2^{\nu+1}}$ and proves the proposition.

Just to be on the safe side we check the conjugation over $\mathbb{C}$. Thus we are given the subgroup $\{\mathbb{C}^* , j^{\varepsilon}\} \in Sl_2(\mathbb{C})$

$$Q = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -z^{-1} \\ z & 0 \end{pmatrix} \;\middle|\; z \in \mathbb{C}^* \right\}.$$

and we take the matrix

$$\left[\begin{array}{c|c} I & -\frac{1}{2}I \\ \hline +I & +\frac{1}{2}I \end{array}\right]$$

which is simplectic since $\begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$ has determinant $+1$ and the above matrix is the Kronecker product. Then conjugation

$$\begin{bmatrix} \frac{1}{2}I & +\frac{1}{2}I \\ -I & I \end{bmatrix} \begin{bmatrix} 0 & -z^{-1} & & \\ z & 0 & & \\ & & 0 & -z^{-1} \\ & & z & 0 \end{bmatrix} \begin{bmatrix} I & -\frac{1}{2}I \\ +I & \frac{1}{2}I \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -z^{-1} & & \\ z & 0 & & \\ & & 0 & -z^{-1} \\ & & z & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ -I & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & -\frac{1}{2}I \\ I & \frac{1}{2}I \end{bmatrix} = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ -I & I \end{bmatrix} \begin{bmatrix} I & \frac{1}{2}I \\ I & -\frac{1}{2}I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

Thus we get both the $\mathbb{Z}_2$ and ΔQ in $Q \times Q$ after conjugation.

Now we know from the étale cohomology spectral sequence that

$$gr\{H^*(BSp_{2n}(\mathbb{F}_8))\} \cong H^*(BSp_{2n}) \otimes H^*(Sp_{2n})$$

is ~~additively~~ as an $H^*(BSp_{2n})$-module at least additively an exterior algebra with generators of degrees $4j-1$, $j=1,\dots,n$. For $n=1$ I know that

$$H^*(BSp_2(\mathbb{F}_8)) \hookrightarrow H^*(BQ_{2^{\nu+1}})$$

has for image the ring $\Lambda[y] \otimes S[x]$ where y, x have degrees $3+4$ respectively. This suggests the following conjecture.

<u>Conjecture:</u> The image of the map $H^*(BSp_{2n}(\mathbb{F}_8)) \longrightarrow H^*(BQ^n_{2^{\nu+1}})$ is contained in the ring $\mathbb{Z}_2[y,x]^{\otimes n}$ and

$$H^*(BSp_{2n}(\mathbb{F}_8)) \cong \Lambda[dc_1,\dots,dc_n] \otimes S[c_1,\dots,c_n]$$

where the c_i are the quaternionic Chern classes and where d is the derivation of degree -1 of $H^*(BQ^n_{2^{\nu+1}})$ which carries the ~~…~~ x_i to y_i and kills other elements of $H^i(BQ_{2^{\nu+1}})$.

Suppose $h(X) = [X, B]$ where B is an infinitely commutative H-space. Then by Boardman we can embed h in a connected cohomology theory h^* with $h = h^0$. Now suppose that $f : Y \to X$ is a covering of degree d. In the suspension category we have maps $f_! : Y \to X$ (this is the map induced by f) and $f^! : X \to Y$ (this is the trace and is defined since f is proper and framed.) The composition

$$X \xleftarrow{f_!} Y \xleftarrow{f^!} X$$

corresponds to the maps on any cohomology theory

$$\{X, Z\} \xrightarrow{f^*} \{Y, Z\} \xrightarrow{f_*} \{X, Z\} .$$

~~By the projection formula~~

$$\cancel{f_*(f^* u) = f_* 1 \cdot u \qquad f_* 1 \in \pi^0(X)}$$

~~the endomorphism $f_! f^!$ in the suspension category is~~

Question: Does $f_! f^!$ induce an isomorphism $\{X, Z\} \to \{X, Z\}$ after inverting d = degree f?

The idea is that if k^* is a multiplicative cohomology theory, then $f_* f^* u = f_* 1 \cdot u$ for $u \in k^0(X)$ and $\varepsilon(f_* 1) = d$ where $\varepsilon : pt \to X$. Thus $f_* 1 - d \in \tilde{k}^0(X)$, so it is nilpotent. Thus if $\frac{1}{d}$ exists in $k^0(pt)$, $f_* 1$ will be a unit in $k^0(X)$. This should also hold for any theory which is a module over k^*. Thus taking $k^*(X) = \{X, S\}[\frac{1}{d}]$, the question should be true in general.

<u>Lemma</u>: Let B be a homotopy symmetric H-space representing a trace theory k. Let $f: Y \longrightarrow X$ be a finite covering of degree d where X is a finite complex of dimension r. Then if $x \in \operatorname{Ker} f^*: k(X) \longrightarrow k(Y)$, we have $d^r x = 0$.

<u>Proof</u>: We use induction on the dimension r of X, the case $r=0$ being trivial since then f has a section. Let $X^{(r-1)}$ be the $(r-1)$-skeleton of X; then there is a cofibration

$$X^{(r-1)} \xrightarrow{i} X \longrightarrow \bigvee_{i \in I} e_i/\dot{e}_i$$

where $\{e_i, i \in I\}$ is the set of r-cells of X. This gives rise to an exact sequence

$$k(X^{(r-1)}) \xleftarrow{i^*} k(X) \xleftarrow{\sum u_i} \bigoplus_{i \in I} k(e_i, \dot{e}_i).$$

By induction hypothesis $d^{r-1} i^* x = i^*(d^{r-1}x) = 0$, hence there are elements $x_i \in k(e_i, \dot{e}_i)$ such that

$$d^{r-1}x = \sum u_i x_i$$

If we pull the covering Y back to e_i, which we think of as a map $e_i \longrightarrow X$, then the covering is trivial of degree d so that the composition

$$k(e_i, \dot{e}_i) \xrightarrow{f^*} k(f^{-1}e_i, f^{-1}\dot{e}_i) \xrightarrow{f_*} k(e_i, \dot{e}_i)$$

is multiplication by d. (More precisely we have this kind of commutative diagram

$$\begin{tikzcd}
k(Y) \arrow[d, shift right, "f_*"'] & k(f^{-1}e_i, f^{-1}\dot{e}_i) \arrow[l, "v_i"'] \arrow[d, shift right, "f_*"'] \arrow[r, phantom, "\cong"] & k_c(Y|e_i) \arrow[d, shift right, "f_*"'] \arrow[r, phantom, "\cong"] & k_c(e_i \times \{1,\dots,d\}) \arrow[dl, shift right, "f_*"'] \\
k(X) \arrow[u, shift right, "f^*"'] & k(e_i, \dot{e}_i) \arrow[l, "u_i"'] \arrow[u, shift right, "f^*"'] \arrow[r, phantom, "\cong"] & k_c(e_i) \arrow[u, shift right, "f^*"'] \arrow[ur, shift right, "f^*"'] &
\end{tikzcd}$$
)

so

$$\begin{aligned}
d^{r}x &= \sum u_i\, dx_i \\
&= \sum u_i\, f_* f^* x_i \\
&= f_* f^*\left(\sum u_i x_i\right) = f_* f^*(d^{r-1}x) = 0
\end{aligned}$$

which was to be proved.

(The good way to rewrite the proof is to put $A = X^{(r-1)}$

$$k(A) \longleftarrow k(X) \longleftarrow k_c(X-A)$$

and then argue that Y is trivial over $X-A$ so that $f_*f^*z = dz$ if $z \in k_c(X-A)$.)

The point is that $Px_{ij} = (c_{p-1}^{1/2})^i x_{ij}$ up to sign since the Bockstein is zero by assumption and the other operations are of too high degree.

Thus setting $\omega = c_{p-1}^{1/2}$ we get that

$$P_X u = \sum_{i=0}^{2q} \boxed{\omega^{-i}} u_i$$

might be algebraic, whence the u_i are algebraic!!

---

in the examples we know that the normalizer of the torus carries the cohomology by sylow theory.

basic question: Injectivity of restriction to the normalizer for a general infinitely commutative H-space?

$$\begin{tikzcd}
BN_{n,\overline{\mathbb{F}}_p} \arrow[r] \arrow[d] & BN_{n,\tilde{\mathbb{Z}}_{(p)}} \arrow[d] & BN_{n,\mathbb{C}} \arrow[l] \\
BGl_{n,\overline{\mathbb{F}}_p} \arrow[r] \arrow[d] & BGl_{n,\tilde{\mathbb{Z}}_{(p)}} \arrow[d] & BGl_{n,\mathbb{C}} \arrow[l] \arrow[d] \\
\mathrm{Spec}\,(\overline{\mathbb{F}}_p) \arrow[r] & \mathrm{Spec}\,\tilde{\mathbb{Z}}_{(p)} & \mathrm{Spec}\,\mathbb{C} \arrow[l]
\end{tikzcd}$$

and now we know that by our cohomological calculation

$$\varinjlim BGl_n(\mathbb{F}_{p^k}) \xrightarrow{\simeq} BGl_{n,\overline{\mathbb{F}}_p}$$

and similarly for N_n. But the point is that $N_n^{(\mathbb{F}_{p^k})} \longrightarrow Gl_n(\mathbb{F}_{p^k})$ descent is always possible at $l \neq p$ once $l \mid p^k - 1$.

February 23, 1970.

It remains to check that $RO_k(G) \oplus RSp_k(G)$ is a λ-ring in such a way that the forgetful homomorphism to $R_k(G)$ is a λ-ring map.

~~Product: Let B = $RO_k(G) \oplus RSp_k(G)$ and consider~~

~~(E,) (E', E')~~

For this aspect of the theory it is probably simpler to use the unitary theory. So we suppose given a ring A with an involution $-$ and an element λ with $\lambda\bar{\lambda} = 1$. A unitary module is a fin. gen. proj A-module E together with a non-degenerate sesquilinear form $B(x,y)$, $E \otimes \bar{E} \longrightarrow A$ such that

$$B(y,x) = \lambda \overline{B(x,y)} \qquad \text{(off 2 perhaps.)}$$

We agree to identify two such E's if the forms differ by a ~~unit~~ real unit of A. Call the resulting ~~theory~~ Grothendieck group ~~KU~~ $KU^\lambda(A)$. I want to define the tensor ~~prod~~ product map

$$KU^\lambda(A) \otimes KU^{\lambda'}(A) \longrightarrow KU^{\lambda\lambda'}(A)$$
$$[E] \otimes [F] \longrightarrow [E \otimes F]$$

For this it suffices to check that the relations go to zero. But if $E' \subset E$ is ~~an~~ isotropic, then so is $E' \otimes F \subset E \otimes F$ and

$$\left(E' \oplus E/E'^{\perp}\right) \otimes F \;\cong\; E'\otimes F \oplus E\otimes F/(E'\otimes F)^{\perp}$$

$$\left(E'^{\perp}/E\right) \otimes F \;\cong\; (E'\otimes F)^{\perp}/E'\otimes F$$

so it's clear.

Next I want to define the λ-operations

$$KU^{\lambda}(A) \longrightarrow 1+\prod_{q\geq 1} t^{q} KU^{\lambda^{q}}(A)$$

$$[E] \longmapsto \sum [\Lambda^{q} E]$$

has sequilinear form

$$B(x_1 \cdots x_q, y_1 \cdots y_q) = \det\{B(x_i, y_j)\}$$

So again we must check that the basic relations hold. Thus suppose $E' \subset E$ is an isotropic direct summand. If you could find a complement Z to $E'^{\perp}$, then we would have

$$E \cong (E' \oplus Z) + (E' \oplus Z)^{\perp} \qquad \text{o.d.s.}$$

and there is no problem with additivity for o.d.s. case.
~~The fact of the point is that if $E' \subset E$ is isotropic, so~~
~~The basic filtration is~~

$$\cancel{0 \subset E' \subset E'^{\perp} \subset E} \qquad ?$$

~~and now~~

Cohomology mod 2 of $BSp_n(\mathbb{F}_q)$ $4 | q-1$.

Form $\bigoplus_{n \geq 0} H_*(BSp_{2n}(\mathbb{F}_q))$ ring. ~~and there is a basic map~~

~~[illegible]~~ Let ~~[illegible]~~ N_{2n} be the normalizer of the torus $\cong \Sigma_n \int N_2$
where
$N_2(\mathbb{F}_q) = \mathbb{Z}_4 \int \mathbb{F}_q^*$

Then I get a surjection by Sylow theory

$$H_*(BN_{2n}(\mathbb{F}_q)) \longrightarrow H_*(BSp_{2n}(\mathbb{F}_q))$$

permitting me to define the ~~[illegible]~~ additive diagonal in

$$\bigoplus_{n \geq 0} H_*(BSp_{2n}(\mathbb{F}_q))$$

Now we know that the composition

$$\underset{\underset{\mathbb{Z}_2[b_i]}{\|}}{S\{H_*(BN_2(\mathbb{F}_q))\}} \longrightarrow H_*(BN_*(\mathbb{F}_q)) \longrightarrow H_*(BSp_*(\mathbb{F}_q))$$

is surjective. ~~so~~

$\mathbb{Z}_2[b_i]$ where b_i is a basis for $H_*(BN_2(\mathbb{F}_q))$

now actually ~~[illegible]~~ since we have a ring homomorphism, ~~we can~~ we can throw away the b_i ~~such the~~ which go to zero in $Sp_2(\mathbb{F}_q)$. Thus we get two kinds

ξ_i dim $4i$ $i \geq 1$
τ_i dim $4i-1$ $i \geq 1$.

February 23, 1970. The Boardman-Vogt theorem.

Given a trace theory $k(X) = [X, B]$, I want to extend it to a generalized cohomology theory. The key step is to define a map $QB \longrightarrow B$, i.e. a family of maps $\Omega^n S^n B \longrightarrow B$. If X is a ~~finite complex~~ C^∞-manifold, then

$$\varinjlim_n [X, \Omega^n S^n B] = \varinjlim_n [S^n X, S^n B]$$

$$= \left\{ \begin{array}{l} \text{cobordism classes of maps } X \xleftarrow{f} Y \xrightarrow{g} B \\ \text{where } f \text{ is proper framed of rel. dim. } 0 \end{array} \right\}$$

Thus we have to define the element $f_* g^*(E_{univ})$ in $k(X)$, where E_{univ} is the canonical element of $k(B)$. In other words we have to extend the trace to framed proper maps of rel. dim. 0.

Now I would like to use ~~...~~ theorems in surgery to do this. ~~...~~ Thus I wish to make f into a homotopy equivalence and apply the (homotopy) inverse of f to pull g down. Unfortunately this method supposes that $f_* f^* = id$ in case f is a (framed) homotopy equivalence for any generalized coh. theory. But if I recall Seattle this needn't be the case, although it might be so when the boundary is non-empty.

~~Unfortunately it~~

To put ourselves in good surgery position we replace X, which is always assumed to be of the homotopy type of a finite complex, by a regular neighborhood in some high Euclidean space. Thus X is a compact C^∞ manifold with boundary with no compact components and $\pi_1(\partial X, x) \xrightarrow{\sim} \pi_1(X, x)$ for all $x \in \partial X$. For simplicity suppose that X is connected. It suffices to define f_*

only for f of degree 1 where the degree is the integer obtain by mapping $pt \to X$ and $B \to pt$. According to Sullivan there is no problem in showing that Y may be surgered until f is a homotopy equivalence and that one may carry the map $g: Y \longrightarrow B$ along with the surgery.

Now the problem becomes whether $f_* 1 = 1$. The idea is that if so, then there is a manifold $W \xrightarrow{h} X$ with $\partial W = X \cup Y$ and $h/X = id$, $h/Y = f$. Actually $W \xrightarrow{h} X \times I$ is given. Now one should be able to surger h keeping it fixed on $h^{-1}(X \times \{0,1\})$ until it's a homotopy equivalence. But then X is diffeomorphic to Y by the h-cobordism theorem and f is equivalent to a diffeomorphism which is extremely unlikely.

<u>Conclusion</u>: Surgery is no good.

February 23, 1970. On Mumford's conjecture

Let k be an algebraically closed field of characteristic p, let G be a reductive algebraic group over k, and let E be a representation of G. Assume that ~~[illegible]~~ E has a 1-dimensional quotient $E \to L$ stable under G. Then for some ~~[illegible]~~ integer $m \geq 1$ ~~≥ 0~~ the map $S_m E \longrightarrow S_m L = L^{\otimes m}$ has an equivariant section.

1.) Replacing E be $E \otimes L^{-1}$ we may suppose that L is a trivial representation and prove that

$$(S_m E)^G \longrightarrow (S_m L)^G$$

is surjective for ~~sufficiently large~~ some m.

2.) Let T be a maximal torus and let N be its normalizer. It is clear that the conjecture is true for N ~~[illegible]~~ (Nagata). Let m be such that

$$(S_m E)^N \longrightarrow (S_m L)^N$$

is surjective. I ~~am going~~ would like to show that m works for G.

3.) ~~It is enough to show~~ Claim that

$$(S_m E)^G = (S_m E)^{G(\mathbb{F}_q)}$$

for some large finite field $\mathbb{F}_q$. This is because $G(\overline{\mathbb{F}}_p)$ is ~~dense~~ dense in G for the Zariski topology.

4) By Chevalley

Question is equivalent to detecting cohomology in unipotent groups. Thus if G is a unipotent alg. group acting on V and we have an element of $H^1(G,V)$, we have to have a criterion which enables us to decide when this is zero or better when the extension

$$0 \dashrightarrow S_{p^\nu} V \longrightarrow S_{p^\nu} 1 \longrightarrow 0$$

is trivial. Note that for the Mumford conjecture if we want

$$S_{p^\nu} V \longrightarrow 1$$

to have a section for a given ν, it suffices to have the element vanish on $H^1(G,V$

Corollary: $$H^1(G,V) \hookrightarrow \varprojlim_{q} H^1(G(\mathbb{F}_q), V)$$

Proof: An element of $H^1(G,V)$ is an exact sequence

$$0 \to V \longrightarrow \tilde{V} \longrightarrow 1 \to 0$$

which is zero iff $\tilde{V}^G \twoheadrightarrow 1$. But by density $\tilde{V}^G = \cap \tilde{V}^{G(\mathbb{F}_q}$

Proof of Mumford conjecture: G alg. gp. reductive over k of char p V representation of G. L 1-dim'l quotient of V. for some n the map $S_nV \longrightarrow S_nL$ has an inv. section.

~~Proof~~ Enough to do for $k = \overline{\mathbb{F}}_p$. One knows that it's true for the normalizer N of a max. torus T. Assume $G = [G,G]$ so that just have to prove invariants are onto.

$$(S_kV)^G = \bigcap_q (S_kV)^{G(\mathbb{F}_q)}$$

$$(S_kV)^{G(\mathbb{F}_q)} \twoheadrightarrow S_kL \qquad \text{all } k \equiv 0 \ (N)$$

~~conj. classes of $GL_n(\mathbb{F}_q)$~~

$$\begin{pmatrix} 0 & a & b \\ & 0 & \\ & & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & & a \\ & 0 & b \\ & & 0 \end{pmatrix} \quad \text{conj.}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} az \\ bz \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} ay+bz \\ 0 \\ 0 \end{pmatrix}$$

non-conj. since image of one is 2 dim & here is 1-dim'l.

$\coprod_{n\geq 0}$ ~~[illegible]~~ $\mathrm{Hom}_{\mathrm{rgs}}\left(R(Gl_n(\mathbb{F}_q)), \mathbb{C}\right)$ is the free commutative monoid (with unit) with generators (f,i) f irreducible monic polynomial of degree ≥ 1 with $f(0)$ a unit (equivalently a Galois conjugacy class of elements of $\overline{\mathbb{F}}_q^*$) and i is an integer $\geq$ ~~0~~ 1 of degree $|(f,i)| = i \deg f$. Thus the generators ~~[illegible]~~ correspond to divisors of G_m rational over $\mathbb{F}_q$ with irreducible suppose, e.g. a closed point of G_m over $\mathbb{F}_q$ with multiplicity ≥ 1.

Indecomposable modules: ~~Suppose~~ Let A be an elementary abelian p gp. over $\mathbb{F}_p$ and let M be an indecomp. $S(A^*)$-module such that $a^p M = 0$ all $a \in A$.

M indecomposable $\Longleftrightarrow$ $\mathrm{End}_{S(A)}(M)$ local ring.

Assume that in ~~[illegible]~~

$$\begin{array}{ccc} \mathrm{Aut}(M) & \longleftarrow & A \\ 1+a & \longleftarrow & a \end{array}$$

is a maximal elementary abelian p-grp. Then we have that if $\theta \in \mathrm{End}_{S(A)}(M)$ and $\theta^p = 0$, then $\theta \in A$.

~~$\mathrm{Hom}_{S(A)}(k, M)$~~ ~~$k[x]/(x^p)$~~. $\left(t_1 X_1 + \cdots + t_r X_r\right)^{l+1} = 0$

understand maximal commutative subalgs of Gl_n
are they all of rank $< l$.

[e]- subgps in $Gl_n\mathbb{Z}$

February 24, 1970: attempt: Mumford conj.

Can you compute $H^*_*(BGl_n(\mathbb{Z}))$? In particular how far off is the map

$$H^*(BGl_n(\mathbb{Z})) \longrightarrow H^*(B\Sigma_n) \qquad (\text{coeff. mod } \ell)$$

from being an isomorphism. (This map is the analogue of the canonical map of stable cohomotopy theory to any gen. coh. theory.)

Idea: Consider the action of $Gl_n(\mathbb{Z})$ on the symmetric space $X = Gl_n(\mathbb{R})/O_n(\mathbb{R}) =$ pos. def. real symm. matrices. I shall assume, until I can check with Borel, that $Gl_n(\mathbb{Z})$ acts nicely enough so that there is no problem with equivariant cohomology. Then since X is contractible

$$E_2 = H^*(BG, H^*(X)) \Longrightarrow H^*_G(X)$$

shows that the equivariant cohomology is what I want. Now form the standard bundle

$$E = Gl_n(\mathbb{R}) \times_{O_n(\mathbb{R})} \mathbb{C}^n \longrightarrow X$$

associated to the standard representation of $O_n(\mathbb{R})$ on $\mathbb{C}^n$. Then I should have that

$$\operatorname{Spec} H_G(X) = \left(\operatorname{Spec} H_G(\operatorname{Flag}(E))\right)_{\Sigma_n}.$$

Actually what I am doing is the following sequence of steps:

~~$H^*(BG)$~~

$$H^*(B\,Gl_n(\mathbb{Z})) \cong H^*(E\,Gl_n(\mathbb{Z}) \times_{Gl_n(\mathbb{Z})} Gl_n(\mathbb{R})/O_n)$$

$$\cong H^*([E(Gl_n\mathbb{Z}) \times E(O_n)] \times_{Gl_n(\mathbb{Z}) \times O_n} Gl_n(\mathbb{R}))$$

$$\cong H^*_{O_n}(\underbrace{Gl_n(\mathbb{R})/Gl_n(\mathbb{Z})}_{\text{space of lattices in } \mathbb{R}^n})$$

Let $\mathcal{L}$ be this space of lattices. I am interested in the elementary abelian l-subgroups of O_n fixing lattices in $\mathbb{R}^n$.

To make things simpler use the similar isomorphism

$$H^*(B\,Gl_n(\mathbb{Z})) \cong H^*_{U_n}(\underbrace{Gl_n(\mathbb{C})/Gl_n(\mathbb{Z})}_{\text{space of lattices in } \mathbb{C}^n})$$

space of lattices in $\mathbb{C}^n$, ie $\mathbb{Z}$-submodules L of $\mathbb{C}^n \ni L \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^n$.

(Embed $Gl_n(\mathbb{Z})$ into $Gl_n(\mathbb{C})$ descend to U_n!) According to my theorems, the spectrum of ~~$H^*(BGl_n\mathbb{Z})$~~ this should reduce to ~~elementary~~ conjugacy classes of elementary abelian l-subgroups ~~~~ of U_n which fix points of ~~~~ of $\mathcal{L}$.

<u>Conclusion</u>: $H^*(B\,Gl_n(\mathbb{Z}), \mathbb{Z}_l) \longrightarrow \varprojlim_{A} S(A^*)$ F iso.

A runs over the ~~conjugacy classes~~ category of elementary abelian l-subgps of $Gl_n(\mathbb{Z})$.

Try to classify the maximal elementary abelian 2-subgroups of $Gl_n(\mathbb{Z})$ up to conjugacy. ~~Thus we wish to~~ Suppose given one $H \subset Gl_n(\mathbb{Z})$. Thus we have an action of H on a free $\mathbb{Z}$-module L and over $\mathbb{Z}[\frac{1}{2}]$ we can form the eigenspaces $L[\frac{1}{2}] = \bigoplus_{\chi} E^{\chi}$ where χ runs over the homomorphisms from H to $\{\pm 1\}$. ~~Because~~ This shows that the rank of H is $\leq n$, ~~I shall suppose that L is indecomposable as an H-module which is completely equivalent to $L \otimes \mathbb{Z}_2$ being an indecomposable $\mathbb{Z}_2[H]$-module. (This is because a splitting would lift to one of $L \otimes \hat{\mathbb{Z}}_{(2)}$~~

since as the representation is faithful the χ for which $E^{\chi} \neq 0$ generate $\check{H}$. So next we consider the representation mod 2^n or better over $\hat{\mathbb{Z}}_{(2)}$. Suppose that $\hat{L}$ is indecomposable.

?

Given $\mathbb{Z}_2^n$ acting on $\mathbb{C}^n$ in the standard way, we want to classify the different invariant lattices under homotopy.

$H^*(BGl_n(\mathbb{Z}), \mathbb{Z}_\ell) \longrightarrow H^*(B\Sigma_n, \mathbb{Z}_\ell)$ can't be an isomorphism because the dimensions are wrong. Thus $Gl_n(\mathbb{Z})$ contains an elementary abelian ℓ-subgroup of rank $\left[\frac{n}{\ell-1}\right]$ coming from the embedding

$$Gl_{\left[\frac{n}{\ell-1}\right]}\left(\mathbb{Z}[\exp 2\pi i/\ell]\right) \hookrightarrow Gl_n(\mathbb{Z})$$

while the ~~...~~ maximal rank of an $[\ell]$-subgroup of Σ_n is

$$\left[\frac{n}{\ell}\right]$$

coming from the product $\Sigma_\ell^{[n/\ell]} \subset \Sigma_n$. Clearly

$$\left[\frac{n}{\ell-1}\right] > \left[\frac{n}{\ell}\right]$$

for suitable n, e.g. $n = \ell - 1$

Proof of assertion at bottom of page 2: To show that the category of $[\ell]$-subgps in $Gl_n(\mathbb{Z})$ is equivalent to the category of pairs (A, λ) where A is an $[\ell]$-subgroup of U_n and $\lambda \in \pi_0(\mathcal{L}^A)$, $\mathcal{L} = Gl_n(\mathbb{C})/Gl_n(\mathbb{Z})$. First step is to go from

$$\{[\ell]\text{-subgps of } Gl_n(\mathbb{Z})\} \longrightarrow \left\{\begin{matrix}\text{transitive } Gl_n(\mathbb{Z}) \text{ sets} \\ \text{whose isotropy groups are } [\ell]\text{-subgps.}\end{matrix}\right\}$$

$$\longrightarrow \left\{\begin{matrix}\text{transitive } Gl_n(\mathbb{C})\text{-spaces} \\ \text{over } Gl_n(\mathbb{C})/Gl_n(\mathbb{Z}) \text{ with} \\ [\ell]\text{-subgroups for isot. groups}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{trans. } Gl_n(\mathbb{C}) \text{ spaces over } \mathcal{L} \\ \text{with } [\ell]\text{-subgroups for isot. gps} \\ \text{homotopy classes of maps}\end{matrix}\right\}$$

$$\longrightarrow \left\{\begin{matrix}\text{trans. } U_n \text{ spaces over } \mathcal{L} \\ \text{with } [\ell]\text{- subgps for isot.} \\ \text{groups + homotopy classes of maps}\end{matrix}\right\}$$

We only have to check that the last step is an equivalence of categories. ~~...~~ The functor backwards is

$$\{X \xrightarrow{f} \mathcal{L}\} \longmapsto \{Gl_n(\mathbb{C}) \times_{U_n} X \longrightarrow Gl_n(\mathbb{C}) \times_{U_n} \mathcal{L} \longrightarrow \mathcal{L}\}$$

and the point to check is that $Gl_n^{(\mathbb{C})} \times_{U_n} \mathcal{L} \longrightarrow \mathcal{L}$ is an equivariant homotopy equivalence for any compact subgroup of $Gl_n(\mathbb{C})$. In effect given $A \subset Gl_n(\mathbb{C})$ and $\lambda \in \pi_0(\mathcal{L}^A)$ one knows how to conjugate A into U_n. ~~Also given the~~ It seems clear.

We know that $Gl_n(\mathbb{Z})$ contains $\mathbb{Z}_2^n$ as diagonal matrices. Suppose $A \subset Gl_n(\mathbb{Z})$ is an elementary abelian 2-subgp of rank n. Then if I think of A as acting faithfully on a free abelian group E I can break up $E[\frac{1}{2}]$ into a sum of eigenspaces

$$E[\tfrac{1}{2}] \cong \bigoplus_{i=1}^n L_i[\tfrac{1}{2}],$$

where $L_i \subset E$ is the corresponding invariant subspace of E. I claim that

$$2E \subset \bigoplus_{i=1}^n L_i \subset E.$$

Only the first has to be proved, so let $e = \sum x_i$, $x_i \in L_i[\frac{1}{2}]$ ~~[illegible]~~. Now choose an element $\sigma_i \in A \ni \sigma_i = +1$ on L_i and -1 on the others. Then

$$(e + \sigma_i e) = 2x_i \in E \cap L_i[\tfrac{1}{2}] = L_i$$

showing that $x_i \in \frac{1}{2}L_i$ for all i, hence that $2e \in \bigoplus L_i$ as claimed.

Note that A acts trivially on $\bigoplus L_i \otimes \mathbb{Z}_2$. Here's how to construct non-conjugate [2]-subgroups of $Gl_n(\mathbb{Z})$ of rank n. Start with with the standard representation of $\mathbb{Z}_2^n$ on $\mathbb{Z}^n = M$. Then any lattice

$$2M \subset L \subset M$$

is stable under $\mathbb{Z}_2^n$. Choosing a basis for L ~~we find~~ ~~...~~ i.e. an isomorphism $\mathbb{Z}^n \xrightarrow{\sim} L$ as $\mathbb{Z}$ modules we obtain a subgroup of $Gl_n(\mathbb{Z})$ isomorphic to $\mathbb{Z}_2^n$. ~~...~~ In this way I get a map

$$L \longmapsto \{\text{conjugacy class of } \mathbb{Z}_2^n \text{ in } Gl_n(\mathbb{Z})\}$$

which is in fact surjective by my previous argument. It seems that the only indeterminacy in going backwards is the ordering of the eigenspaces. This leads to the following.

<u>Conclusion:</u> Conjugacy classes of $\mathbb{Z}_2^n$ in $Gl_n(\mathbb{Z})$ are in 1-1 correspondence with ~~subspaces~~ Σ_n-orbits of non-zero subspaces V of $\mathbb{Z}_2^n$ such none of the coordinate vectors $(0,\cdots,1,\cdots,0)$ belong to V.

<u>Conclusion:</u> ℓ-rank of $Gl_n(\mathbb{Z})$ equals $\left[\frac{n}{\ell-1}\right]$.

<u>Remark:</u> A similar analysis of a maximal $[2]$-subgroup A of $Gl_n(\mathbb{Z})$ of lower rank runs into trouble because we can't show that $2E \subset$ sum of eigenspaces.

~~Question: Is $H^i(G(\mathbb{F}_q), V)$ a Galois module.~~

Green's paper & the cohomology mod p of $H^*(BGl_n(\mathbb{F}_q))$.
character theory:

$$\bigoplus_n R_*(Gl_n(\mathbb{F}_q)) \qquad \text{becomes a ring.}$$

Conjugacy classes: $\mathrm{Hom}(\mathrm{Spec}, \coprod_n \mathrm{Spec}\, R(Gl_n(\mathbb{F}_q)))$ graded monoid

a point in this is a conjugacy class and the generators are the irreducible Jordan blocks.

~~each field contributes~~

Suppose A semi-simple $d\times d$ indecomposable matrix ~~with Then the eigenvalues of A are~~ irreducible equation of degree d

So let γ_d be the number of **irred. equations** of degree d. Then you get one semi-simple irred. class. Thus conj. classes are ~~of the form~~ indexed by ~~[illegible]~~ pairs d, j where d stands for eigenvalue field and j for index of nilpotency.

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conj. cl. of $Gl_n(\mathbb{F}_q)$ are ~~[illegible]~~ sequences $(f_1, \varepsilon_1) \cdots (f_r, \varepsilon_r)$

$\ni$ f_i irreducible monic equation and

$$n = \sum \varepsilon_i \deg f_i$$

February 24, 1970: Projects:

1. Spectrum of an equivariant cohomology ring: It is necessary to add $p=0$, the localization + fixpoint theorems. I propose to rewrite this paper along the following lines: First separate out the dimension theorem + examples (Σ_n, $Gl_n(\mathbb{Z})$, $Gl_n(\mathbb{F}_q)$) as part I. Then organize part II by first computing what happens for $H^*_T(X)$ and then reducing to this. ~~[illegible]~~ Need:

$$\operatorname{Spec}\{A[X_1,\dots,X_n]/(\sigma_i(X)=a_i)\}_{\Sigma_n} \longrightarrow \operatorname{Spec} A$$

is a universal homeomorphism (and an F-isom. in char. p.) It might be nice to write this up in such a way that the analogues for $K_G(X)$ are clear.

~~[illegible]~~

Adams' conjecture:

2. ~~Cohomology of the finite classical groups with applications to and the Adams' conjecture.~~ I propose to write possibly ~~[illegible]~~ a complete proof of the Adams' conjecture without etale cohomology. At the moment I need to check ~~the passage from fields~~ that the ~~[illegible]~~ fact that $BGl_\infty(\mathbb{F}_p) \longrightarrow BU$ induces $\simeq$ coh. ~~[illegible]~~ mod $l \neq p$ implies that the Adams' conjecture is true. The problem is knowing how to deal with these infinite complexes. I also need the orthogonal Brauer theory — some minor technical lemmas remain.

3. Cohomology of $BG(\mathbb{F}_q)$. At the moment you have a general theorem which gives good information mod l for l odd. For Sp + O some problems remain at 2. General question about restriction to T is unclear.

4. Symmetric groups. Nowhere near in writable form. The main theorem is the computation of $\bigoplus_{n \geq 0} H_*(E\Sigma_n \times_{\Sigma_n} X^n)$ for any space X, possibly also with twisted coefficients. ~~[illegible]~~ X is superfluous I think. The idea was to construct Steenrod ops.

$$H(X) \longrightarrow \operatorname{Hom}_{\text{rgs}}(R, H(X))$$

and then use the Klein groups to put elements in R. ~~[illegible]~~ I wanted to produce by Steenrod ~~[illegible]~~ a multiplicative operation

$$H(X) \xrightarrow{\gamma} \mathbb{Z}_p[\xi_\nu, \tau_\nu]_{\nu \geq 0} \otimes H(X) \qquad \begin{array}{ll} \xi_\nu & 2p^\nu - 2 \\ \tau_\nu & 2p^\nu - 1 \end{array}$$

$$\gamma(x) = \sum_{\nu \geq 0} \tau_\nu (\beta x)^{p^\nu} + \xi_0 x$$

$$\gamma(\beta x) = \sum_{\nu \geq 0} \xi_\nu (\beta x)^{p^\nu} \qquad x \in H^1(X)$$

which ~~[illegible]~~ would ~~[illegible]~~ prove that $\mathbb{Z}_p[\xi_\nu, \tau_\nu] \longrightarrow R_{univ}$ was free. Then I hoped to find an argument showing this map had to be an isomorphism. Finally I hoped to go back and put the symmetric groups into shape.

5. The Boardman-Vogt theorem and KADL-theory: I have now a formulation of BV thm. and what I think might be a lousy proof using Milgram + Nakaoka to show $\coprod_n E\Sigma_n \times_{\Sigma_n} X^n \longrightarrow QX$ induces an isomorphism on classifying space cohomology.

6. Higher K-theory: I have a definition of $K_i(R)$ $i > 0$ and can hopefully compute for $R = \mathbb{F}_q$, at least the off char. part. And I have supporting evidence for symmetric groups

~~January 7, 1970 (still groggy).~~

~~I want to compute the K-A-D-L operations for the additive structure of BO.~~

Following are ideas of last night

1. Look again at Mumford conjecture
2. symplectic cob.
3. Is Chern subring of $H^*(G, \mathbb{Z}_p)$ F-equivalent?

4. If X is a G-manifold oriented, then X/G is a rational homology mainfold, ~~so following ...~~ but the quadratic form doesn't have to have det $+1$ so there are more invariants of the signature type.

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5. $\mathbb{Z}_\ell + \bigoplus_{k\geq 1} H_*(GL_k(\mathbb{Z}_q), \mathbb{Z}_\ell)$ ℓ prime $\ell \nmid q$.

odd cohomology of the symmetric groups

6. infinite loop space and the map

$$\varinjlim_{k\geq 1} X^k_{\Sigma_k} \longrightarrow \varinjlim_{n} \Omega^n \Sigma^n X$$

for a pointed space. Is this a (weak) homotopy equivalence (look at semi-simplicially to get formula for QX)

7. Can one construct interesting classes in $H^*(BPL)$ using the KADL method.

8. Higher order K-groups as ~~...~~ homotopy of infinite general linear group.

Take $U^*_{G,mod}(X)$ modified so as to have a f. grp. law.

Does

$$U^*_{G,mod}(X) \otimes_{Laz_G} H^*_G$$

give the same spectral results?

February 27, 1970. Brauer modular theory of orthogonal and symplectic representations.

Let G be a group and let K be a field. By an orthogonal (resp. symplectic) representation of G over K we mean a representation of G in a finite dimensional vector space V endowed with ~~an~~ homothety class of invariant non-degenerate symmetric (resp. skew-symmetric) bilinear forms. Thus the forms B and λB, $\lambda \in K^*$, define the same quadratic structure on V.

Introduce Grothendieck groups $RO_K(G)$ and $RSp_K(G)$ by the relations

(i) $V = V' \oplus V''$ orth. direct sum $\Rightarrow [V] = [V'] + [V'']$

(ii) $W \subset W^{\perp} \subset V \Rightarrow [V] = [W + V/W^{\perp}] + [W^{\perp}/W]$.

and we check that there are maps

(*)
$$\begin{tikzcd} & RO_K(G) \arrow[dr, "\text{forget}"] & \\ R_K(G) \arrow[ur, "\text{hyperbolic}"'] \arrow[dr] & & R_K(G) \\ & RSp_K(G) \arrow[ur, "\text{forget}"'] & \end{tikzcd}$$

All the above should work for a commutative ringed topos, and also in the unitary theory.

Then there should exist two further aspects

1.) λ-ring structure on $RO_K(G) \oplus RSp_K(G)$.

2.) decomposition (or specialization) homomorphism

$$d : RO_{A[t^{-1}]}(G) \longrightarrow RO_{A/tA}(G)$$

defined when t is a regular element of A (and maybe A regular?)

Check compatibility of this structure with $(*)$.

<u>Trick</u>: How to use the decomposition homomorphism to define the λ-operations. The point is to check that the Λ^g are compatible with the equivalence relations. Thus I am given $0 \subset W \subset W^\perp \subset V$ and I want to show that

$$[\Lambda^n V] = [\Lambda^n ((W + V/W^\perp) \oplus W^\perp/W)]$$

But consider the polynomial ring $K[T]$ and $t = T$. Then in $V \otimes K[T, T^{-1}]$ I have two "lattices"

$$L_1 = V \otimes K[T] \quad , \quad (W \cdot T^{-1} + W^\perp + V \cdot T) \cdot K[T] = L_2$$

and

$$L_1 \otimes_{K[T]} K = V$$

$$L_2 \otimes_{K[T]} K = W \oplus W^\perp/W \oplus V/W^\perp$$

To see the last equation use the exact sequence

$$\begin{tikzcd}[row sep=small]
 & V[T] \\
W^\perp[T] \arrow[ur, "\text{inc.}"] \arrow[r, "T"'] & \oplus \\
\oplus & W^\perp[T] \arrow[r] & L_2 \arrow[r] & 0 \\
W[T] \arrow[ur, "\text{inc.}"] \arrow[r, "T"'] & \oplus \\
 & W[T]
\end{tikzcd}$$

which on killing T yields the desired result. Note that L_2 has a non-degenerate form (split the flag + check). Now since there is a decomposition theorem we know that $\Lambda^n L_1$, $\Lambda^n L_2 \subset \Lambda^n V[T, T^{-1}]$ give the same element in $RO_K(G)$ which is what we want to have proved.

<u>Question:</u> Is $RO_A(G) \longrightarrow RO_{A[t^{-1}]}(G)$ onto?

For this you probably need A regular since this is required for R.

<u>Outline of simplest path:</u> assume K field char $\neq 2$, large.

1.) definition of $RO_K(G) \oplus RSp_K(G)$
+ maps ~~to R_K~~ from and to $R_K(G)$

2.) ~~assume K large + char ≠ 2, then~~ By induction on the length of a quadratic repn. get explicit description of $RO_K(G)$ and $RSp_K(G)$ as submodules of $R_K(G)$ and we get basic exact sequence.

$$R_K(G) \xrightarrow{*-1} R_K(G) \longrightarrow RO_K(G) \oplus RSp_K(G) \longrightarrow \cdots$$

3) $k \leftarrow A \rightarrow K$ A d.v.r. then you check that

$$\begin{array}{ccc} RO_K(G) & \hookrightarrow & R_K(G) \\ \downarrow & & \downarrow d \\ RO_k(G) & \hookrightarrow & R_k(G) \end{array}$$

and also for Sp.

4) now introduce λ-ring structure on $RO_K(G) \oplus RSp_K(G)$ and check well-defined, actually you know this because to show that

$$[\Lambda^i E] = [\Lambda^i((W+W/W^\perp) \oplus W^\perp/W)]$$

it is enough to know same in $R_K(G)$. To show λ-ring identities hold enough to use

$$RO_K(G) \oplus RSp_K(G) \hookrightarrow R_K(G \times \mathbb{Z}_2)$$

The only thing that remains is 3) namely to prove the dotted arrow exists. So I take an orthogonal representation E of G over K and choose an invariant lattice L. It is necessary to show that the semi-simple $k[G]$-module associated to $L\otimes_A k$ carries a non-degenerate symmetric bilinear form.

Example: 1). Suppose we can find an invariant lattice L such that $\pi\check{L}\subset L\subset\check{L}$, where $\check{L}=\{x\in E \mid B(x,L)\subset A\}$. Then

$$0\longrightarrow \check{L}/L \xrightarrow{\ \pi\ } L\otimes k \longrightarrow \check{L}\otimes k \longrightarrow \check{L}/L \longrightarrow 0$$

is exact. Now $L\otimes k\longrightarrow \check{L}\otimes k \simeq (L\otimes k)^*$ is the restriction of B to L so the image of this map carries a non-degenerate symmetric form. It remains to produce one for $\check{L}/L$. But $B:\check{L}\otimes\check{L}\longrightarrow A\pi^{-1}$ (since $\check{L}\subset\pi^{-1}L$) and the reduction of this has L for kernel so it induces a non-degenerate symmetric form on $\check{L}/L$.

~~2). Suppose there exists an invariant lattice L with $\pi^2\check{L}\subset L\subset\check{L}$.~~

$$\cancel{L\supset\pi\check{L}\cap L\supset\pi L+\pi^2\check{L}\supset\pi L}$$

~~and~~ $\cancel{\{x\in L \mid B(x,L)\subset\pi A\}=L\cap\pi\check{L}}$

$$\cancel{\{x\in\pi\check{L}\cap L \mid B(x,\pi\check{L}\cap L)\subset\pi^2 A\}=\pi\check{L}\cap L\cap(\pi L+\pi^2\check{L})}$$

$$\cancel{\pi L+\pi^2\check{L}}$$

superfluous

2.) Suppose there is an invariant lattice L with $\pi^n L^\vee \subset L \subset \pi^{n-1} L^\vee$ for some integer n. Then setting $L = \pi^i L_1$ we have

$$\pi^{n-i} L_1^\vee \subset \pi^i L_1 \subset \pi^{n-1-i} L_1^\vee$$

so if we set $n = 2i + \varepsilon \qquad \varepsilon = 0, 1$ we have

$$\pi^\varepsilon L_1^\vee \subset L_1 \subset \pi^{\varepsilon - 1} L_1^\vee$$

and hence setting $L_2 = L_1$ if $\varepsilon = 1$ and $L_2 = L_1^\vee$ if $\varepsilon = 0$ we have an invariant lattice with

$$\pi L_2^\vee \subset L_2 \subset L_2^\vee$$

and we are reduced to the preceding case.

3.) Next suppose there is an invariant lattice L such that $\pi^2 L^\vee \subset L \subset L^\vee$. Then

$$\pi \underbrace{\left(L + \pi L^\vee\right)}_{L_1}{}^\vee = \pi\left(L^\vee \cap \pi^{-1} L\right) = \pi L^\vee \cap L$$

and

$$\underbrace{L + \pi L^\vee}_{L_1} \underset{\text{trivial}}{\supset} \underbrace{\pi L^\vee \cap L}_{\pi L_1^\vee} \underset{\text{hypothesis}}{\supset} \underbrace{\pi L + \pi^2 L^\vee}_{\pi}$$

Thus if $L_1 = L + \pi L^\vee$ we have that

$$\pi L_1^\vee \subset L_1 \subset L_1^\vee$$

and so are reduced to case 1).

4.) Now we can always produce an invariant lattice L satisfying $\pi^g \check{L} \subset L \subset \check{L}$ for some positive integer g. Assume g least >0 and let $g = 2j - \varepsilon$ with $\varepsilon = 0, 1$, so that $\pi^{2j} \check{L} \subset L \subset \check{L}$. Repeat preceding argument with π^j instead of π. Thus

$$\left(L + \pi^j \check{L}\right)^\vee = \check{L} \cap \pi^{-j} L \supset L + \pi^j \check{L}$$

$$\pi^j \left(L + \pi^j \check{L}\right)^\vee = \pi^j \check{L} \cap L \subset L + \pi^j \check{L}$$

and so we get an invariant lattice with g replaced by j. Now $j < g$ impossible so $j = g = \varepsilon$ so $j = g = 1$.

<u>Conclusion:</u> It is always possible to find a lattice L invariant under G such that $\pi \check{L} \subset L \subset \check{L}$