December 4, 1970

Tate conjectures that

$$\varprojlim_n H^1(\mathrm{Gal}(\bar{F}/F), \mu_{\ell^n}^{\otimes 2}) = \begin{array}{c}\text{cyclic group}\\ \text{dealing with}\\ \text{roots of 1}\end{array} \times \mathbb{Z}_\ell^{r_2}$$

but believes it and related results are as deep as Iwasawa conjectures.

Tate tells ~~...~~ me that $\mathrm{Gal}(\bar{F}/F)$ is of cd $= 2$. so the only things of interest are

$$c_i : K_a \longrightarrow H^{2i-a}(\mu^{\otimes i})$$

$$\begin{array}{ll} a = 2i-1 & H^1(\mu^{\otimes i}) \\ a = 2i & H^2(\mu^{\otimes(i+1)}) \end{array}$$

just like in the local case.

Basic corollary of Hilbert thm. 90

$$0 \to H^2(\mu_\ell) \longrightarrow H^2(F^*) \xrightarrow{\ \ell\ } H^2(F^*)$$

exact while

$$H^1(\mu_\ell) \xleftarrow{\sim} F^*/(F^*)^\ell .$$

For a number field F

$$0 \longrightarrow Br(F) \longrightarrow \bigoplus_p Br(F_p) \longrightarrow Q/\mathbb{Z} \longrightarrow 0$$

is exact, last map being given by sum of local invariants.

Hence

$$0 \to H^2(\mu_\ell) \longrightarrow \bigoplus_{\text{all } p} \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

will be exact. (Assume for simplicity $\mu_\ell \subset F$, ℓ odd so that no real places $\exists$). Then

$$H^1(\mathbb{Z}/\ell\mathbb{Z}) \cong F^*/(F^*)^\ell \otimes \mu_\ell^{\otimes(-1)}$$

$$H^2(\mathbb{Z}/\ell\mathbb{Z}) \cong {}_\ell Br(F) \otimes \mu_\ell^{\otimes(-1)}$$

and the other cohomology groups are zero. Thus it seems that everything should be computable in principle. Cup product should be given by the Hilbert symbol

$$(a,b) \longmapsto (a,b)_\ell \in {}_\ell Br(F) \otimes \mu_\ell$$

which is defined in Serre.

Your conjecture that $H^*(GL(F))$ injects on torus and is generated by Chern class components seems to ignore the cohomology in

$$H^*(Gal(\bar{F}/F), \mu^{\otimes *})$$

not lying in the subring generated by $H^1(\mu^{\otimes 1})$. In effect the first Chern class has only two components: the geometric one in $H^0(\mu_\ell^{\otimes 1})$ and the other c_1'' with coefficients in $H^1(\mu_\ell^{\otimes 1})$. Thus the Chern classes of the standard repn. of $(F^*)^n$ lie in $H^*((F)^{*n}, S.)$ where $S.$ is subring generated by $H^1(Gal(\bar{F}/F)) = F^*/(F^*)^\ell$. So it would be extremely nice to know that $H^*(Gal(\bar{F}/F))$ is generated by $H^1(Gal(\bar{F}/F))$. For example is ${}_\ell Br(F) \otimes \mu_\ell$ generated by Hilbert symbols $(a,b)_\ell$?

material related to paper on normal p-complements
(thm. proved Dec. 9, paper finished Jan 4, '71
conj formulated Nov 13–14

December 4, 1970

~~Discussion~~ on normal p-complements:

Tate has proved that if $H^1(G) \xrightarrow{\sim} H^1(P)$ (coeffs. in $\mathbb{Z}/p\mathbb{Z}$, G finite gp, P Sylow p-subgp.) then G has a normal p-complement. Tate proves by induction on r that $P/\Gamma_r^{(p)}P \xrightarrow{\sim} G/\Gamma_r^{(p)}G$ as follows. Assume this true for a given r and denote by Q this quotient gp and by $M = \Gamma_r^{(p)}G$ and $N = \Gamma_r^{(p)}P$. Then by Hochschild-Serre

$$\begin{tikzcd}
0 \arrow[r] & H^1Q \arrow[r] & H^1P \arrow[r] & (N/(P,N)N^p)^\vee \arrow[r] & H^2Q \arrow[r] & H^2P \\
0 \arrow[r] & H^1Q \arrow[r] \arrow[u, equal] & H^1G \arrow[r] \arrow[u, "\cong"] & (M/(P,M)M^p)^\vee \arrow[r] \arrow[u] & H^2Q \arrow[r] \arrow[u, equal] & H^2Q \arrow[u, hook]
\end{tikzcd}$$

five lemma gives isom. in middle, so done as $N/(P,N)N^p = \Gamma_r^{(p)}P/\Gamma_{r+1}^{(p)}P$, etc.

On the other hand Atiyah has proved that if $H^n(G) \xrightarrow{\sim} H^n(P)$ $\forall\, n \geq n_0$ then G has a normal p-complement. Atiyah uses the spectral sequence relating $R(G)$ and $H^*(G,\mathbb{Z})$. Precisely one has a convergent spectral sequence ~~starting~~ with $H^*(G,\mathbb{Z}_p)$ and converging to $R(G)\otimes\mathbb{Z}_p$. The hypothesis implies that $R(G)\otimes\mathbb{Z}_p \longrightarrow R(P)\otimes\mathbb{Z}_p$ becomes an isom. after $\otimes\, \mathbb{Q}_p$, i.e. $R(G)_p$ and $R(P)_p$ are free $\mathbb{Z}_p$-modules of the same rank. But one knows $R(G)_p$ is a free module over $\mathbb{Z}_p$ of rank = no. of conjugacy class of p unipotent elements, so there is no fusion of elements of P in G. One knows then by transfer theory that

$$P^{ab} \longrightarrow G^{ab} \longrightarrow P^{ab}$$

is an isomorphism, so done e.g. by Tate as $H^1(G) \xrightarrow{\sim} H^1(P)$.

From the point of view of the theorems on the spectrum one would like to know if Atiyah's result can be improved in the following way: (Yes paper written)

<u>Conjecture</u>: If p is odd and if $H^*(G) \longrightarrow H^*(P)$ is an F-isomorphism, then G has a normal p-complement.

For $p=2$ one takes ~~the~~ $G = (\mathbb{Z}/3\mathbb{Z}) \tilde{\times}$ (quaternion gp order 8), where $\mathbb{Z}/3\mathbb{Z}$ permutes i, j, k cyclically. Then this cyclic permutation acts trivially on the only elementary abelian 2-subgroup, hence $H^*(G) \longrightarrow H^*(P)$ is an F-isomorphism. So one must assume p odd.

Suppose $P \triangleleft G$ in which case we are trying to show that the p'-elements of G centralize P. However Thompson has shown that P possesses a characteristic subgroup C which is an extension of one elem. ab. p-group by another such that a non-trivial p'-auto θ of P acts non-trivially on C. Moreover $(P, C) \subset {}_pZ(C)$.

My theorem on the spectrum implies that the functor $\underline{A}(P) \longrightarrow \underline{A}(G)$ is an equivalence of categories, hence as P acts trivially on ${}_pZ(C)$ ~~so~~ must θ. Next given $c \in C$ we have $c^p = 1$, hence

If $\mathrm{Cent}_p(C)/D > 1$, then $\mathrm{Cent}_p(C)/D \cap {}_pZ(P/D) \neq 1$

so $\mathrm{Cent}_p(C)/D \cap C/D = \mathrm{Cent}_p(C)/D \cap \mathrm{Cent}_p(D)/D \cap \ldots$
$\neq 1$ contrad.

$\theta c = pcp^{-1}$ showing θ acts trivially on $C/{}_pZ(C)$. But θ being a p'-auto is thus trivial on C hence also on P.

Proof of Thompson's thm. (Gorenstein book)
Let D be a maximal char. ab. subgroup of P and let C be $\ni$

$$C/D = {}_pZ(P/D) \cap \mathrm{Cent}_p(D)/D$$

Then (i) $(P,C) \subset D$ ✓ (ii) C/D is a $[p]$-group ✓ (iii) $\mathrm{Cent}_p(C) = D$. Indeed

$$\mathrm{Cent}_p(C) \cap C = Z(C) = D$$

Moreover $\mathrm{Cent}_p(C) \subset \mathrm{Cent}_p(D)$

by max of D. and $\mathrm{Cent}_p(C)/D$ is a normal subgroup of P/D. Use that a non-trivial normal subgroup N of a p-group P $\ni$ ${}_pZ(P) \cap N \neq 1$ (let P act on N). * Conclude $\mathrm{Cent}_p(C) = D$ proving (iii). Now let a group A act on P. Then

$$(C,(A,P)) \subset ((C,A),P)\,(A,(C,P))$$

so if $(A,C) = 1$ we have $(C,(A,P)) = 1$ so $(A,P) \subset D$ and $(A,D) = 1$, so A stabilizes the series $P \supset D \supset 1$. Thus if A is a p'-gp acting trivially on ${}_pC$, then A acts trivially on P.

Finally replace C by its subgroup ${}_pC$ of elements of order $|p$ when p is odd. Then a p'-auto acting trivially on ${}_pC$ acts trivially on D and on C/D so is trivial

Conclusion: The conjecture is true if $P \triangleleft G$.

Another special situation. The subgroup ${}_pZ(P)$ is weakly closed in fact any normal elementary abelian p-subgroup A is, because given $gAg^{-1} \subset P$ have $\exists p$ with $gAg^{-1} = pAp^{-1} = A$. Consequently by the Grün theorem $H^*(G) \xrightarrow{\sim} H^*(N)$ where N is the normalizer of ${}_pZ(P)$. So one can assume that ${}_pZ(P) \triangleleft G$, as by Tate N has normal p-comp. $\Rightarrow$ same for G.

If you combine Frobenius and Thompson, you find that ~~if~~ every special p-subgroup of P is centralized by the p'-elements of its normalizer, then $\exists$ normal p-complement. Here special means either elementary abelian or a group C with $\Phi(C) = Z(C)$ elementary abelian. Now what you need is a variant where special is replaced by elementary abelian but where you must take the category, the point being that for a p'-auto. θ of an extra-special p-group, there are not enough θ-stable $[p]$-gps.

It appears necessary to learn the Alperin fusion theorems, as these are probably the correct way to understand that impossible Burnside theorem.

December 5, 1970: Haefliger's classifying space.

Topological category = category whose objects set and morphism set are spaces ∋ relevant maps (source, target, identity, composition) are continuous.

Equivalently a semi-simplicial space

$$\cdots X_3 \;\overset{\longleftarrow}{\underset{\longrightarrow}{\equiv}}\; X_2 \;\overset{\longleftarrow}{\underset{\longrightarrow}{\equiv}}\; X_1 \;\overset{\longleftarrow}{=}\; X_0$$

where $\quad X_2 = X_1 \times_{X_0} X_1 \qquad \cdot\!\longrightarrow\!\cdot\!\longrightarrow\!\cdot$

$\qquad\quad X_3 = X_1 \times_{X_0} X_1 \times_{X_0} X_1 \qquad \longrightarrow\longrightarrow\longrightarrow\cdot$

Example: A topological group

$$G\times G\times G \;\overrightarrow{\equiv}\; G\times G \;\overrightarrow{\equiv}\; G \;\overrightarrow{=}\; e$$

The classifying space of a topological category is the geometric realization of the semi-simplicial space in sense of Segal. At least this gives the good thing for a topological group.

Let Γ be a pseudo group of homeos. of a space Z. Then it gives a topological groupoid (morphisms are isos.; equivalently (after Groth.) the semi-simplicial space is a (strict) simplicial space.) such that the source and target maps are local homeomorphisms

$$\cdots \Gamma \times_Z \Gamma \;\overrightarrow{\equiv}\; \Gamma \;\underset{\text{étale}}{\overset{\text{étale}}{\rightrightarrows}}\; Z \qquad \text{~~étale~~}$$

It is customary to require that the different elements of γ give distinct germs of homeos, but this isn't a necessary feature of the definition. For example if a discrete group G acts on Z we get a (strict) simplicial space

$$\cdots G\times G\times Z \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; G\times Z \;\substack{\longrightarrow\\\longrightarrow}\; Z$$

where the source and target are etale, even though G might act trivially.

<u>Conjecture 1</u>: Haefliger's classifying space $B\Gamma$ is the realization (as with Segal) of the simplicial space

$$\text{Nerve}(\Gamma): \qquad \Gamma\times\Gamma \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \Gamma \;\substack{\xleftarrow{id}\\\longrightarrow\\\longrightarrow}\; Z$$

Here's the evidence: A Haefliger structure on X is defined as a cocycle $\gamma_{ij}: U_i\cap U_j \longrightarrow \Gamma$ with

$$\gamma_{ii}(x) = id\{\text{source } \gamma_{ii}(x)\}$$

$$\gamma_{ij}(x)\,\gamma_{jk}(x) = \gamma_{ik}(x).$$

modulo coboundaries

In other words a Haefliger structure on X is an element of

$$\check{H}^1(X,\Gamma) = \varinjlim_{\mathcal{U}} \pi_0\, \underline{\text{Hom}}(\text{Nerve } \mathcal{U}, \text{Nerve } \Gamma)$$

Now it should be so that $|\text{Nerve}(\Gamma)|$ maps to $B\Gamma$ and $|\text{Nerve}(\Gamma)|$ should be the closest representable functor to $\check{H}^1(X,\Gamma)$. But Haefliger's amazing result is that

$$\check{H}^1(X,\Gamma) \twoheadrightarrow [X, B\Gamma]$$

(at least if Γ = diffeos. of $\mathbb{R}^q$.)

with equivalence relation induced by homotopy (two structure on X equivalent if come from one on $X \times I$), so $|\text{Nerve}(\Gamma)|$ ought to $= B\Gamma$.

Now suppose Γ_q = local diffeomorphisms of $\mathbb{R}^q$, and let $\Gamma_q^{(0)}$ be the group of local diffeos. preserving origin. Then we have map

$$B\Gamma_q^{(0)} \longrightarrow B\Gamma_q$$

(inclusion of a full subcategory). Note that in spite of the fact that every point of $\mathbb{R}^q$ is equivalent to any other, the top. groupoid Γ_q is not equivalent to a group. However if we discretize $\mathbb{R}^q$ and Γ_q it is. The above map may be interpreted as the discretization

$$\begin{array}{ccc} \Gamma_q^{\text{discrete}} & \longrightarrow & \Gamma_q \\ \wr\!\!\| \text{ equivalence} & & \\ \Gamma_q^{(0)} & & \end{array}$$

$\Gamma \rightrightarrows Z$

so given ~~[illegible]~~ $\varphi_i : U_i \longrightarrow Z$

compose with

<u>Kan problem:</u> $\underline{Z \longrightarrow \Gamma}$

$\Gamma_q =$ ~~[illegible]~~ pseudo grp of diffeos. of $\mathbb{R}^q$

$\Gamma_q \longrightarrow O_q$

associate to ~~[illegible]~~ a germ γ.

the pair source etc. and tangent space

etc. etc.

what is the map

$GL_q(\mathbb{R}) \longrightarrow \Gamma_q$?

namely takes discrete group of autos. of $\mathbb{R}^q$

$GL_q(\mathbb{R}) \longrightarrow$

$GL_q(\mathbb{R}$

~~[illegible]~~ germs of diffeos.

<u>diffeomorphism of $\mathbb{R}^q$</u> <u>fixing origin</u>

$\boxed{\text{diffeos. fixing origin.}}$

first of all we have the whole affine group with discrete topology mapping with continuous topology.

$\boxed{\text{Thus if one has}}$

Take Γ
how about discontinuous transfs.

x

<u>diffeo. nbd. of origin</u>

cohomology classes

The point ~~of~~ to understand is ~~why~~ ~~the map~~ whether the DR style Chern classes

$$H^{2i}\left(GL_q(\mathbb{R}),\, H^*_{DR}(\mathbb{R}/\mathbb{Q})\right)$$

might extend to Γ_q via the canonical map

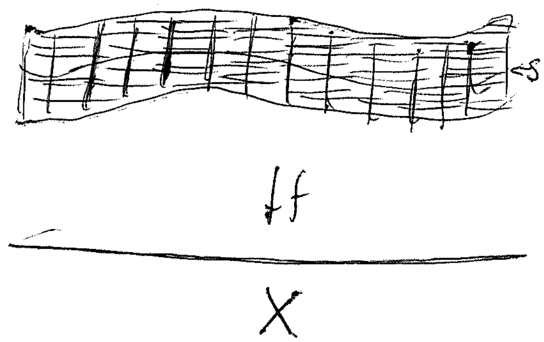
$$GL_q(\mathbb{R}) \longrightarrow \Gamma_q$$

micro-bundles: interpretation as a topological category. point is to make germs of homeos. into a "space" inductive limit etc. Thus the fibres of Γ over Z which normally have discrete topology are given a topology which renders things equivalent to a group again.

December 11, 1970: On Haefliger structures:

Suppose X is a manifold. Consider the following data i) a manifold E with a submersion $f: E \longrightarrow X$ of relative dimension q ii) a q-codiml. foliation $\mathcal{F}$ on E transversal to the fibres of f. iii) a continuous (not necessarily differentiable) section s of f.

To such a triple $(E, \mathcal{F}, s)$



is associated a Haefliger structure in the following way. For each x we can choose a germ of an isomorphism from a nbd. of $s(x)$ to $X \times \mathbb{R}^q$ compatible with the foliation. (Actually this means that we are considering $X \times \mathbb{R}^q \longrightarrow X$ with its obvious foliation transversal to the fibres.) Such a germ is defined over $s(U)$ for some U. So we get a covering U_i of X and a local isomorphism

$$V_i \hookrightarrow U_i \times \mathbb{R}^q$$

where V_i is a nbd of $s(U_i)$, the isomorphism being compatible with foliation and projection

Let X be a manifold and let $E \xrightarrow{\pi} X$ be a q-dimensional vector bundle over X endowed with a q codim. foliation transversal to the fibres such that the zero section is an integral leaf. I claim that then one gets a homomorphism

$$\pi_1(X) \longrightarrow \Gamma_q^{(0)} = \text{germs of diffeos. of } \mathbb{R}^q \text{ preserving origin.}$$

In effect one first chooses an isomorphism of $E_{x_0} \cong \mathbb{R}^q$. Now given an arc α in X, as S^1 is compact one can integrate out the horizontal curves in E covering α which start out close to zero. Thus existence of ordinary DE.'s gives a definite local autom. of $\mathbb{R}^q$ near the origin.

The next point is to understand just what the equivalence relation is on these animals

December 15, 1970:

Computation of max. $[l]$-subgps in $GL_n(\Lambda)$ where $\mu_l \subset \Lambda$, $l^{-1} \in \Lambda$ and Λ Dedekind domain

Let Λ be a comm. ring which is an algebra over $\mathbb{Z}_{(l)}$, l a prime number. Let A be an $[l]$-group and let A act on a finitely generated projective Λ-module E. Suppose

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is an exact sequence of representations of A over Λ. Then by the Maschke theorem there is an equivariant splitting

$$H^1(A, \operatorname{Hom}_\Lambda(E'', E')) = 0$$

as $\operatorname{Hom}_\Lambda(E'', E')$ is a $\mathbb{Z}_{(l)}$-module.

Thus E can be written as a direct sum of indecomposable representations.

To keep things simple suppose now that Λ is a Dedekind domain, and let K be its residue field. Then if E is indecomposable, $E \otimes_\Lambda K$ will be an irreducible representation of A over K. Indeed given a subrep. $0 < V < E \otimes_\Lambda K$, set $E' = E \cap V$; then $E/E' \hookrightarrow E \otimes_\Lambda K/V$ will be a torsion-free f.g. Λ-module, hence projective, so we get a non-trivial exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

contradicting indecomposability by the above.

Let $K[\mu_l]$ be cyclotomic extension of K (in some alg. closure $\bar{K}_s$ of K for simplicity). Then if A acts

non-trivially,
$\operatorname{End}_K(E\otimes_\Lambda K) \simeq K[\mu_\ell]$ with A action via a homomorphism $\chi: A \to \mu_\ell$ which is surjective. Put another way

$$(*) \qquad E\otimes_\Lambda K \simeq K[\mu_\ell]$$

with A acting through the character χ.

Relative to such an isomorphism $(*)$, E corresponds to a non-zero $\Lambda[\mu_\ell]$-submodule of $K[\mu_\ell]$. ~~Now $\Lambda[\mu_\ell]$ is the integral closure of~~ By the following lemma $\Lambda[\mu_\ell]$ is a Dedekind domain with quotient field $K[\mu_\ell]$, so E being a torsion-free f.g. rank 1 $\Lambda[\mu_\ell]$-module is an invertible $\Lambda[\mu_\ell]$-module and so we conclude

<u>Proposition:</u> Any non-trivial indecomposable representation E of A over Λ is isomorphic to an invertible $\Lambda[\mu_\ell]$-module with A acting via a surjection $\chi: A \to \mu_\ell$.

It remains to prove the

<u>Lemma:</u> $\Lambda[\mu_\ell]$ is the integral closure of Λ in $K[\mu_\ell]$.

Proof. As $\mathbb{Z}_{(\ell)} \to \mathbb{Z}_{(\ell)}[\mu_\ell] = \mathbb{Z}_{(\ell)}[T]/(1+\cdots+T^{\ell-1})$ is smooth so is

$$\Lambda \longrightarrow \Lambda[T]/(1+\cdots+T^{\ell-1})$$

and, hence the latter ring is regular and thus is the product of the localizations at its minimal primes. $\Lambda[\mu_\ell]$ is one of these localizations hence is regular hence a Dedekind domain. Thus

$\Lambda[\mu_\ell]$ is integrally closed, so being finite over Λ it is the integral closure.

Suppose now for simplicity that $\mu_\ell \subset \Lambda$. Then given a representation E we have

$$E = \bigoplus E_\chi$$

where $\chi \in \mathrm{Hom}(A, \mu_\ell)$ and

$$E_\chi = \{ e \in E \mid a \cdot e = \chi(a) e \}.$$

is a finitely generated projective Λ-module. Thus we see that

$$R_\Lambda(A) = K_0(\Lambda) \otimes [\mathrm{Hom}(A, \mu_\ell)]$$

because the category of representations is equivalent to the category of graded f.g. proj. Λ-modules $E = \oplus E_\chi$, grading over $\mathrm{Hom}(A, \mu_\ell)$.

Now take $GL_n(\Lambda)$ and a maximal elementary abelian p-subgroup A. Let E denote the repn. on Λ^n. and decompose it

$$E = \bigoplus_{\chi \in \hat{A}} E_\chi \qquad \hat{A} = \mathrm{Hom}(A, \mu_\ell)$$

Then each E_χ is of rank 1, because any f.g. projective is a sum of invertible modules. Now μ_ℓ^n acts on $\oplus E_\chi$ in evident fashion so we see that the $\chi_1, \ldots, \chi_n$ such that $E\chi_i \neq 0$ form a base for $\hat{A}$. (Precisely $(\chi_i): A \longrightarrow \mu_\ell^n$ which is injective, as A acts faithfully, and surjective by maximality.) So first of all we have

<u>Proposition</u>: If $\mathrm{Pic}(\Lambda) = 0$, then the diagonal subgroup μ_ℓ^n of $GL_n(\Lambda)$ is the only maximal [p]-subgroup up to conjugacy.

Now in general we have a [redacted] direct sum decomposition

$$\Lambda^n = L_1 \oplus \cdots \oplus L_n$$

where the L_i are invertible Λ-module and we take A to be μ_ℓ^n acting in the obvious way. More precisely given a maximal A, its eigenspaces determine such a decomposition so A is conjugacy to the [redacted] [p]-subgroup determined by the [redacted] decomposition. Now the only invariant of such a decomposition is the isom. classes of the L_i up to order so we have

<u>Proposition</u>: (Λ [redacted] Dedekind domain containing μ_ℓ and $1/\ell$). Then all [redacted] max. [p]-subgroups of $GL_n(\Lambda)$ have rank n and the conjugacy classes are in one-one correspondence with divisors $\sum_{i=1}^{n} [redacted] z_i$ of degree n with $z_i \in \mathrm{Pic}(\Lambda)$ and such that

$$\sum_{i=1}^{n} z_i = 0 \quad \text{in } \mathrm{Pic}(\Lambda).$$

This is analogous to conjugacy classes of [2]-groups in $O_n(\mathbb{F}_q)$, q odd. The analogue of $\mathrm{Pic}(\Lambda)$ in this case is the group $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2$ classifying one-dimensional quadratic spaces. Each maximal A determines an orthogonal direct sum decomposition

$$\mathbb{F}_q^n = \bigoplus_{i=1}^{n} L_i$$

where the L_i are one-dimensional quadratic spaces.

December 17, 1970

Let A be a d.v.r. with quotient field K and residue field k. Suppose $l \neq \mathrm{char}\, k$ and $\mu_l \subset K$. Want to compute mod l cohomology of $SL_2(K)$ using Serre's tree. It tells me that

$$SL_2(K) \xleftarrow[\sim]{(i,j)} SL_2(A) *_{\Gamma} SL_2(A)$$

where

$$i\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$j\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \pi^{-1}b \\ \pi c & d \end{pmatrix} \qquad \pi \text{ unif.}$$

and Γ is the intersection

$$\Gamma = \left\{ \begin{pmatrix} * & * \\ c & * \end{pmatrix} \;\middle|\; c \equiv 0 \quad (\pi) \right\}.$$

Suppose k finite, A discrete, and that we expect cohomology to respect the topology. Then if T is diagonal matrices have coh. isoms.

$$SL_2(A) \longrightarrow SL_2(k)$$

$$\Gamma \longrightarrow B(k) \longleftarrow T(k)$$

and hence an exact sequence of Mayer-Vietoris

$$\xrightarrow{\delta} H^i(SL_2(K)) \longrightarrow H^i(SL_2(k)) \oplus H^i(SL_2(k)) \longrightarrow H^i(T(k)) \xrightarrow{\delta}$$

$$x, y \longmapsto \mathrm{res}\, x + \mathrm{res}\, y.$$

Since res from $SL_2(k)$ to $T(k)$ is injective onto symmetric invariants for $\ell \neq 2$ we have

$$\boxed{0 \to H^{i-1}(SL_2(k)) \xrightarrow{res} H^{i-1}(T(k)) \xrightarrow{\delta} H^i(SL_2(K)) \xrightarrow{res} H^i(SL_2(k)) \to 0}$$

		1	1
1	1	0	0
0	y	*	0
0	x	* *	c_2''
c_2''	xy	*	c_2'
c_2'	x^2	0	0
0	x^2y	*	0
0	x^3	* *	$(c_2'c_2'')$
$c_2'c_2''$	x^3y	*	$(c_2')^2$
		0	

So it appears that $H^*(SL_2(K))$ has periodic cohomology of period 4, periodicity probably given by geometric c_2'.

Now we have conjectured that this cohomology injects to the torus and is the Weyl invariants

$$H^*(K^*) \cong H^*(\mathbb{Z} \times k^*) = \mathbb{Z}/\ell\mathbb{Z}[z, y, x]$$

z, y degree 1

x deg 2.

and the Weyl group $\mathbb{Z}_2$ acts by inverse on K^*

$$\sigma(x) = -x$$
$$\sigma(y) = -y$$
$$\sigma(z) = -z$$

$H^*(K^*)$	$H^*(K^*)^W$
1	1
z, y	0
x, zy	zy
xz, xy	xz, xy
x^2, xzy	x^2
x^2z, x^2y	0
x^3, x^2zy	x^2zy

$c = x^2$

$a = xz$, $b = xy$

$e = zy$

So it seems to be OKAY. The formula is

$$H^*(SL_2(K)) = H^*(K^*)^W = \mathbb{Z}/\ell\mathbb{Z}\,[\overset{4}{c}, \overset{3}{a}, \overset{3}{b}, \overset{2}{e}] \Big/ \begin{matrix}(a^2 = b^2 = 0) \\ ab = ce\end{matrix}$$

(This really (almost certainly) can be made into a proof because one has bounded the size of $H^*(SL_2(K))$ and on the other hand Chern classes should give the required elements in the cohomology.)

So your conjecture is verified for $\ell | q-1$, ℓ odd, and $SL_2(K)$.

Obsolete ~~see~~ Dec 25, 1970

December 19, 1970:

Let N be the normalizer of a maximal elem. ab. p-subgroup A of a finite group G and let C be the centralizer, so that all elements of order p of C are in A. I want to show that $\mathfrak{p}$ is a prime ideal in $H^*(C)$ with support A and if $\mathfrak{q}$ is the inverse image in $H^*(N)$, then

$$H^*(N)_{\mathfrak{q}} \xrightarrow{\sim} H^*(C)_{\mathfrak{p}}^{N/C}$$

not quite correct as there should be a whole N/C-orbit of $\mathfrak{p}$ lying over $\mathfrak{q}$.

Again I formulate this for N-spaces

$$(*) \qquad H^*_N(X)_{\mathfrak{q}} \xrightarrow{\sim} H^*_C(X)_{\mathfrak{p}}^{N/C}$$

and again I can restrict to having all isotropy groups elementary abel. p-groups, since invariants under a group is left exact.

Now the point will be to use Galois descent. I claim that $H^*_C(X)_{\mathfrak{p}}$ is an induced N/C-module. Consequently in an exact sequence bounded below like Mayer-Vietoris

$$0 \to H^0_C(U \cup V)_{\mathfrak{p}} \to H^0_C(U)_{\mathfrak{p}} \oplus H^0_C(V)_{\mathfrak{p}} \to H^0_C(U \cap V)_{\mathfrak{p}} \to \cdots$$

on applying the N/C-invariants we will get an exact sequence. Therefore for a nice N-space X the formula $(*)$ will reduce to orbits $X = N/B$ where B is a $[p]$-subgp. of N. If $(N/B)^A \neq \phi$ i.e. $A = n^{-1}An \subset B$, then maximality of $A \Rightarrow A = B$ whence

$$H_N^*(N/A) = H_A^*$$

$$H_C^*(N/A) = H_A^*(N/C) = H_A^* \otimes H^*(N/C)$$

as A acts trivially on N/C, and so $*$ is an isomorphism without localization. If $(N/B)^A = \phi$, then both sides are zero by localization. This proves $(*)$ modulo the claim.

<u>Lemma</u>: Let a finite group W act on a commutative ring R in such a way that it acts freely on the set $R(\Omega)$ of geometric points with values in any algebraically closed field Ω. Then if M is an R-module with a compatible action of W, we have

$$M \xleftarrow{\sim} M^W \otimes_{R^W} R.$$

More generally $M \longmapsto M^W$ and $N \longmapsto N \otimes_{R^W} R$

are quasi-inverse equivalences of the categories of R-modules with compatible W-action and the catagories of R^W- modules, R is a finitely generated projective R^W-module, and $M \longmapsto M^W$ is exact.

This is faithfully flat descent. I have only to show that R is a finitely generated ~~[illegible]~~ projective R^W- module of rank $|W|$ and that

$$R \otimes_{R^W} R \xrightarrow{\ \sim\ } \prod_{w \in W} R$$

$$r, s \longmapsto \left(r\, w(s)\right)_{w \in W}$$

because "compatible W-action" is just descent data. ~~[illegible]~~ I want to choose elements $s_w \in R$ such that the matrix ~~[illegible]~~ $w, w' \mapsto w(s_{w'})$ is invertible ~~[illegible]~~ near a prime m. For if this can be done then I have

$$\begin{tikzcd}
(R^W) \otimes_{\mathbb{Z}} \mathbb{Z}^n \arrow[r] \arrow[d] & R \otimes_{\mathbb{Z}} \mathbb{Z}^n \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \left(\prod_{w \in W} R\right) \otimes_{\mathbb{Z}} \mathbb{Z}^n \arrow[d, "\cong"] \\
R \arrow[r] & \prod_{W} R \arrow[r, shift left] \arrow[r, shift right] & \prod_{W \times W} R
\end{tikzcd}$$

In more detail, set $A = R^W$ and enumerate W: $\sigma_1, \dots, \sigma_n$. Suppose I can find $r_1, \dots, r_n \in R$ such that $\|\sigma_i(r_j)\|$ is invertible. Then given $x \in R$ there are unique $\lambda_j \in R$ such that

$$\sigma_i(x) = \sum_j \sigma_i(r_j)\lambda_j$$

Apply τ

$$\tau\sigma_i(x) = \sum_j (\tau\sigma_i)(r_j)\ \tau\lambda_j$$

and using that the $\{\tau\sigma_i\}$ is the same as $\{\sigma_i\}$, we see $\tau\lambda_j = \lambda_j$ by uniqueness. Hence taking $\sigma_i = 1$

$$x = \sum_{j=1}^{n} r_j \lambda_j$$

showing the r_i generate R as an A-module. Moreover they are a basis since

$$\sum r_j\lambda_j = 0 \implies \sum \sigma_i(r_j)\lambda_j = 0 \quad \text{all } i$$
$$\implies \lambda_j = 0 \quad \text{all } j.$$

The above is the Artin argument.

So now it is only necessary to produce such elements locally near a prime $\mathfrak{p}$ of A, hence we can assume A local and so by integrality argument that R is semi-local. Then by the Nakyama lemma (the matrix is invertible iff so after reduction mod $rad(R)$) we can look at ~~what~~ $R/rad(R)$ as an extension of $A/\mathfrak{p} = k$. But the hypothesis that W acts freely on $R(\Omega)$ shows that $R/rad(R)$ is a product

$$R/rad\,R = \prod_{G/H} K$$

where K is a Galois extension with group $H = Gal(K/k)$. So everything reduces to a Galois field extension, hence it's all clear.

December 20, 1970: On p-groups with $\Omega_1 P \subset Z(P)$.

If in my setup one thinks of $[p]$-groups as being primes, then one must think of p-groups P with $\Omega_1 P \subset Z(P)$ as being primary. Now we have the following due to Thompson (see Huppert book).

Lemma: If $\Omega_1 P \subset Z(P)$, then $P/\Omega_1(P)$ has the same property.

Proof: Let $H/\Omega_1 P \subset P/\Omega_1 P$ be a maximal normal $[p]$-subgroup. Now if $x \in P$ and $h \in H$, then $h^p \in \Omega_1 P$ so

$$(*) \quad h^p = g^{-1} h^p g = (g^{-1}hg)^p = (h(h,g))^p = h^p (h,g)^p ((h,g),h)^{\binom{p}{2}}$$

where we have used that $((h,g),h) \in \Omega_1 P$ hence centralizes h and (h,g), and the identity

$$(xy)^n = x^n y^n (y,x)^{\frac{n(n-1)}{2}} \qquad \text{if } (y,x) \in Z\langle x,y\rangle.$$

The formula $(*)$ implies (since p odd so $\frac{p(p-1)}{2} \equiv 0 \ (p)$) that $h^p = h^p (h,g)^p \implies (h,g) \in \Omega_1 P$. Thus $H/\Omega_1 P$ is in the center of $P/\Omega_1 P$, and as $H/\Omega_1 P$ is max. elem. ab. $\implies \Omega_1(P/\Omega_1 P) \subseteq H \subset Z(P/\Omega_1 P)$. q.e.d.

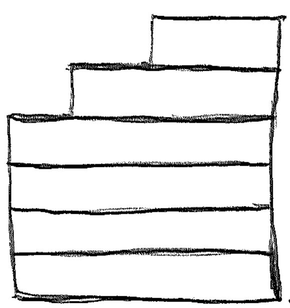
This has the implication that such a group admits a filtration

$$1 \subset F_1 \subset F_2 \subset F_3 \cdots\cdots \subset F_n = P$$

where each $F_i \lhd P$, F_i/F_{i-1} is an elementary abelian p-subgroup of the center of P/F_{i-1}, and multiplication by p

$$F_i/F_{i-1} \hookrightarrow F_{i-1}/F_{i-2}$$

is injective. Thus we have the familiar picture of an abelian group:



(so $\operatorname{rank} P/\Phi(P) \leq \operatorname{rank} \Omega_1(P)$)

The mod p cohomology of such a group is computable just like an abelian group

$$H^*(P) \cong \Lambda(P/\Phi(P))^\# \otimes S(\Omega_1 P)^\#$$

Proof: By induction on the length of the filtration

I would like to understand the cohomology of such a p-group P. First consider the situation where the group looks like

that is, the filtration is of length n and $x \longmapsto x^{p^{n-1}}$ induces an isomorphism of $P/\Phi(P) \xrightarrow{\sim} \Omega_1 P$. Then the cohomology looks like that of an abelian

NO group of type $(p^n, \ldots, p^n)$, and this can be proved by induction on n. I claim that if $V \subset H^2(P)$ is complementary to $\mathrm{Im}\{\Lambda^2 H^1(P) \longrightarrow H^2(P)\}$, then

$$H^*(P) \simeq \Lambda H^1(P) \otimes S(V)$$

and moreover V restricts isomorphically to a complement of $\Lambda^2 H^1(B)$ in $H^2(B)$ where $B = \Omega_1 P$. This is true for a $[p]$-group and by induction we assume it is true for P/B (of height $n-1 \geq 1$). Then consider the Hochschild-Serre s.s. for

$$0 \longrightarrow B \longrightarrow P \longrightarrow P/B \longrightarrow 0$$

and

$$d_2 : H^1(B) \longrightarrow H^2(P/B)$$

This is injective (as $H^1(P) \to H^1(B)$ is zero clearly, $n \geq 2$) and the image has rank $= d = \dim B$. Given an element x of $\Omega_1(P/B)$ one knows that

$$\begin{tikzcd}
H^1(B) \arrow[r, "d_2"] \arrow[d, equal] & H^2(P/B) \arrow[d] \\
H^1(B) \arrow[r, "d_2"] & H^2(\mathbb{Z}/p)
\end{tikzcd}$$

the map $\rightarrow\downarrow$ is non-zero as the extension of $\mathbb{Z}/p$ by B is non-trivial. The point is that d_2 as an element of $H^2(P/B) \otimes B = H^2(P/B, B)$ is the class of the extension. Therefore $W = \mathrm{Im}(d_2)$ is complementary to $\mathrm{Im}\, \Lambda^2 H^1(P/B) \subset H^2(P/B)$. More precisely I know that the image of $W \subset H^2(P/B) \longrightarrow H^2(\Omega_1(P/B)) \longrightarrow \beta H^1(\Omega_1(P/B))$ is surjective and both ends have same dimensions. By induction

$$H^*(P/B) = \Lambda[H^1(P/B)] \otimes S[W]$$

$$H^*(B) = \Lambda[H^1(B)] \otimes S[\beta H^1(B)].$$

Now $d_2 : H^1(B) \xrightarrow{\sim} W$, so by exactness of the Koszul sequence we have

$$E_3 = \Lambda H^1(P/B) \otimes S[\beta H^1(B)].$$

Now

$$d_3(\beta\lambda) = \beta(d_2\lambda) \qquad \text{mod image of } d_2$$

and so the only thing left to see is why $\beta(W) \subset W \cdot H^1(P/B)$ where $H^3(P/B) = \Lambda^3 H^1(P/B) \oplus W \cdot H^1(P/B)$.

Now given P/B, $d_2 \in H^2(P/B)\otimes B$ determines the extension and by induction

$$H^2(P/B) \cong \Lambda^2 H^1(P/B) + \underline{H^1(P/B)}.$$

What is the Bockstein $\beta : H^2(P/B) \longrightarrow H^3(P/B)$
$\beta : H^1(P/B) \longrightarrow H^2(P/B)$?

If $P/B = Q$ is a $[p]$-group, then the first β is injective and the second is zero on βH^1 and map $\Lambda^2 H^1$ into $H^1 \cdot \beta H^1$. Thus if $W \subset H^2(Q) = \Lambda^2 H^1(Q) \oplus \beta H^1(Q)$ is chosen carefully maybe $\beta W \not\subset W \cdot H^1(Q)$.

Carefully now: let $e_1, \dots, e_d$ be a basis for $H^1(Q)$ and let

$$\omega_i = \sum_{jk}{}' \alpha^i_{jk}\, e_j \wedge e_k + \beta e_i \qquad \alpha_{jk} = -\alpha_{kj} \in \mathbb{Z}/p$$

Then

$$\beta \omega_i = \sum_{jk} \alpha^i_{jk} \left(\beta e_j \wedge e_k - e_j \wedge \beta e_k\right)$$

$$= 2 \sum_{jk}{}' \alpha^i_{jk}\, e_j \wedge \beta e_k$$

Now does it follow that $\beta\omega_i \in H^1(Q)\cdot W$? If

$$\beta \omega_i = \sum_{\mu,\nu}{}' \delta^i_{\nu\mu}\, e_\nu \cdot \omega_\mu$$

$$2\sum_{jk}\alpha^i_{jk}\, e_j \wedge \beta e_k = \sum_{\mu,\nu}\delta^i_{\nu\mu}\, e_\nu\Big(\sum_{jk}\alpha^\mu_{jk}\, e_j e_k + \beta e_\mu\Big)$$

$$= \sum_{\nu j k} e_\nu e_j e_k\Big(\sum_\mu \delta^i_{\nu\mu}\alpha^\mu_{jk}\Big)$$

$$+ \sum_{\nu,\mu}\delta^i_{\nu\mu}\cdot e_\nu \cdot \beta e_\mu$$

Therefore
$$2\alpha^i_{\nu\mu} = \delta^i_{\nu\mu}$$

and so
$$\sum_{\nu j k}\Big(\sum_\mu \alpha^i_{\nu\mu}\,\alpha^\mu_{jk}\Big) e_\nu e_j e_k = 0 \qquad \text{all } i$$

when made anti-symmetric in $\nu j k$ must be zero for each i

$$\alpha^i_{\nu\mu}\alpha^\mu_{jk} + \alpha^i_{j\mu}\alpha^\mu_{k\nu} + \alpha^i_{k\mu}\alpha^\mu_{\nu j} = 0$$

Try to produce a counterexample with $d=3$

$$\omega_1 = \beta e_1$$
$$\omega_2 = \beta e_2$$
$$\omega_3 = \beta e_3 + \rho e_1\wedge e_2 + \sigma e_1\wedge e_3 + \tau e_1 \wedge e_3$$

Better
$$W \subset \Lambda^2 H \oplus \beta H \qquad H = H^1$$

think of as a map $T : \beta H \to \Lambda^2 H$: $\quad W = \{b + Tb \mid b \in \beta H\}$.

We should look at this as follows. The subspace $W \subset H^2(Q) = \Lambda^2 H \oplus \beta H$, $H = H^1(Q)$, is of the form

$$W = \{ \beta h + Th \mid h \in H \}$$

where $T: H \longrightarrow \Lambda^2 H$. Extend T to a degree one derivation of $\Lambda^* T$. Thus $T: \Lambda^2 H \longrightarrow \Lambda^3 H$ is given by

$$T(h_1 \cdot h_2) = Th_1 \cdot h_2 - h_1 \cdot Th_2 .$$

Then one knows that $T^2 : H \longrightarrow \Lambda^3 H$ is zero iff T is the transpose of a Lie algebra structure on the dual of H.

Suppose that $\beta(W) \subset H \cdot W \subset H^3(Q) \cong \Lambda^3 H + H \otimes W$. If

$$T(h) = \sum h_i' \wedge h_i''$$

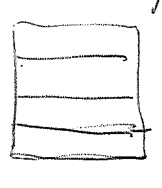
then

$$\beta T(h) = \sum \beta h_i' \wedge h_i'' - h_i' \wedge \beta h_i''$$

$$\equiv - \sum Th_i' \wedge h_i'' + h_i' \wedge Th_i'' \qquad \text{mod } H \cdot W$$

so $\beta(W) \subset H \cdot W \iff T^2 = 0$. Therefore one sees that H^* is a Lie algebra iff $\beta(W) \subset H \cdot W$.

<u>Conclusion:</u> Even among the p-group P with $\Omega_1 P \subset Z(P)$ of shape



The cohomology needn't have the nice form. However if the group P of height n admits an extension to one of height $n+1$, equivalently there is a subspace V of $H^2(P)$ restricting isomorphically to a complement of $\Lambda^2 H^1(\Omega,P)$ in $H^2(\Omega,P)$, then the cohomology should be ~~is~~ of the ~~[illegible]~~ nice form. (proof by spectral sequence)

The groups of height 2 are given by the commutator pairing $\Lambda^2 gr_1 P \longrightarrow gr_2 P \cong gr_1 P$. This extends to one of height 3 iff the pairing satisfies the Jacobi identity. Presumably there are higher obstructions to extendability, since if one can extend indefinitely one obtain an analytic pro-p-group with a good coordinate system (Lazard's book), and ~~[illegible]~~ the reduction mod p of ~~[illegible]~~ the Lie algebra should be of a rather restricted type of Lie algebras.

Note that there are groups P with $\Omega_1 P \subset Z(P)$ such that rank $P/\Phi(P) <$ rank $\Omega_1(P)$, and so it seems the only general fact about such groups is the Thompson inequality. Example: ~~[illegible]~~

$$0 \longrightarrow (\mathbb{Z}/p)^3 \longrightarrow P \longrightarrow (\mathbb{Z}/p)^2 \longrightarrow 0$$

$$(\mathbb{Z}/p)^3 = H_2(Q), \qquad (\mathbb{Z}/p)^2 = Q$$

~~[illegible]~~ "versal extension"

localization theorems

<u>December 25, 1970</u>

I want to show that if A is a maximal elementary abelian p-subgroup of G, $C = C_G(A)$, $N = N_G(A)$, then the part of $H_G^*(X)$ lying over the stratum V_A can be computed from $H_C^*(X)$ and N/C. Precisely I want to show that the canonical map

$$H_G^*(X)_{\mathfrak{y}} \xrightarrow{\sim} \{S^{-1}H_C^*(X^A)\}^{N/C}$$

is an isomorphism, where $\mathfrak{y} \in V_A$ and $S^{-1} = H_C^* - \bigcup \mathfrak{q}$ with $\mathfrak{q}$ running over the primes with support A lying over $\mathfrak{y}$.

Second reduction: Can assume $G = \overbrace{N}^{\text{and } X = X^A}$! Indeed we know already that

$$H_G^*(X)_{\mathfrak{y}} = H_N^*(X^A)_{\mathfrak{y}'}$$

where $\mathfrak{y}' \subset H_N^*$ is the unique prime with support A lying over $\mathfrak{y}$.

First reduction: Can assume X has only $[p]$-groups for isotropy groups.

$$\begin{array}{ccccc}
H_G^*(X)_{\mathfrak{y}} & \longrightarrow & H_G^*(X\times F)_{\mathfrak{y}} & \rightrightarrows & H_G^*(X\times F\times F)_{\mathfrak{y}} \\
\downarrow & & \downarrow & & \downarrow \\
S^{-1}H_C^*(X)^{N/C} & \longrightarrow & S^{-1}H_C^*(X\times F)^{N/C} & \rightrightarrows & S^{-1}H_C^*(X\times F\times F)^{N/C}
\end{array}$$

Thus can assume A normal in G, G/A acts freely on X.

<u>Special case</u>: Assume that the principal G/A-bundle: $X \longrightarrow X/G$ is trivial, i.e.

$$X = G/A \times Y \qquad Y \simeq X/G$$

Then

$$H^*_G(X) = H^*_A(Y) = H^*_A \otimes H^*(Y)$$

$$H^*_C(X) = H^*_{C\times A}(G \times Y) = H^*_A(G/C \times \underset{\text{A-trivial}}{Y})$$

$$\cong H^*_A \otimes H^*(G/C) \otimes H^*(Y)$$

~~But $H^*(G/C) = \mathrm{Map}(G/C, \mathbb{Z}/p\mathbb{Z})$ so really as an G/C-module~~

$$\cancel{H^*_{CA}(G/C) \simeq \mathrm{Map}(G/C, H}$$

The point is that

$$H^*_C(X) = H^*_C(G/A) \otimes H^*(Y)$$

and that

$$H^*_C(G/A) \xrightarrow{\sim} H^0(G/C, H^*_A)$$

is an induced module. The isomorphism is clear in this cases without localization.

Assume for the moment the following:

(H): Let $S \subset H^*_G$ be a multiplicative system such that $S^{-1}(H^*_G/\mathfrak{y}) \neq 0 \Longrightarrow \mathfrak{y} \subset V_A$ for example $S = H^*_G - \mathfrak{y}$. Then any $S^{-1}H^*_C$ module M with equivariant G/C-action is cohomologically trivial, i.e., for all subgroups $H \subset G/C$

$$H^+(H, M) = 0.$$

Assuming this suppose given a G-space X which is parac. + $cd_p(X) < \infty$, and such that G/A acts freely on X. Then since the spectral sequence

$$E_2^{st} = H^s(X/G, \mathcal{H}^t_G) \Longrightarrow H^{s+t}_G(X)$$

has only finitely many non-zero columns, we can localize and obtain a spectral sequence

$$\begin{tikzcd} S^{-1}E_2^{rs} = H^s(X/G, (S^{-1}\mathcal{H}^*_G)^t) \arrow[r, Rightarrow] \arrow[d] & (S^{-1}H_G)^{s+t}(X) \arrow[d] \\ S^{-1}E_2^{rs} = H^s(X/C, (S^{-1}H_A)^t) \arrow[r, Rightarrow] & (S^{-1}H_C)^{s+t}(X) \end{tikzcd}$$

and similarly one for C. In more detail the spectral sequence should be thought of in terms of its columns

$$d_r : E_r^{s*}(G,X) \longrightarrow E_r^{s+r,*-r+1}(G,X)$$

$$E_{r+1}^{s,*}(G,X) = \frac{\mathrm{Ker}\{d_r : E_r^{s,*} \longrightarrow \quad\}}{\mathrm{Im}\{d_r : \quad \longrightarrow E_r^{s*}\}}$$

$$F_s H_G^*(X) / F_{s+1} H_G^*(X) = E_\infty^{s,*-s}(G,X)$$

so there is no problem in localizing, i.e.

$$(S^{-1}M)^* = \varinjlim_{s \in S} M^{*+\deg(s)} .$$

Now our claim is that for each r, s, t.

$$S^{-1} E_r^{st}(G,X) \xrightarrow{\ \sim\ } \{S^{-1} E_r^{st}(C,X)\}^{G/C}$$

The points to check are

(i) true for $r = 2$.

(ii) exactness is preserved

But (ii) follows from (H) because every term of the $S^{-1}E(C,X)$ spectral sequence including Z_r, B_r, etc. are $S^{-1}H_C^*$ modules with equivariant G/C-action, hence are cohomologically trivial.

~~For (i) we have that $(S^{-1}H_A)^t$ is an induced G/C module and that G/C acts freely on X/C, hence H^0 $H^s(X/C,$~~

To prove (i) consider Cartan-Leray for the covering $\pi: X/C \longrightarrow X/G$.

$$E_2 = H^s(G/C, H^t(X/C, \pi^*M)) \Longrightarrow H^{s+t}(X/G, M).$$

(here $M = (S^{-1}H_G^*)^t$ on X/G and $\pi^* M = (S^{-1}H_A)^t$ is constant). But in the case we are interested in $H^t(X/C, \pi^*M)$ will be cohomologically trivial, so we have an isom

$$H^*(\cancel{X}G, G) \xrightarrow{\sim} H^*(X/C, \pi^*M)^{G/C}.$$

So our theorem will follow once (H) is proved.

(I have used implicitly that if M is an H_C^*-module, then localizing with resp. to $\mathfrak{g} \subset H_G^*$ or $H_C^* - \cup\, \mathfrak{q}$ is the same. But this is clear since both localizations will sit over same part of spectrum with the same stalks.)

In proving (H) we can therefore replace S by the multiplicative system in H_C^* it generates, and so we can replace it by a smaller multiplicative system. Now recall

$$H^*_C \longrightarrow H^*_A \longrightarrow S(A^\#)$$

is an F-isomorphism since A is a max. elem. ab. p-subgroup of C and it is contained in the center. This implies I can find a section of the form

$$\begin{tikzcd} & S(A^\#) \arrow[dl, "\theta"'] \arrow[d, "x \mapsto x^{p^n} = F^n"] \\ H^*_C \arrow[r, "\rho"] & S(A^\#) \end{tikzcd}$$

for some integer n. (Take a basis λ_i of $A^\#$ and lift λ_i^q, q some large power of p.) Next by raising to some higher power of p, I can arrange θ to be equivariant under the finite group G/C. $(\rho(\theta g\lambda - g\cdot\theta\lambda)=0 \Rightarrow F^b\theta g = gF^b\theta)$. Finally we can assume S is generated by $\theta\left(\prod_{\lambda\in A^\#-0}\lambda\right)$.

Now any G/C-equiv. $S^{-1}H^*_C$ is an equivariant $S(A^\#)[e_A^{-1}]$ module via θ, and as G/C acts freely on $S(A^\#)[e_A^{-1}](\Omega)$ one knows by Galois descent that any equivariant module M over this ring is cohomologically trivial.

(see page 8 for argum.)

<u>Conclusion:</u> Assume X as in the main theorem, let A be a max. $[p]$-subgp of G with centralizer C and Weyl group $W = N_G(A)/C_G(A)$. Then the canonical map

$$H^*_G(X) \longrightarrow H^*_C(X^A)^W$$

is an isomorphism over the stratum V_A. In down to earth terms if $\mathfrak{p}$ is a prime ideal in V_A and $\mathfrak{q}$ is a prime in H^*_C with support A lying over $\mathfrak{p}$, then

$$H^*_G(X)_{\mathfrak{p}} \xrightarrow{\sim} S_{\mathfrak{p}}^{-1} H^*_C(X^A)^W$$

where
$$S_{\mathfrak{p}} = H^*_G - \bigcup_{w \in W} w(\mathfrak{q})$$

(and vanish on all $A' < A$)

<u>Remark:</u> Let $e \in H^*_A$ be invariant under W_A. Then for some $q = p^\nu$

$$e^q = \mathrm{res}(f) \qquad f \in H^*_G.$$

This I proved already – 5.17 of first draft. More agreeable then to state above thm. as

$$H^*_G(X)[f^{-1}] \xrightarrow{\sim} H^*_C(X^A)[f^{-1}]^W$$

Assume that G acts freely on $A(\Omega)$ and $B = A^G$, and M is an equivariant G-module. Then Galois descent says that

$$M = A \otimes_B M^G$$

and that the complex

$$\begin{array}{ccccc} M & \rightrightarrows & Map(G, M) & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} & Map(G \times G, M) \;\substack{\to \\ \to \\ \to \\ \to} \cdots \\ \| & & \| & & \| \\ M & & A \otimes_B M & & A \otimes_B A \otimes_B M \end{array}$$

is acyclic. Thus **M** is cohomologically trivial.

Put another way A is cohomologically trivial, hence $M = A \otimes_B M^G$ is also cohomologically trivial.

Suppose we want to find an element $u \in H_G^*$ such that $u|A = e_A^g$ and $u|A' = 0$ if $A \not\to A'$. This possible iff compatible with **all** morphisms. But given any A' of rank

December 27, 1970. $H_*(GL(\Lambda))$ $\begin{cases} \Lambda & \text{ring of } S\text{-units} \\ \Lambda \supset \mu_p, \frac{1}{p}. \end{cases}$

I want to compute the mod p cohomology of $GL_n(\Lambda)$ where Λ is the Dedekind domain of S-units such that $\Lambda \supset \mu_p, p^{-1}$. I can compute the different classes of p-subgroups and know they all are of rank n. Conjecture is that the ring of mod p cohomology has no embedded components so that by your results on the spectrum a non-zero class is detected on the centralizer of some maximal A.

[This conjecture is equivalent to the geometric Euler class c_n being a non-zero divisor.] NO must replace c_n by discriminant class

One might conjecture further that $H^*(GL_n(\Lambda))$ is a free module over $H^*(B\underline{GL}_n) = \mathbb{Z}/p[c_1, \ldots, c_n]$. This implies $H^*(GL_n(\Lambda))$ is a Cohen-Macaulay ring. One possible method of attacking this is ~~to get~~ one's hands on the fiber X of $BGL_n(\Lambda) \longrightarrow B\underline{GL}_n = BGL_n(\mathbb{C})$. One can consider the map of spectral sequences associated to

$$\begin{array}{ccccc} X & \longrightarrow & BGL_n(\Lambda) & \longrightarrow & B\underline{GL}_n \\ \| & & \uparrow & & \uparrow \\ X & \longrightarrow & \boxed{?} & \longrightarrow & B\mu_p^n \end{array}$$

and observe the latter sp. seq. is free over the first. Then perhaps X is identifiable and like for finite fields has a natural $\underline{GL}_n$ and a twisted action which coincide on μ_p^n. Unfortunately we need a natural candidate for X.

$\Lambda =$ ring of S-units in K, $[K:\mathbb{Q}] < \infty$. Then have by Kummer theory

$$
\begin{aligned}
0 \longrightarrow \mu_p \longrightarrow \Lambda^* \xrightarrow{\ p\ } \Lambda^* &\longrightarrow \\
H^1(\Lambda, \mu_p) \longrightarrow \mathrm{Pic}(\Lambda) \xrightarrow{\ p\ } \mathrm{Pic}(\Lambda) &\longrightarrow \\
H^2(\Lambda, \mu_p) \longrightarrow \mathrm{Br}'(\Lambda) \xrightarrow{\ p\ } \mathrm{Br}'(\Lambda) &\cdots
\end{aligned}
$$

and by Groth. Gp Brauer III, §2.

$$0 \longrightarrow \mathrm{Br}'(\Lambda) \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{\mathfrak{p} \subset \Lambda} \mathbb{Q}/\mathbb{Z} \longrightarrow H^3(\Lambda, \mathbb{G}_m) \longrightarrow 0$$

$$H^i(\Lambda, \mathbb{G}_m) \xrightarrow{\sim} H^i(K, \mathbb{G}_m) = 0 \qquad i > 3$$

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{\mathfrak{p} \text{ in } K} \mathbb{Q}/\mathbb{Z} \xrightarrow{\text{sum}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

(all p odd $\Rightarrow$ K tot. imag.)

So I conclude that

$$H^i(\Lambda, \mathbb{G}_m) = 0 \qquad i \geq 3$$

$$0 \longrightarrow H^2(\Lambda, \mathbb{G}_m) \longrightarrow \bigoplus_{\mathfrak{p} \in S} \mathbb{Q}/\mathbb{Z} \xrightarrow{\text{sum}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Now one has exact sequence

$$\bigoplus_{p\in S} \mathbb{Z} \longrightarrow \operatorname{Pic}\Lambda_0 \longrightarrow \operatorname{Pic}\Lambda \longrightarrow 0$$

$\Lambda_0 = \bar{\mathbb{Z}}$ in K, hence $\operatorname{Pic}\Lambda$ is finite as it is a quotient of $\operatorname{Pic}(\Lambda_0)$. So

$$H^1(\Lambda, \mathbb{Z}/p) \underset{\text{not can}}{\cong} \Lambda^*/(\Lambda^*)^p \oplus {}_p(\operatorname{Pic}\Lambda)$$

$$\text{rank } \Lambda^*/(\Lambda^*)^p = \underset{\mu_{p^r}}{1} + \underset{\Lambda_0^*/\text{tors}}{(r_1'' + r_2 - 1)} + \text{card } S$$

$$\text{rank } H^2(\Lambda, \mathbb{Z}/p) = \underbrace{\text{card } S - 1}_{{}_p Br(\Lambda)} + \text{rank}(\operatorname{Pic}\Lambda)/p.$$

$$H^i(\Lambda, \mathbb{Z}/p) = 0 \qquad i > 2.$$

Important idea emerging from this calculation – stable classes cannot distinguish p'-elements of the ideal class group. Precisely suppose we have an invertible L such that $\underbrace{L + L \cdots}_{h} \simeq \underbrace{\Lambda \oplus \Lambda \cdots}_{h}$ where $(h, p) = 1$. Then we have two non-conjugate $[p]$-subgroups of $GL_2(\Lambda)$ obtained by letting μ_p^2 act on $L \oplus L^{-1} \simeq \Lambda\oplus\Lambda$. However if Θ is an exponential class for representations over Λ, then these two representations E, E' have same

Θ values since $hE \simeq hE' \Rightarrow \theta(E)^h = \theta(E')^h$
$\Longrightarrow \theta(E) = \theta(E')$ as $(h,p)=1$.

This shows that $H^*(GL(\Lambda)) \longrightarrow H^*(GL_n(\Lambda))$ is not surjective, when $\mathrm{Pic}(\Lambda)$ has p'-elements.

Critical computation: Assume that $\mathrm{Pic}(\Lambda)$ has no p-torsion ("regular"). In this case the mod p cohomology has fairly standard form

$$\text{rank } H^1 = r_2 + \text{card } S$$
$$\text{rank } H^2 = (\text{card } S) - 1$$

and the cup product structure is probably easy to understand i.e. $H^1 = \Lambda^*/(\Lambda^*)^p$, cup product given by Hilbert symbols $(\mu,\nu) \longmapsto \left(\left\{\frac{\mu,\nu}{\mathfrak{p}}\right\}_p\right)_{\mathfrak{p}\in S}$

subject to the single relation $$\prod_{\mathfrak{p}} \left\{\frac{\mu,\nu}{\mathfrak{p}}\right\} = 1.$$

This all checks out nicely. Now it should be possible to compute the characteristic classes of Chern type and so test your conjectures. The same thing should be possible locally.

Assume $\mathrm{Pic}(\Lambda) = 0$ for simplicity; then there is a unique conjugacy class of [p]-subgroups in $GL_n(\Lambda)$ namely the diagonal group μ_p. First form of conjecture reads

$$(*) \qquad S\{H^*_*(\Lambda^*)\} \xrightarrow{\sim} H_*\{GL(\Lambda)\}$$

In any case the map should be surjective because

$$H_*((\Lambda^*)^n) \twoheadrightarrow H_*(GL_n(\Lambda))$$

by your conjecture that there are no embedded components.

Another way of stating (*) is that to give ~~[illegible]~~ an exponential char. class for representations over Λ is the same as giving one for one-dimensional representations. In general if θ is ~~[illegible]~~ an exponential class for Λ-representations, then to each invertible Λ-module L we denote by $\theta(L)$ the class in $H^0(\Lambda^*, S.)$ given by the canonical representation of Λ^* on L. Now these elements $\theta(L)$ are not independent. Indeed we have

$$L \oplus L' = \Lambda \oplus (L \otimes L') \qquad (\Lambda^* \text{ acting as scalars})$$

so

$$\theta(L)\theta(L') = \theta(\Lambda)\theta(L \otimes L')$$

and hence

$$L \longmapsto \theta(L)\cdot\theta(\Lambda)^{-1}$$

is a homomorphism $\mathrm{Pic}(\Lambda) \longrightarrow H^0(\Lambda^*, S.)^*$

Perhaps the best way to put it is that there is a homomorphism

$$(*) \qquad K_0(\Lambda) \longrightarrow H^0(\Lambda^*, S_\circ)^\times$$

which assigns to a projective module E, the class of $\theta(E)$ where Λ^* acts in obvious way on E. Thus conjecture asserts the canonical map

$$\mathrm{Hom}_{rgs}\left(H_\circ(GL(\Lambda)), S_\circ\right) \longrightarrow \mathrm{Hom}_{ab}\left(K_0\Lambda, H^0(\Lambda^*, S_\circ)^\times\right)$$

is an isomorphism.

Since $(*)$ is a homomorphism, the p'-prime of the ideal class group is irrelevant to the mod p cohomology. Now what is mystifying is the fact that at least when $\mathrm{Pic}(\Lambda) = 0$ the formula for $H_*(GL(\Lambda))$ predicted by this conjecture

$$S\{H_*(\Lambda^*)\}$$

has a form which depends on Λ^* hence seems not to use the fact that $H^2(\Lambda, \mathbb{Z}/p\mathbb{Z})$ has (card $S - 1$) for its rank.

What you have to check is that the map $S\{H_*(\Lambda^*)\} \longrightarrow H_*(GL(\Lambda))$ is injective using Chern classes. This being a map of Hopf algebras, the ideal has a special form.

Summary

Conjectures and problems:

1) $H^*(GL_n(\Lambda))$ free over $H^*(BGL_n(\mathbb{C}))$.

2) $S\{H_*(\Lambda^*)\} \xrightarrow{\sim} H_*(GL(\Lambda))$ if $Pic(\Lambda) = 0$?

Accessible problem: Prove the homomorphism 2) is injective by constructing exponential classes with prescribed effect on rank one representation.

Surjectivity of 2) can be formulated in this way: Any exponential class vanishing on rank one representations vanishes identically. I would <u>not</u> be surprised if such things existed in the number field case.

Another accessible problem: Computation in local case for p prime to residual characteristics, to check irrelevance of $H^2(\Lambda)$.

Significance of Bate-Tate kernel

$$K_2(F) \longrightarrow \bigoplus_v K_2(F_v).$$

<u>December 28, 1970.</u> central $[p]$- subgroups.

<u>Lemma 1</u>: Let $E_2^{st} \Rightarrow H^{s+t}$ be a standard first quadrant multiplicative spectral sequence and suppose $u \in H^d$, $d>0$, is such that multiplying by its image $\bar{u} \in E_2^{od}$ gives isomorphisms

$$E_2^{st} \xrightarrow{\sim} E_2^{s,t+d} \qquad t \geq 0, \text{ all } s$$

Then u is a non-zero divisor in H^*.

<u>Proof:</u> By induction we show that

$$\begin{array}{ccc} E_r^{st} & \xrightarrow{\sim} & E_r^{s,t+d} \\ z & \longmapsto & z \cdot \bar{u} \end{array} \qquad d_r(z\bar{u}) = d_r z \cdot \bar{u}$$

is an isom. for $t \geq 0$, and all s. We have

$$\begin{tikzcd} E_r^{s-r,t+r-1} \arrow[r, "d_r"] \arrow[d, "\cong"] & E_r^{st} \arrow[r, "d_r"] \arrow[d, "\cong"] & E_r^{s+r,t-r+1} \arrow[d, hook] \\ E_r^{s-r,t+r-1+d} \arrow[r] & E_r^{s,t+d} \arrow[r, "d_r"] & E_r^{s+r,t-r+1+d} \end{tikzcd} \quad *$$

and one has an isomorphism at the right unless $t-r+1<0$ in which case $E_r^{s+r,t-r+1}=0$. In either case the map is injective, so the homology in the middle is the same, i.e.

$$E_{r+1}^{st} \xrightarrow{\sim} E_{r+1}^{s,t+d} \qquad t \geq 0$$

completing the induction step.

By convergence $E_\infty^{st} \xrightarrow{\sim} E_\infty^{s,t+d}$. Now suppose $v \in H^m$ and $u \cdot v = 0$. Let s be chosen such that

$v \in F_s H^m$ and its image $\bar{v}$ in $F_s H^m / F_{s+1} H^m = E_\infty^{s,m-s}$ is non-zero. Then $u \cdot v \in F_s H^{m+d}$ and $\bar{u} \cdot \bar{v} = 0$ contradicting the isomorphism $E_\infty^{s,m-s} \xrightarrow{\sim} E_\infty^{s,m-s+d}$. q.e.d.

<u>Remark</u>: One may weaken the hypotheses slightly and assume only that

$$E_r^{st} \xrightarrow{\sim} E_r^{s,t+d} \qquad t > 0$$
$$\hookrightarrow \qquad t = 0.$$

The point being that in $*$ one has for $t>0$ again

$$\cong \quad \cong \quad \downarrow \quad \Longrightarrow \quad \cong \text{ on } E_{r+1}$$

and for $t=0$ one has

$$\cong \quad \downarrow \quad \downarrow \quad \Longrightarrow \quad \hookrightarrow \text{ on } E_{r+1}.$$

<u>Lemma 2</u>: Let Z be a central cyclic subgroup of order p of a compact Lie group G and let X be a G-space on which Z acts trivially. Let $u \in H_G^{2i}(X)$ be an element which for each map $(Z, pt) \longrightarrow (G, X)$ given by the different components of X restricts to a generator of H_Z^{2i}. Then u is a non-zero divisor in $H_G^*(X)$.

Proof. We can assume G/Z acts freely on X (by replacing X by $P(G/Z) \times X$). Then

$$PG \times^{G} X = (PG/Z) \times^{G/Z} X$$

is a fibre bundle over $X/(G/Z)$ with fibre $PG/Z = BZ$ and so we have a Leray spectral sequence

$$E_2^{st} = H^s_{G/Z}(X, H^t_Z) \Longrightarrow H^{s+t}_G(X)$$

Now since Z is central in G, G/Z acts trivially on H^*_Z so

$$E_2 = H^*_{G/Z}(X) \otimes H^*_Z \Longrightarrow H^*_G(X)$$

Now ~~the cohomology of a cyclic group is periodic with~~ by hypotheses $\bar{u} \in E_2^{0,2i} = H^0_{G/Z}(X) \otimes H^{2i}_Z$ will give rise to an isomorphism

$$E_2^{st} \xrightarrow{\sim} E_2^{s,t+2i} \qquad t \geq 0$$

hence we can apply lemma 1.

<u>Remark</u>: Assume for simplicity that X is connected. Do not assume Z central, only normal. Then

$$E_2^{0t} = H^0_{G/Z}(X, H^t_Z) = (H^t_Z)^{G/Z}$$

is still periodic.

$$1 \quad 0 \quad 0 \quad \cdot \quad 0 \quad 2i{-}1 \xrightarrow{\beta} 2i \quad 0 \cdots\cdots \qquad i = p-1$$

It is clear that E_2^{st} is periodic also, hence in the preceding lemma it should be possible to assume only that Z is normal.

Now suppose that X is a connected G-space and A is an elementary abelian subgroup of the center of G such that A acts trivially on X and A is a maximal $[p]$-subgroup of each isotropy group. (Example: Starting with a general G-space X, let (A,c) be maximal and replace (G,X) by $(C(A,c), X^{A,c})$.)

Let $\chi : A \twoheadrightarrow \mu_p \subset \mathbb{C}^*$ be a non-trivial character of A and let V be an irreducible complex representation of G whose restriction to A is purely of type χ. Then take $u = e(V) \in H^{ev}_G(X)$ and take Z to be a cyclic subgroup of A such that $\chi(Z) \neq 0$. Then we can apply the lemma 2 and conclude that $e(V)$ is a non-zero divisor in $H^*_G(X)$. Now the Gysin sequence

$$\longrightarrow H^*_G(X) \xrightarrow{e(V)} H^*_G(X) \longrightarrow H^*_G(X \times SV) \xrightarrow{\delta} \cdots$$

shows us that

$$H^*_G(X \times SV) = H^*_G(X) / e(V)\, H^*_G(X).$$

But $X \times SV$ is connected and $\operatorname{Ker}\chi \subset A$ acts trivially on it, $\operatorname{Ker}\chi \subset G$ central $[p]$-subgroup, $\operatorname{Ker}\chi$ is a maximal

[p]-subgroup of each isotropy group of $X \times SV$. Moreover the rank has gone down by 1, so by induction we have proved the following theorem:

<u>Theorem:</u> Let A be a central [p]-subgroup of G of rank r, and let X be a connected G/A-space such that A is maximal in each isotropy group. Then $H_G^*(X)$ is Cohen–Macaulay, more precisely if $V_1, \ldots, V_r$ ~~are~~ a representations of G such that $V_i | A = \chi_i^{\oplus n_i}$ where χ_i are a basis for $\hat{A}$, then the sequence $e(V_1), \ldots, e(V_r)$ is regular for $H_G^*(X)$ and

$$H_G^*(X)/(e(V_1), \ldots, e(V_j))\, H_G^*(X) \xrightarrow{\sim} H_G^*(X \times SV_1 \times \cdots \times SV_j)$$

for $0 \leq j \leq r$.

Better statement:

<u>Theorem:</u> Let A be a central [p]-subgroup of G of rank r, and let X be a G/A-space. Let $V_1, \ldots, V_r$ be representations of G such that

$$V_i \,|\, A = \chi_i^{\oplus n_i} \qquad 1 \leq i \leq r \qquad n_i > 0$$

where $\chi_1, \ldots, \chi_r$ are a basis for the character group $\hat{A}$. Then $e(V_1), \ldots, e(V_r)$ is a regular sequence for $H_G^*(X)$ and

$$H_G^*(X)/(e(V_1), \ldots, e(V_j))\, H_G^*(X) \xrightarrow{\sim} H_G^*(X \times SV_1 \times \cdots \times SV_j)$$

for $0 \leq j \leq r$. In particular depth $H_G^*(X) \geq r$.

8. MISCELLANEOUS PRELIMINARY LEMMAS.

Lemma 8.1. *If X is a π-group, and $\mathfrak{C}$ is a chain $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n = 1$, then the stability group A of $\mathfrak{C}$ is a π-group.*

Proof. We proceed by induction on n. Let $A \in A$. By induction, there is a π-number m such that $B = A^m$ centralizes X_1. Let $X \in X$; then $X^B = XY$ with Y in X_1, and by induction, $X^{B^r} = XY^r$. It follows that $B^{|X_1|} = 1$. ∎

Lemma 8.2. *If P is a p-group, then P possesses a characteristic subgroup C such that*

(i) $cl(C) \leq 2$, *and $C/Z(C)$ is elementary.*

(ii) $\ker(\mathrm{Aut}\ P \xrightarrow{res} \mathrm{Aut}\ C)$ *is a p-group. (res is the homomorphism induced by restricting A in Aut P to C.)*

(iii) $[P,C] \subseteq Z(C)$ *and $C(C) = Z(C)$.*

Proof. Suppose C can be found to satisfy (i) and (iii). Let $K = \ker$ res. In commutator notation, $[K,C] = 1$, and so $[K,C,P] = 1$. Since $[C,P] \subseteq C$, we also have $[C,P,K] = 1$ and 3.1 implies $[P,K,C] = 1$, so that $[P,K] \subseteq Z(C)$. Thus, K stabilizes the chain $P \supseteq C \supseteq 1$ so is a p-group by Lemma 8.1.

If now some element of $SCN(P)$ is characteristic in P, then (i) and (iii) are satisfied and we are done. Otherwise, let A be a maximal characteristic abelian subgroup of P, and let C be the group generated by all subgroups D of P such that $A \subset D$, $|D : A| = p$, $D \subseteq Z(P \bmod A)$, $D \subseteq C(A)$. By construction, $A \subseteq Z(C)$, and C is seen to be characteristic. The maximal nature of A implies that $A = Z(C)$. Also by construction $[P,C] \subseteq A = Z(C)$, so in particular, $[C,C] \subseteq Z(C)$

and $cl(C) \leq 2$. By construction, $C/Z(C)$ is elementary.

We next show that $C(C) = Z(C)$. This statement is of course equivalent to the statement that $C(C) \subseteq C$. Suppose by way of contradiction that $C(C) \nsubseteq C$. Let E be a subgroup of $C(C)$ of minimal order subject to (a) $E \triangleleft P$, and (b) $E \nsubseteq C$. Since $C(C)$ satisfies (a) and (b), such an E exists. By the minimality of E, we see that $[P,E] \subseteq C$ and $D(E) \subseteq C$. Since E centralizes $\overset{C}{\cancel{A}}$, so do

?✓ $[P,E]$ and $D(E)$, so we have $[P,E] \subseteq A$ and $D(E) \subseteq A$. The minimal { since $A = Z(C)$.

?✓ nature of E guarantees that $E/E \cap C$ is of order p. Since $E \cap C =$ { since chief factors of a p-gp are cyclic.

$E \cap A$, $E/E \cap A$ is of order p, so EA/A is of order p. By construction of C, we find $EA \subseteq C$, so $E \subseteq C$, in conflict with (b). Hence, $C(C) = Z(C)$, and (i) and (iii) are proved. ▌

Lemma 8.3. Let X be a p-group, p odd, and among all elements of SCN(X), choose A to maximize $m(A)$. Then $\Omega_1(C(\Omega_1(A))) = \Omega_1(A)$.

Remark. The oddness of p is required, as the dihedral group of order 16 shows.

Proof. We must show that whenever an element of X of order p centralizes $\Omega_1(A)$, then the element lies in $\Omega_1(A)$.

If $X \in C(\Omega_1(A))$ and $X^p = 1$, let $B(X) = B_1 = \langle \Omega_1(A), X \rangle$, and let $B_1 \subset B_2 \subset \cdots \subset B_n = \langle A, X \rangle$ be an ascending chain of subgroups, each of index p in its sucessor. We wish to show that $B_1 \triangleleft B_n$. Suppose $B_1 \triangleleft B_m$ for some $m \leq n-1$. Then B_m is generated by its

modular law { normal abelian subgroups B_1 and $B_m \cap A$, so B_m is of class at most two, so is regular. Let $Z \in B_m$, Z of order p. Then $Z = X^k A$, A in A, k an integer. Since B_m is regular, $X^{-k}Z$ is of order 1 or p.

Hence, $A \in \Omega_1(A)$, and $Z \in B_1$. Hence, $B_1 = \Omega_1(B_m)$ char $B_m \lhd B_{m+1}$, and $B_1 \lhd B_n$ follows. In particular, X stabilizes the chain $A \supseteq \Omega_1(A) \supseteq \langle 1 \rangle$.

It follows that if $D = \Omega_1(C(\Omega_1(A)))$, then D' centralizes A. Since $A \in SCN(X)$, $D' \subseteq A$. We next show that D is of exponent p. Since $[D,D] \subseteq A$, we see that $[D,D,D] \subseteq \Omega_1(A)$, and so $[D,D,D,D] = 1$, and $cl(D) \leq 3$. If $p \geq 5$, then D is regular, and being generated by elements of order p, is of exponent p. It remains to treat the case $p = 3$, and we must show that the elements of D of order at most 3 form a subgroup. Suppose false, and that $\langle X, Y \rangle$ is of minimal order subject to $X^3 = Y^3 = 1$, $(XY)^3 \neq 1$, X and Y being elements of D. Here we have

$$\begin{aligned}(XY)^3 = XYXYXY &= X^2Y[Y,X]YXY \\ &= X^2Y\,[Y,X]XY[Y,X]Y \\ &= X^2YX[Y,X][Y,X,X]Y[Y,X]Y \\ &= X^3Y[Y,X]^2[Y,X,X]Y[Y,X]Y.\end{aligned}$$

Now $[Y,X] = Y^{-1}.X^{-1}YX$ is of order three, since $\langle Y, Y^X \rangle \subset \langle X, Y \rangle$. Hence, $[Y,X]$ is in $\Omega_1(A)$, and so $[Y,X]$ is centralized by both X and Y. It follows that $(XY)^3 = X^3Y^3[Y,X]^3 = 1$, so D is of exponent p in all cases.

If $\Omega_1(A) \subset D$, let $E \lhd \overset{P}{\cancel{X}}$, $E \subseteq D$, $|E : \Omega_1(A)| = p$. Since $\Omega_1(A) \subseteq Z(E)$, E is abelian. But $m(E) = m(A) + 1 > m(A)$, in conflict with the maximal nature of A, since E is contained in some element of $SCN(P)$ by 3.9.

Lemma 8.4. Suppose p is an odd prime and X is a p-group.

(i) If $SCN_3(X)$ is empty, then every abelian subgroup of X is generated by two elements.

(ii) If $SCN_3(X)$ is empty and A is an automorphism of X of prime order q, $p \neq q$, then q divides $p^2 - 1$.

Proof. (i) Suppose A is chosen in accordance with Lemma 8.3. Suppose also that X possesses an elementary subgroup E of order p^3. Let $E_1 = C_E(\Omega_1(A))$, so that E_1 is of order p^2 at least. But by Lemma 8.3, $E_1 \subseteq \Omega_1(A)$, a group of order at most p^2, and so $E_1 = \Omega_1(A)$. But now Lemma 8.3 is violated since E centralizes E_1.

(ii) Among the A-invariant subgroups of X on which A acts non trivially, let H be minimal. By 3.11, H is a special p-group. Since p is odd, H is regular, so 3.6 implies that H is of exponent p. By the first part of this lemma, H possesses no elementary subgroup of order p^3. It follows readily that $m(H) \leq 2$, and (ii) follows from the well known fact that q divides $|\mathrm{Aut}\ H/D(H)|$.

Lemma 8.5. If X is a group of odd order, p is the smallest prime in $\pi(X)$, and if in addition a S_p-subgroup of X possesses no elementary subgroup of order p^3, then X possesses a normal p-complement.

Proof. Let P be a S_p-subgroup of X. By Lemma 8.4(i), if H is a subgroup of P, then $SCN_3(H)$ is empty. Application of Lemma 8.4(ii) shows that $N_X(H)/C_X(H)$ is a p-group for every subgroup H of P. We apply Theorem 14.4.7 in [13] to complete the proof.

Application of Lemma 8.5 to the group G implies that if p is the smallest prime in $\pi(G)$, then G possesses an elementary subgroup of order p^3. In particular, if $3 \in \pi(G)$, then G possesses an elementary subgroup of order 27.

Lemma 8.6. Let X be a p-group, and suppose that X possesses a subgroup A of order p, such that $C(A) = A \times B$, where B is cyclic. Then $m(C) \leq p$ for every normal abelian subgroup C of X.

Proof. Suppose C is a normal abelian subgroup of X. Then $\Omega_1(C)$ char $C \triangleleft X$, and if $|\Omega_1(C)| = p^d$, then $d = m(C)$.

If $\mathfrak{A} = \langle A \rangle$ then A normalizes $\Omega_1(\mathfrak{C})$ and with respect to a suitable basis of $\Omega_1(\mathfrak{C})$ has a matrix $\mathrm{diag}(J_{n_1}, \ldots, J_{n_t})$, $n_1 \leq \cdots \leq n_t \leq p$. Since $\Omega_1(\mathfrak{C}(\mathfrak{A}))$ is of type (p,p), $t \leq 2$. If $t = 1$, then $d = n_1 \leq p$. If $t = 2$, then $\mathfrak{A} \subseteq \Omega_1(\mathfrak{C}(\mathfrak{A})) \subseteq \mathfrak{C} \subseteq \mathfrak{C}(\mathfrak{A})$, and $\mathfrak{C}$ is generated by $2 \leq p$ elements, as desired.

✓<u>Lemma 8.7.</u> <u>If A is a p'-group of automorphisms of the</u> p-group <u>$\mathfrak{P}$, if A has no fixed points on $\mathfrak{P}/\mathfrak{D}(\mathfrak{P})$, and A acts trivially on $\mathfrak{D}(\mathfrak{P})$ then $\mathfrak{D}(\mathfrak{P}) \subseteq \mathfrak{Z}(\mathfrak{P})$.</u>

Proof. In commutator notation, we are assuming $[\mathfrak{P},\mathfrak{A}] = \mathfrak{P}$, and $[\mathfrak{A},\mathfrak{D}(\mathfrak{P})] = 1$. Hence, $[\mathfrak{A},\mathfrak{D}(\mathfrak{P}),\mathfrak{P}] = 1$. Since $[\mathfrak{D}(\mathfrak{P}),\mathfrak{P}] \subseteq \mathfrak{D}(\mathfrak{P})$, we also have $[\mathfrak{D}(\mathfrak{P}),\mathfrak{P},\mathfrak{A}] = 1$. By the three subgroups lemma, we have $[\mathfrak{P},\mathfrak{A},\mathfrak{D}(\mathfrak{P})] = 1$. Since $[\mathfrak{P},\mathfrak{A}] = \mathfrak{P}$, the lemma follows. ▌

<u>Lemma 8.8.</u> <u>Suppose $\mathfrak{Q}$ is a q-group,</u> q <u>is odd,</u> A <u>is an automorphism of $\mathfrak{Q}$ of prime order</u> p, $p \equiv 1 \pmod q$, <u>and $\mathfrak{Q}$ possesses a subgroup $\mathfrak{Q}_0$ of index</u> q <u>such that</u> $SCN_3(\mathfrak{Q}_0)$ <u>is empty. Then</u> $p = 1 + q + q^2$ <u>and $\mathfrak{Q}$ is elementary of order</u> q^3.

Proof. Since $p \equiv 1 \pmod q$ and q is odd, p does not divide $q^2 - 1$. Since $\mathfrak{D}(\mathfrak{Q}) \subseteq \mathfrak{Q}_0$, Lemma 8.4(i) implies that A acts trivially on $\mathfrak{D}(\mathfrak{Q})$.

Suppose that A has a non trivial fixed point on $\mathfrak{Q}/\mathfrak{D}(\mathfrak{Q})$. We can then find an A-invariant subgroup $\mathfrak{M}$ of index q in $\mathfrak{Q}$ such that A acts trivially on $\mathfrak{Q}/\mathfrak{M}$. In this case, A does not act trivially on $\mathfrak{M}$, and so $\mathfrak{M} \neq \mathfrak{Q}_0$, and $\mathfrak{M} \cap \mathfrak{Q}_0$ is of index q in $\mathfrak{M}$. By induction, $p = 1 + q + q^2$ and $\mathfrak{M}$ is elementary of order q^3. Since A acts trivially on $\mathfrak{Q}/\mathfrak{M}$, it follows that $\mathfrak{Q}$ is abelian of order q^4. If $\mathfrak{Q}$ were elementary, $\mathfrak{Q}_0$ would not exist. But if $\mathfrak{Q}$ were not elementary, then A would have a fixed point on $\Omega_1(\mathfrak{Q}) = \mathfrak{M}$, which is not possible. Hence A has no fixed points on $\mathfrak{Q}/\mathfrak{D}(\mathfrak{Q})$, so by Lemma 8.7, $\mathfrak{D}(\mathfrak{Q}) \subseteq \mathfrak{Z}(\mathfrak{Q})$.

Next, suppose that A does not act irreducibly on $Q/D(Q)$. Let $N/D(Q)$ be an irreducible constituent of A on $Q/D(Q)$. By induction, N is of order q^3, and $p = 1 + q + q^2$. Since $D(Q) \subset N$, $D(Q)$ is a proper A-invariant subgroup of N. The only possibility is $D(Q) = 1$, and $|Q| = q^3$ follows from the existence of Q_o.

If $|Q| = q^3$, then $p = 1 + q + q^2$ follows from Lemma 5.1. Thus, we can suppose that $|Q| > q^3$, and that A acts irreducibly on $Q/D(Q)$, and try to derive a contradiction. We see that Q must be non abelian. This implies that $D(Q) = Z(Q)$. Let $|Q : D(Q)| = q^n$. Since $p \equiv 1 \pmod q$, and $q^n \equiv 1 \pmod p$, $n \geq 3$. Since $D(Q) = Z(Q)$, n is even, $Q/Z(Q)$ possessing a non singular skew-symmetric inner product over integers mod q which admits A. Namely, let C be a subgroup of order q contained in Q' and let C_1 be a complement for C in Q'. This complement exists since Q' is elementary. Then $Z(P \bmod C_1)$ is A-invariant, proper, and contains $D(Q)$. Since A acts irreducibly on $Q/D(Q)$, we must have $D(Q) = Z(Q \bmod C_1)$, so a non-singular skew-symmetric inner product is available. Now Q is regular, since $cl(Q) = 2$, and q is odd, so $|\Omega_1(Q)| = |Q : \mho^1(Q)|$, by [15]. Since $cl(Q) = 2$, $\Omega_1(Q)$ is of exponent q. Since $|Q : \mho^1(Q)| \geq |Q : D(Q)| \geq q^4$, we see that $|\Omega_1(Q)| \geq q^4$. Since Q_o exists, $\Omega_1(Q)$ is non abelian, of order exactly q^4, since otherwise $Q_o \cap \Omega_1(Q)$ would possess an elementary subgroup of order q^3. It follows readily that A centralizes $\Omega_1(Q)$, and so centralizes Q, by 3.6. This is the desired contradiction.

Lemma 8.9. *If P is a p-group, if $SCN_3(P)$ is non empty and A is a normal abelian subgroup of P of type (p,p), then A is contained in some element of $SCN_3(P)$.*

Proof. Let E be a normal elementary subgroup of P or order p^3, and let $E_1 = C_E(A)$. Then $E_1 \lhd P$, and $\langle A, E_1\rangle = F$ is abelian.

If $|F| = p^2$, then $A = E_1 = F \subset E$, and we are done, since E is contained in an element of $SCN_3(P)$. If $|F| \geq p^3$, then again we are done, since F is contained in an element of $SCN_3(P)$.

If X and Y are groups, we say that Y is involved in X provided some section of X is isomorphic to Y.

Lemma 8.10. Let P be a S_p-subgroup of the group X. Suppose that $Z(P)$ is cyclic and that for each subgroup A in P of order p which does not lie in $Z(P)$, there is an element $X = X(A)$ of P which normalizes but does not centralize $\langle A, \Omega_1(Z(P))\rangle$. Then either $SL(2,p)$ is involved in X or else $\Omega_1(Z(P))$ is weakly closed in P.

Proof. Let $D = \Omega_1(Z(P))$. Suppose $E = D^G$ is a conjugate of D contained in P, but that $E \neq D$. Let $D = \langle D\rangle$, $E = \langle E\rangle$. By hypothesis, we can find an element $X = X(E)$ in P such that X normalizes $\langle E, D\rangle = F$, and with respect to the basis (E, D) has the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Enlarge F to a S_p-subgroup P^* of $C_X(E)$. Since $E = D^G$, $P^G \subseteq C_X(E)$, so P^* is a S_p-subgroup of X, and $E \subseteq Z(P^*)$. Since $Z(P^*)$ is cyclic by hypothesis, we have $E = \Omega_1(Z(P^*)$. By hypothesis, there is an element $Y = Y(D)$ in P^* which normalizes F and with respect ot the basis (E, D) has the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Now $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ generate $SL(2,p)$ [6, sections 262 and 263], so $SL(2,p)$ is involved in $N_X(F)$, as desired.

✓Lemma 8.11. If A is a p-subgroup and B is a q-subgroup of X, $p \neq q$, and A normalizes B, then $[B,A] = [B,A,A]$.

Proof. By 3.7, $[A,B] \triangleleft AB$. Since $AB/[A,B]$ is nilpotent, we can suppose that $[A,B]$ is elementary. With this reduction, $[B,A,A] \triangleleft AB$, and we can assume that $[B,A,A] = 1$. In this case, A stabilizes the chain $B \supseteq [B,A] \supseteq 1$, so $[B,A] = 1$ follows from Lemma 8.1 and $p \neq q$.

Lemma 8.12. Let p be an odd prime, and E an elementary subgroup of the p-group P. Suppose A is a p'-automorphism of P which centralizes $\Omega_1(C_P(E))$. Then $A = 1$.

Proof. Since $E \leq \Omega_1(C_P(E))$, A centralizes E. Since E is A-invariant, so is $C_P(E)$. By 3.6 A centralizes $C_P(E)$, so if $E \leq Z(P)$, we are done. ?

If $C_P(E) \subset P$, then $C_P(E)D(P) \subset P$, and by induction A

centralizes $D(P)$. Now $[P,E] \leq D(P)$ and so $[P,E,\langle A\rangle] = 1$. Also, $[E,\langle A\rangle] = 1$, so that $[E,\langle A\rangle,P] = 1$. By the three subgroups lemma, we have $[\langle A\rangle,P,E] = 1$, so that $[P,\langle A\rangle] \leq C_P(E)$, and A stabilizes the chain $P \supset C_P(E) \supset 1$. It follows from Lemma 8.1 that $A = 1$.

Lemma 8.13. Suppose P is a S_p-subgroup of the solvable group S, $SCN_3(P)$ is empty and S is of odd order. Then S' centralizes every chief p-factor of S.

Proof. We assume without loss of generality that $O_{p'}(S) = 1$. We first show that $P \triangleleft S$. Let $H = O_p(S)$, and let C be a subgroup of H chosen in accordance with Lemma 8.2. Let $W = \Omega_1(C)$. Since p is odd and $cl(C) \leq 2$, W is of exponent p.

Since $O_{p'}(S) = 1$, Lemma 8.2 implies that $\ker(S \longrightarrow \mathrm{Aut}\ C)$ is a p-group. By 3.6, it now follows that $\ker(S \xrightarrow{\alpha} \mathrm{Aut}\ W)$ is a p-group. Since P has no elementary subgroup of order p^3, neither does W, and so $|W : D(W)| \leq p^2$. Hence no p-element of S has a minimal polynomial $(x-1)^p$ on $W/D(W)$. Now (B) implies that $P/\ker\alpha \triangleleft S/\ker\alpha$, and so $P \triangleleft S$, since $\ker\alpha \leq P$.

Since $P \triangleleft S$, the lemma is equivalent to the assertion that if L is a $S_{p'}$-subgroup of S, then $L' = 1$. If $L' \neq 1$, we can suppose that L' centralizes every proper subgroup of P which is normal in S. Since L is completely reducible on $P/D(P)$, we can suppose that $[P,L'] = P$ and $[D(P),L'] = 1$. By Lemma 8.7 we have $D(P) \subseteq Z(P)$ and so $\Omega_1(P) = K$ is of exponent p and class at most 2. Since P has no elementary subgroup of order p^3, neither does K. If K is of order p, L' centralizes K and so centralizes P by 3.6, thus $L' = 1$. Otherwise, $|K : D(K)| = p^2$ and L is faithfully represented as automorphisms of $V = K/D(K)$. Since L is odd, $L' = 1$.

Lemma 8.14. If S is a solvable group of odd order, and $SCN_3(P)$ is empty for every S_p-subgroup P of S and every prime p, then S' is nilpotent.

Proof. By the preceding lemma, S' centralizes every chief factor of S. By 3.2, $S' \subseteq F(S)$, a nilpotent group.

~~Then~~ But $A \not\to A' \iff \mathscr{I}_A \not\supset \mathscr{I}_{A'}$. If A'_i $i = 1, \dots, n$ are representatives of the conj. classes of A' not dominated by A, and $f_i \in \mathscr{I}_{A'_i} - \mathscr{I}_{A_i}$, then

$$e = \prod_i f_i^p$$

is desired element. ($f_i \in \mathscr{I}_{A'_i} \Longrightarrow f_i^p | A'_i = 0$.)

Proof of 2). Assume known in case isotropy gps. of X are all in $\mathcal{A}_G$. Note formula

$$(*) \qquad H_N^*(X)_{\mathfrak{q}} \simeq H_N^*(X^A)_{\mathfrak{q}}$$

by 1). Applying H_G^*, H_N^* to

$$X \longleftarrow X \times F \rightleftarrows X \times F \times F$$

leads to exact rows; localization exact; use $(*)$:

$$\begin{array}{ccccc}
H_G^*(X)_{\mathscr{I}} & \longrightarrow & H_G^*(X \times F)_{\mathscr{I}} & \rightrightarrows & H_G^*(X \times F \times F)_{\mathscr{I}} \\
\downarrow & & \downarrow\cong & & \downarrow\cong \\
H_N^*(X^A)_{\mathfrak{q}} & \longrightarrow & H_N^*(X^A \times F^A)_{\mathfrak{q}} & \rightrightarrows & H_N^*(X^A \times F^A \times F^A)_{\mathfrak{q}}
\end{array}$$

isomorphisms due to $X \times F$, $X \times F \times F$ having all isot. gps in $\mathcal{A}_G$. done by 5 lemma.

Case where X has isot. gps in $\mathcal{A}_G$.

$$H_G^*(X)_{\mathscr{I}} \simeq H_G^*(GX^A)_{\mathscr{I}}$$

N/A acts freely on X^A, since isot gp in $\mathcal{Q}_G$ and it contains A and A is maximal. Thus

$$G X^A = G \times^N X^A$$

so by induction formula

$$H_G^*(G X^A)_{\mathfrak{p}} \simeq H_N^*(X^A)_{\mathfrak{p}}$$

Lemma. $R \to R'$ finite, $\mathfrak{p} \in \operatorname{Spec} R$, $\mathfrak{q}_i \in \operatorname{Spec} R'$ $i = 1, \dots, n$ the primes over it. M an R'-module $\ni$ $M_{\mathfrak{q}_j} = 0$ $j = r+1, \dots, n$. Then

$$M_{\mathfrak{p}} \xrightarrow{\sim} S^{-1} M$$

where $S = R' - (\mathfrak{q}_1 \cup \dots \cup \mathfrak{q}_r)$. (Proof: p. 5)

Let $\mathfrak{q}_i$ $i = 1, \dots, n$ be the primes of H_N^* lying over $\mathfrak{p}$, $\mathfrak{q} = \mathfrak{q}_1$, the only one with support A. If $\mathfrak{q}_i$ $i > 1$ has support A', then $AA' > A$ as rank A' = rank A. So by 1)

$$H_N^*(X^A)_{\mathfrak{q}_i} = H_N^*((X^A)^{A'})_{\mathfrak{q}_i} = H_N^*(X^{AA'})_{\mathfrak{q}_i} = 0$$

because $X^{AA'} = \phi$ as A is max. in $\mathcal{Q}_G$ and hypoth. on X. Applying lemma conclude

$$H_N^*(X^A)_{\mathfrak{p}} = H_N^*(X^A)_{\mathfrak{q}}$$

concluding proof.

"Important" corollary:

$$H^*_{G,\mathfrak{p}} \xrightarrow{\sim} H^*_{N,\mathfrak{q}} \qquad \mathfrak{p} \text{ with max. supp.}$$

Significance lies in the fact that in good cases maybe

$$H^*_G \hookrightarrow \prod_i H^*_{G,\mathfrak{p}_i} \qquad \left(\begin{matrix}\text{no embedded}\\ \text{components}\end{matrix}\right)$$

where $\mathfrak{p}_i$ run over minimal prime ideals of H^*_G whence one would have some hold on H^*_G.

Generalized versions. Y a G-space (as in main thm + $\pi_0 Y^A$ finite for each $A \in \mathcal{a}_G$), so have available theorems on the spectrum. $(A,c) \in \mathcal{a}(G,Y)$. If $X \to Y$ is a G-space, let $X^{A,c}$ be part of X^A lying over component c.

1′). (X as in main thm.) If $\mathfrak{p} \subset H_G(Y)$ has supp. (A,c)

$$H^*_G(X)_{\mathfrak{p}} \xrightarrow{\sim} H^*_G(G\,X^{(A,c)})_{\mathfrak{p}}$$

2′). Let $N = \text{Norm}\,(A,c)$ and let $\mathfrak{q} \subset H^*_N(c)$ be unique prime with support (A,c) lying over $\mathfrak{p}$. Then

$$H^*_G(X)_{\mathfrak{p}} \xrightarrow{\sim} H^*_N(X^{A,c})_{\mathfrak{q}}$$

Proofs by same method: You check ——

<u>Lemma:</u> Let $R \to S$ be finite and M an S module. Let $\mathfrak{p}$ be a prime ideal of R and $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ the primes in S lying over $\mathfrak{p}$. If $M_{\mathfrak{q}_i} = 0$ $i = r+1, \dots, n$, then

$$M_{\mathfrak{p}} \xrightarrow{\sim} (S - (\mathfrak{q}_1 \cup \dots \cup \mathfrak{q}_r))^{-1} M.$$

<u>Proof.</u> We may assume that $M = M_{\mathfrak{p}}$ as $R - \mathfrak{p} \subset S - \mathfrak{q}_i$ for each i. ~~We must show that we may suppose also that~~ It's then a question of showing that for $s \in S - (\mathfrak{q}_1 \cup \dots \cup \mathfrak{q}_r)$ ~~that~~ multiplication by s is an isomorphism of M. We can thus replace R by $R_{\mathfrak{p}}$, etc, whence the $\mathfrak{q}_i$ are the maximal ideals of S. ~~Now the hypotheses are inherited by submodules (and quotient modules) of M, so we can assume M is finitely generated in fact of the form S/𝔞 for some ideal 𝔞. So forming the exact sequence~~

$$\cancel{0 \to {}_sM \to M \xrightarrow{s} M \to M/sM \to 0.}$$

~~we see that the kernel & cokernel are killed by localizing at all the~~ $\cancel{\mathfrak{q}_i}$~~'s!!~~ The hypotheses hold for submodules and quotient modules of M, hence as localization commutes with inductive limits we can suppose M is finitely generated, then as localization is exact that $M \simeq S/I$ for some ideal I of S. The hypothesis says that $I \not\subset \mathfrak{q}_i$ for $i = r+1, \dots, n$. Hence

$$Ss + I \not\subset \mathfrak{q}_i \qquad i = 1, \dots, n.$$

As the $\mathfrak{q}_i$ are the maximal ideals of S, it follows that $Ss + I = S$, so s is a unit mod I. q.e.d.

December 30, 1970: Central [p]-subgroups (th. on p 10)

Suppose G is a p-primary group of rank r; this means G is a p-group with $\Omega_1(G) < Z(G)$. Then if $V_1, \dots, V_r$ are irreducible complex representations whose characters on the center form a basis for $\Omega_1(G)^\wedge$, then I know that $e(V_1), \dots, e(V_r)$ is a regular sequence in $H^*(G)$ and

$$H^*(G)/(e(V_1), \dots, e(V_r)) \simeq H^*_G(SV_1 \times \dots \times SV_r)$$

$$\simeq H^*((SV_1 \times \dots \times SV_r)/G)$$

I claim that if $G' < G$, then the restriction homomorphism

$$H^*(G) \longrightarrow H^*(G')$$

is not injective. Indeed this is clear if $G' \not\supset \Omega_1 G$, by considering dimensions, so assume $G' \supset \Omega_1 G$. Then it will suffice to show that

$$H^*((SV_1 \times \dots \times SV_r)/G) \longrightarrow H^*((SV_1 \times \dots \times SV_r)/G')$$

kills the fundamental top diml. cohomology class. But in general if $f: Y \to X$ is a cyclic covering of compact (connected) oriented manifolds, then the top class ν_X is $i_* 1$, i inclusion of a point. Thus $f^* \nu_X = f^* i_* 1 = p \cdot i_* 1 = p \cdot \nu_Y = 0$.

Question: $\quad P.S.\left\{H_G^*(SV_1 \times \cdots \times SV_r)\right\} = \alpha(t)$

so that

$$P.S.\left\{H_G^*\right\} = \frac{\alpha(t)}{(1-t^{2d_1})\cdots(1-t^{2d_r})} \qquad d_r = \dim_{\mathbb{C}} V_r$$

Does this have a nice limit as $t \to -1$, e.g. for an abelian p-group rank r

$$P.S.\left\{H_G^*\right\} = \frac{(1+t)^r}{(1-t^2)^r} = \frac{1}{(1-t)^r} \longrightarrow \frac{1}{2^r}$$

~~Definition: primary pair (G,X): X connected and for each x the elements of order p in G_x form the same subgroup A of G and~~

~~Definition: (G,X) primary for A if $A \subset Z(G)$ and A acts trivially on X and A is maximal~~

Definition: (G,X) primary if X is connected and if for each $x \in X$, $\Omega_1 G_x$ is a central subgroup of G which is independent of X.

Thus if $\Omega_1 G_x \subset Z(G) \Longrightarrow \Omega_1 G_x \subset Z(G_x) \Longrightarrow$ $\Omega_1 G_x$ is a $[p]$-subgroup. Thus $A \subset Z(G)$ and $X = X^A$ and $A = \Omega_1 G_x$ for all x.
(Example: (A,c) maximal for (G,X) and $(C(A,c), X^{A,c})$).
(Example: G acts freely on X)

Conjecture: Given (G,X) we consider maps $(G',X') \longrightarrow (G,X)$ where (G',X') is primary, and by such maps we can detect all classes in $H^*_G(X)$.

Special case: Suppose $A = \mathbb{Z}/2\mathbb{Z}$ and X is a smooth manifold. Let $\tilde{X}$ be the blow-up of X along X^A. Then one knows that

$$H^*_A(X) \longrightarrow H^*_A(\tilde{X})$$

is injective, hence to prove the conjecture I can assume that X^A is of codimension 1 in X. Then let Y be the manifold with boundary obtained by separating the sides of X^A in X. Then A acts freely on Y and

the map $Y \longrightarrow X$ is equivariant, an isomorphism of $Y - \partial Y \xrightarrow{\sim} X - X^A$ and finally $\partial Y/A \xrightarrow{\sim} X^A$.
Then consider

$$
\begin{tikzcd}
H^*_A(Y,\partial Y) \arrow[r, "\delta"] & H^*_A(Y) \arrow[r] & H^*_A(\partial Y) \\
H^*_A(X, X^A) \arrow[r] \arrow[u, "\cong"'] & H^*_A(X) \arrow[r] \arrow[u] & H^*_A(X^A) \arrow[u] \\
H^*(X/A, X^A) \arrow[r] \arrow[u, "\cong"'] \arrow[uu, bend left, "\cong"] & H^*(X/A) \arrow[r] \arrow[u] & H^*(X^A) \arrow[u] \arrow[uu, bend right, "\cong"']
\end{tikzcd}
$$

Better one has Mayer-Vietoris

$$
\begin{tikzcd}
 & & & H^*(X^A) \arrow[d] & \\
\arrow[r, two heads] & H^*_A(\partial Y) \arrow[r] & H^*_A(X) \arrow[r] & H^*_A(Y) \oplus H^*_A(X^A) \arrow[r] & H^*_A(\partial Y)
\end{tikzcd}
$$

showing that the cohomology $\underbrace{}_{H^*_A(X)}$ is detected by the primary pairs (A, Y), (A, X^A).

December 31, 1970.

Primary detection:

Suppose that C is a primary group of dim. 1 with $\Omega_1 C = A$ and let X be a G-space. Then again assuming X is a manifold I consider the Mayer-Vietoris sequence

$$\xrightarrow{\delta} H^*_{\mathbb{C}}(X) \longrightarrow H^*_C(X^A) \oplus H^*_C(Y) \longrightarrow H^*_C(\partial Y) \longrightarrow$$

where Y is the manifold with boundary whose interior in $X - X^A$ and ∂Y = sphere of normal bundle of X^A in X. Another interpretation is $H^*(Y) = H^*_{\mathbb{C}}(X - X^A)$ and $H^*_{\mathbb{C}}(\partial Y)$ = equiv. cohomology of $X - X^A$ ~~...~~ supported at infinity.

Now $\partial Y \longrightarrow X^A$ ~~is an A-equivariant sphere bundle ...~~ is a C-equivariant sphere bundle. It is the sphere bundle associated to the normal bundle ν of X^A in X, so as ν is a "continuous" family of representations of A over X^A not containing the trivial repn., there is a unique complex structure on ν such that on breaking ν down into eigenbundles for A ~~the characters~~

$$\nu = \bigoplus_{\chi \in A^\wedge} E_\chi$$

the characters χ carry a given generator $a \in A$ into

$$\chi(a) = \exp \frac{2\pi i j}{p} \qquad 1 \leq j \leq \frac{p-1}{2}.$$

Thus ν is orientable and moreover it is clear that C doesn't change the orientation, so we have an Euler class

$$e(\nu) \in H_C^{ev}(X^A)$$

and this class restricts to a generator in $H_A^*(pt)$ for all components of X^A. Thus we know $e(\nu)$ is a non-zero-divisor and that

$$H_C^*(X^A) / e(\nu) H_C^*(X^A) \xrightarrow{\sim} H_C^*(\partial Y)$$

so $H_C^*(X^A) \twoheadrightarrow H_C^*(\partial Y)$ and the Mayer-Vietoris sequence breaks up into short exact sequences

$$0 \to H_C^*(X) \to H_C^*(X^A) \oplus H_C^*(X - X^A) \to H_C^*(S\nu) \to 0.$$

which proves the cohomology of (C, X) is detected by the spaces (C, X^A) and $(C, X - X^A)$.

Now each of these spaces has to be broken down into connected pieces. Thus if G acts freely on Z on divides $\pi_0 Z$ up into G-orbits and so can assume $G/H \xrightarrow{\sim} \pi_0 Z$; then

$$Z = G \times^H Z_0$$

so we get to the primary situation (H, Z_0). Next for a G/A-space X with A central in G and $X = X^A$ with A maximal in each isotropy group, we look at G acting on $\pi_0 X^A$ we again can cut G down until we reach a primary situation.

Let G now be a primary group of arbitrary dimension and let X be a G-manifold, compact possibly with boundary. We want to use the same method to show that classes in $H_G(X)$ can be detected by primary pairs (G', X').

Choose a maximal A such that $X^A \neq \phi$. First we show the Mayer-Vietoris sequence leads to exact sequences

$$0 \longrightarrow H_G^*(X) \longrightarrow H_G^*(X - X^A) \oplus H_G^*(X^A) \longrightarrow H_G^*(S\nu) \longrightarrow 0$$

since $X - X^A$ is of same homotopy type as a compact G-manifold (with ∂). (Because $A \triangleleft G$, in fact central, X^A is G-stable) Again everything comes to showing that $e(\nu) \in H_G(X^A)$ is a non-zero divisor. Breaking ν down into eigenbundles for A we can write $e(\nu) = e(\nu_1) \cdots e(\nu_r)$ and each $e(\nu_i)$ is a non-zero divisor because there is a central cyclic subgroup Z of G acting trivially on X^A such that $e(\nu_i)$ restricts to a generator of H_Z^* over each component.

The above doesn't use maximality of A. But now we want to see that $H_G^*(X^A)$ decomposes into primaries.

Again letting G act on $\pi_0 X^A$ and taking orbits, we can reduce to the case where X^A is connected. But then A maximal implies primary.

<u>Proposition:</u> Let A be a central $[p]$-subgroup of G, and let X be a G-manifold and let ν be the normal bundle of X^A in X. Then there are short exact sequences

$$0 \longrightarrow H^*_G(X) \longrightarrow H^*_G(X - X^A) \oplus H^*_G(X^A) \longrightarrow H^*_G(S\nu) \rightarrow 0$$

Proof. Let U be an open tubular nbd. of X^A in X which is invariant under G, so $\exists$ an equivariant diffeomorphism $U \simeq \nu$. Then

$$X = (X - X^A) \underset{U - X^A}{\cup} U$$

and $U - X^A \sim S\nu$, so the proposition follows from the Mayer-Vietoris sequence and surjectivity of $H^*_G(X^A) \rightarrow H^*_G(S\nu)$. ν being a G-bundle over X^A has an Euler class $e(\nu) \in H^{\dim\nu}_G(X^A)$, provided it is equivariantly oriented.

Step 1. ν is equivariantly orientable. trivial for $p=2$ The fibre ν_x over $x \in X^A$ is a representation of A not containing the trivial representation. Divide up $A^\wedge_{\;\;\;1}$ $= S \sqcup S^*$ into its orbits under conjugation. Then there is a unique complex structure on ν_x such that the irreducible constituents are in

S. Hence there is a unique complex structure on ν such that only eigenbundles are for the characters in S. Since G preserves $\hat{A} = S \amalg S^*$, G leaves this complex structure invariant.

(e(ν) is a non-zero divisor.)

Step 2: Let $\nu = \sum_{\chi \in S} \nu_\chi$ be the eigenbundle decomposition. Since G centralizes A, ν_χ is stable under G, hence gives rise to

$$e(\nu_\chi) \in H_G^{2d_\chi}(X^A) \qquad d_\chi = \dim_{\mathbb{C}} \nu_\chi .$$

So it is enough now to worry about $e(E)$ where E is a G-bundle over X^A whose restriction to (A,x) is a sum of copies of χ. Choose now a subgroup $Z < A$ which will be central in G. Then $e(X) \in H_G(X^A)$ restricts to a generator of H_Z^* at each x so I know by earlier work that $e(X)$ is a non-zero divisor, i.e. the s.s.

$$H^*_{G/Z}(X^A) \otimes H^*_Z \Longrightarrow H^*_G(X^A).$$

Alternative ways of putting the proposition

$$\begin{tikzcd}
0 \arrow[r] & H^*_G(X, X-X^A) \arrow[r] & H^*_G(X) \arrow[r, "j^*"] \arrow[d, "i^*"] & H^*_G(X-X^A) \arrow[r] & 0 \\
& H^*_G(X^A) \arrow[u, "\cong", "\text{Thom isom}"'] \arrow[ur, "i_*"] \arrow[r, "e(\nu)"] & H_G(X^A) & &
\end{tikzcd}$$

So the good statement is the diagram

$$\boxed{\begin{tikzcd} 0 \arrow[r] & H^*_G(X^A) \arrow[r, "i_*"] \arrow[dr, "\cdot e(\nu)"'] & H^*_G(X) \arrow[r, "j^*"] \arrow[d, "i^*"] & H^*_G(X-X^A) \arrow[r] & 0 \\ & & H^*_G(X^A) & & \end{tikzcd}}$$

if A is a central $[p]$-subgroup of G

So let G be a p-primary group.
Then we have a stratification

$$X = \coprod_{B\subset A} X^{(B)}, \quad A = \Omega_1 G, \qquad X^B = \coprod_{B'\subset B} X^{(B')}$$

$x \in X^{(B)} \iff B = \Omega_1(G_x)$. Now the point is that there is an isomorphism

$$\boxed{\begin{array}{c} gr\, H^*_G(X) \cong \bigoplus_{B\subset A} H^*_G(X^{(B)}) \\ \text{of } H^*_G\text{-modules} \end{array}}$$

whose nature will become clear from the proof. To prove this one uses induction on the number of subgroups B for which $X^B \neq \phi$. If B is a maximal one of these, then $X^{(B)} = X^B$. ($X^{(B)} \subset X^B$ always and $B \subset G_x$, B max $\Rightarrow B = \Omega_1 G_x$).

The above exact sequence gives

$$0 \longrightarrow H^*_G(X^B) \longrightarrow H^*_G(X) \longrightarrow H^*_G(X - X^B) \longrightarrow 0$$

hence

$$gr\, H^*_G(X) = H^*_G(X^B) \oplus H^*_G(X - X^B)$$

But $X - X^B$ has fewer subgroups $B' \subset \Omega_1 G$ with $(X - X^B)^{B'} \neq \phi$.

Here is a better arrangement of the same ideas: Introduce a filtration on X by setting

$$x \in F_p X \iff \operatorname{rank}(\Omega_1 G_x) \geq p.$$

Then

$$F_p X - F_{p+1} X = \coprod_{\operatorname{rank}(B) = p} X^{(B)}$$

Moreover in $X - F_{p+1}X$, the $X^{(B)}$ with $\operatorname{rank}(B) = p$ are closed and disjoint, hence

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{rg(B)=p} H^*_G(X^{(B)}) \arrow[r, "i_*"] \arrow[dr, bend right, "e(\nu)"'] & H^*_G(X - F_{p+1}X) \arrow[r, two heads, "j^*"] \arrow[d, "i^*"] & H^*_G(X - F_p X) \arrow[r] & 0 \\
& & H^*_G(F_p X - F_{p+1} X) \arrow[d, phantom, "\parallel"] & & \\
& & \bigoplus_{rg(B)=p} H^*_G(X^{(B)}) & &
\end{tikzcd}$$

$\mathcal{K}$ fibred category of ~~[illegible]~~ virtual coverings
an object of $\mathcal{K}$ over X is a kind of cycles. It is a formal sum $\sum m_Y[Y]$ where $m_Y \in \mathbb{Z}$ and ~~Y~~ Y runs over the finite covering spaces of X. The concept of isomorphism class of such things is clear. And now — an isomorphism ~~between two such~~ or cobordism should be what

Difference between this and algebraic geometry is that a cycle does not admit automorphisms so there is no isomorphism question involved with the equality of two cycles.

Cobordism of cycles: you ask for one on the product $X \times I$. This is same as equality

---

$\mathcal{K}$ fibred category in groupoids over $\mathcal{S}$
then if $f_0, f_1 : X \longrightarrow Y$ are homotopic, then the homotopy gives rise to an isomorphism $f_0^* y \cong f_1^* y$.

check fibre axiom.

Suppose $\theta : \mathcal{K} \longrightarrow E$ is a natural transformation
i.e. for each $x \in \mathcal{K}(X)$ have $\theta_x : X \longrightarrow E$ and for each $x \xrightarrow{\lambda} y$ over $X \longrightarrow Y$ have

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "\theta_x"] \arrow[dr, phantom, shift right=3, "\theta_\lambda \Downarrow"] & \\
Y \arrow[r, "\theta_y"'] & E
\end{tikzcd}$$

satisfying evident compatibility conditions.

now you should also notice that there is a natural map

$$\cancel{k(G \overset{G'}{\amalg} G) \times k(G \overset{G'}{\amalg} G) \to k(G \overset{G'}{\amalg} G)}$$

$$k(G \overset{G'}{\amalg} G) \times_{k(G)} k(G \overset{G'}{\amalg} G) \longrightarrow k(G \overset{G'}{\amalg} G)$$

Suppose you have an element of $k(G)$ i.e. a integral linear combination of ~~coverings~~ irreducible G-sets; write it as difference

$$S_0 - S_1$$

of positive disjoint G-sets. Now ~~if you~~ if this goes to zero in $k(G')$, then S_0 and S_1 are isomorphic. Choosing an isomorphism one gets an element of $k(G \overset{G'}{\amalg} G)$.

$$d(S_0, S_1, u) \in A$$

and $d(S_1, S_2, v)$

I want $\quad d(S_0, S_2, uv) = d(S_0, S_1, u) + d(S_1, S_2, v)$

so you need an algebraic K-theory to associate to the algebraic general linear groups over a scheme S, i.e. $B\underline{GL}_S$. What happens prime to the characteristic tends to be clear? Thus one tends to understand completely what happens prime to the characteristic, i.e. one gets that

$$\pi_{2q}(B\underline{GL}_S) = T^{\otimes q} \qquad T = \text{Tate motive.}$$

T is an inverse system of finite étale group schemes over S. hence this is the relative situation. Now you must understand what gives when you start taking sections of this **animal** and more generally "integrating it" over a ~~map~~ map f_*

This style intuition is that alg. K generalizes cohomology. Kummer theory

$$0 \longrightarrow BGL(\mathbb{F}_q) \longrightarrow B\,\underline{GL}_S \xrightarrow{\Psi^q - id} B\underline{GL}_S \longrightarrow 0 \qquad ?$$

If S is a topological space, then $B(\underline{GL}_S) = S \times BU$ so the ~~sections~~ sections are space $\underline{Hom}(S, BU)$ whose homotopy groups are $K_*(S)$

Idea is that $\underline{GL}$ should be defined cat. of annele space and gives something over any annele topos. Even non-commutative ring. Something like a generalization of a sheaf.

functor from simplicial sets to simplicial sets which hopefully will give $\underline{\underline{k}}$ by truncation say.

Let us assume that we have a bisimplicial set $A_{\cdot\cdot}$ and we consider the functor from s. sets to s. sets represented by this:

$$X \longmapsto \underbrace{\operatorname{Hom}(X_{\cdot}, A_{\cdot,\cdot})}_{} = Q(X).$$

Then I can define

$$k_i(X) = \pi_{i+1}(Q(X))$$

and I can still ask for the required universal property

---

so even if you can solve the category problem, it still remains to define $\underline{\underline{k}} \longrightarrow \underline{\underline{E\Psi^8}}$. Thus if X is a space and if we are given a word

Thus to a space X associate the Picard category $\underline{\underline{E\Psi}}^8(X)$ defined by rigidifying BU, etc. Then given an $\mathbb{F}_8$-vector bundle E over X choose a classifying map $f: X \longrightarrow BGL_n(\mathbb{F}_8)$ for E ie a pair

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d] & EGL_n(\mathbb{F}_8) \arrow[d] \\ X \arrow[r, "f"] & BGL_n(\mathbb{F}_8) \arrow[r, "\text{chosen}", "\text{already}"'] & \underline{\underline{E\Psi}}^8 \end{tikzcd}$$

and associate to an exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of bundles

---

better start with $\quad BGL_n(\mathbb{F}_q) \xrightarrow{u_n} E\Psi^q$

and for n, m

$$\begin{tikzcd}
 & BGL_m(\mathbb{F}_q) \times BGL_n(\mathbb{F}_q) \arrow[r, "u_m \times u_n"] \arrow[d, phantom, "\Downarrow"] & E\Psi^q \times E\Psi^q \arrow[d, "\mu"] \\
B\Gamma_{mn}(\mathbb{F}_q) \arrow[ur, "(p_1, p_2)"] \arrow[r, "(i_n)"'] & BGL_{m+n}(\mathbb{F}_q) \arrow[r, "u_{m+n}"] & E\Psi^q
\end{tikzcd}$$

so choose such a homotopy. must check transitivity hence have to worry about the uniqueness of the homotopy

$$\begin{array}{cc}
 & u \\
 & \downarrow \Psi^q - id \\
 & U \\
 & \downarrow \\
BG & E\Psi^q \\
 & \downarrow
\end{array}$$

but this should be OKAY again by Atiyah's theorem.

Thus by universal property of the functor $\underline{\underline{k}}$ this part should be OKAY, hence if my universal property holds I get a map ~~[illegible]~~ $\quad B \longrightarrow E\Psi^q$ of spaces

<u>mult. char. classes</u>

~~State~~ to simplify suppose A ~~k~~ is a field and suppose ~~k finite~~ $R = \bigoplus_{n\ge 0} R_n$ is a graded anti-commutative A-algebra. Then by a R-valued mult. char. class

$$\Theta(E) = \sum \Theta_n(E) \qquad \Theta_n(E) \in H^n(X)\otimes R_n$$

$\exists$ ....

Somehow there ~~is~~ should be a universal one represented by $\mathbb{Z}[GQ]$? ~~Yes~~

i.e. $\Gamma = \mathbb{Z}[GQ]$ is a simplicial ring

and in particular a DG ring ~~and~~ To each bundle $E \longrightarrow X$ we ~~get~~ want a coh. class

$$X \longrightarrow \Gamma$$

and to each exact sequence I want a <u>homotopy</u> class of maps!!!!!

If R is a ~~differential~~ simplicial graded ring

$$\cdots \longrightarrow R_2 \rightrightarrows R_1 \rightrightarrows R_0$$

not nec. commutative, then by a multiplicative charac. class with values in R I mean ultimately a hom.
$$\mathbb{Z}[GQ] \longrightarrow R$$

or
$$GQ \longrightarrow R^\times$$

or
$$Q \xrightarrow{\tau} R^\times. \qquad Q \longrightarrow \overline{W}R^\times$$

basic question which you would really like to solve is to produce something over $G(\mathbb{Q})$, ideally a perfect complex, which would possess Chern classes.

Thus suppose you form cokernel of

$$B\Gamma_{m,n} \rightrightarrows BGL_{m+n} \longrightarrow C$$

or better you have X with E' E'' over and an exact sequence $0 \to E' \to E \to E'' \to 0$.
and you want to make $E = E' \oplus E''$.
So you want to map $G_n \times G_m \longrightarrow \Gamma_{m,n}$ an equivalence.

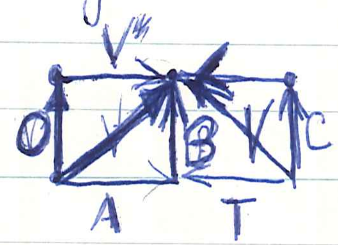
~~ge~~

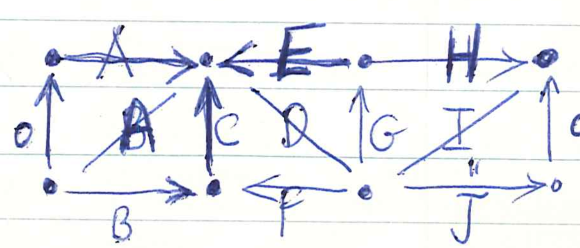
$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

$$\cancel{B\Gamma}\ G_m \times G_n \to \Gamma_{mn}.$$

---

form a 1-cocycle on $I \times X$





In this setup what I get is

$$A - E + H \qquad B - F + J$$

and a reason for ~~these~~ these being the same

~~Thus suppose X comp~~

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & W \\ Z \arrow[ur, dotted] & \end{tikzcd}$$

$\pi_1 W = 0$

The first obstruction to extending to Z ~~is in~~ are in $H^*(Z, X; \pi_*(W))$ so

This works as long as ~~$\pi_*(W)$ simple~~ W simple.

~~$K(\pi_2 W$~~

~~qed~~

$H^*(Z, X; \pi_*(W))$.

$$\begin{tikzcd} & K(\pi_2 W, 2) \arrow[d] & \\ X \arrow[r] \arrow[d] & W_2 \arrow[d] & X \arrow[d] \\ Z \arrow[r] \arrow[ur] \arrow[rr, bend left] & W_1 = pt. \arrow[r] & K(\pi_2 W, 3) \end{tikzcd}$$

$~~= K(\pi_2 W, 2)~~$

$$\overset{0}{\parallel} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{0}{\parallel}$$

$$H^3(Z/X, \pi_2 W) \longleftarrow H^2(X, \pi_2 W) \overset{\sim}{\longleftarrow} H^2(Z, \pi_2 W) \longleftarrow H^2(Z/X, \pi_2 W)$$

H^4

$$\begin{tikzcd} & K(\pi_3 W, 3) \arrow[d] & \\ X \arrow[r] \arrow[d] & W_3 \arrow[r] \arrow[d] & \cdot \\ Z \arrow[r] \arrow[ur, dashed] & W_2 \arrow[r] & K(\pi_3 W, 4) \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[rr] \arrow[d] & & W \\ Z \arrow[urr, dashed] \arrow[d] & & \\ Z/X & \vee S^2 & \end{tikzcd}$$

so Z/X is contractible because it has no cohomology and is 1 connected since it has only 2+3 cells.

Suppose W is a 1-connected $\overbrace{}^{H\text{-space}}$ $\pi_1(W) = 0$

then one knows that

$$[Z,W] \longrightarrow [X,W]$$

~~Suppp~~ Suppose W is a space, then

~~$[Z, W] \ [$~~

$$\begin{tikzcd} & & 0 \arrow[d, phantom, "\shortparallel"] & & \\ {[Z/X,\Omega W] = 0} \arrow[d] & & & & \\ {[Z,\Omega W]} \arrow[r] & {[X,\Omega W]} \arrow[r] & {[Z/X,W]} \arrow[r] & {[Z,W]} \arrow[r] & {[X,W]} \end{tikzcd}$$

hence $[Z,\Omega W] = [X,\Omega W]$

argument should be modifiable

killing fdl. gp. without changing H^*.

Suppose $H_1^{*}(X) = 0$ X connected

attach 2 cells to kill $\pi_1(X)$.

then get Y & $H_q(Y) = H_q(X)$

except $H_2(Y) = H_2(X) +$ free ab grp gen. by attached cells.

but $\pi_1(Y) = 0$ so these new elements are spherical so attaching 3 cells can obtain space Z with same homology.

~~If you want to~~

Is this construction functorial up to homotopy at least. Thus given

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & W \\ Y \arrow[ur, dotted] & \end{tikzcd} \qquad \text{with } \pi_1(W) = 0$$

$$[\vee S^2, W] \longrightarrow [Y, W] \longrightarrow [X, W] \longrightarrow [\vee S^1, W]$$

so unique if $\pi_2(W) = 0$, similarly need $\pi_3(W) = 0$ to get a unique extension to Z.

absolutely canonical

$\boxed{k \text{ field then } \mu^{\otimes i}.}$

If k a field of char p, ~~then~~ and suppose one consider the maximal cyclotomic subfield k_0 say finite with q elements. Then

$$\mu^{\otimes i}_{q^i-1} \quad \text{is canonical.}$$

e.g. Galois group invariant

because

$$\begin{tikzcd} k_0 \arrow[r] \arrow[d] & \bar{k}_0 \arrow[d] \\ k \arrow[r] & \bar{k} \end{tikzcd}$$

$$\mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{Gal}(\bar{k}_0/k_0)$$

gen. by Frobenius. $x \mapsto x^q$

$$[\sigma(c_i), S^{2i-1}] = [c_i, S^{2i}]$$
$$= (i-1)!$$

$$\begin{tikzcd}
S^{2i-1} \arrow[r] \arrow[rrd, bend left] & U \arrow[d] \arrow[rd, "\sigma(c_i)"] & \\
& E \arrow[r, "\Phi(c_i)"] \arrow[d] & K(\mathbb{Z}_p, 2i-1) \\
& BU &
\end{tikzcd}$$

question: How bad is $(i-1)!$ in comparison with $q^i - 1$

answer: is very bad since as i goes to infinity

$$v_\ell((i-1)!) \sim (i-1)\left(\frac{1}{\ell} + \frac{1}{\ell^2} + \cdots\right)$$

$$\frac{i-1}{\ell}\left(\frac{1}{1-\frac{1}{\ell}}\right) = \frac{i-1}{\ell-1}$$

It is linear in i with slope $\dfrac{1}{\ell-1}$

while $\quad v_\ell(q^i - 1) = i\, v_\ell(q-1)$

This is linear in i with slope $v_\ell(q-1)$. ~~Completely~~

Conclusion is that the ~~[illegible]~~ map

$$K_{2i-1}(\mathbb{F}_q) \longrightarrow \varprojlim_n H^1(Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q), \mu_\ell^{\otimes i}) = \mu_{q^i-1}^{\otimes i}$$

$\cong$

$\mu_{q^i-1}^{\otimes i}$ is multiplication by $(i-1)!$

I have to produce an independent proof of the existence of the c_i' and c_i'' in part II

X polyhedron ~~topsa~~ topoes

Suppose you form inductive limit of various topoi associated to finer & finer triangulations then do you get all sheaves. First take the usual limit and

$U \subset X$ open, then $\mathbb{Z}_U = \varinjlim_V \mathbb{Z}_V$

V open subset of an open subpolyhedron so it seems clear that one must obtain all sheaves.

Existence of c_i' c_i'' topologically

$$\begin{tikzcd} & E\Phi^\delta \arrow[d] \\ BG \arrow[ur] & BU \arrow[d] \\ & BU \end{tikzcd}$$

so you produce elements c_i' $c_i'' \in$ $H^*(E\Phi^\delta)$ by you method

so you **must know** how these behave under sums

$$\begin{tikzcd} E\Phi^\delta \times E\Phi^\delta \arrow[r] \arrow[d] & E\Phi^\delta \arrow[d] \\ BU \times BU \arrow[r] \arrow[d, shift right] \arrow[d, shift left, "(\Phi^\delta \times \Phi^\delta)"] & BU \arrow[d, shift right, "\Phi^\delta"'] \arrow[d, shift left, "id"] \\ BU \times BU \arrow[r] & BU \end{tikzcd}$$

~~And the problem I run into is uniqueness of the lifting.~~

$$\begin{tikzcd}
k \arrow[dr] & & K(\pi_n X, n) \arrow[d] \\
& B \arrow[r, shift left] \arrow[r, shift right] & X_{(n)} \arrow[d] \\
& & X_{(n)}
\end{tikzcd}$$

~~Let us first suppose that the difference of these two maps from B to $K(\pi_n X, n)$ is an H-space map, i.e.~~

and the problem is commutativity of

$$\begin{tikzcd}
& K(\pi_n X, n) \arrow[d] \\
k \arrow[r] \arrow[d] & X_{(n)} \arrow[d] \\
B \arrow[ur] \arrow[r] & X_{(n-1)}
\end{tikzcd}$$

point is that we don't yet know that

$$\mathrm{Hom}(k, K(\pi_n X, n)) \longrightarrow \mathrm{Hom}(k, X_{(n)}) \longrightarrow \mathrm{Hom}(k, X_{(n-1)})$$

is exact.

Thus suppose for each $E \longrightarrow Y$ you give a homotopy class.

Maybe you have to formulate the desired condition 2-universally, e.g.

The idea is that a cartesian functor

$$\begin{tikzcd} \underline{\underline{k}} \arrow[rr] \arrow[dr, "\substack{\text{univ} \\ \text{one}}"'] & & \underline{\underline{E\Phi^{q}}} \\ & \underline{\underline{B}} \arrow[ur] & \end{tikzcd}$$

what you have already is a functor, $k \longrightarrow [E\Phi^{q}]$ but you need a one of fibred cats.

The other thing you want to be able to handle is cohomology, and for this you must know that

$$\pi_0 \, \text{Homcart}\left(\underline{\underline{k}}, \underline{\underline{K}}(A, q)\right) = \text{Hom}\left(k, H^q(\ , A)\right).$$

Modulo these two unknowns the theory should work.

Back to simplicial setup. Thus we study

$Q(X)_1$ = set of R-v. bundles over X.

but instead we take $\qquad B_0 = B\{0\}$

$= \text{Hom}(X, B_1) \qquad B_1 = B\{\mathbf{Cat}(V)\}$

$Q(X)_2$ = set of exact sequences

instead take

$= \text{Hom}(X, B_2) \qquad B_2 = B\{\text{Cat}(\text{exact sequences})\}$.

and similarly for all $Q(X)_n$. Thus get a scratch:

terrifically confusing.

I have been looking at category of R-vector bundles over X, which is not a functor of X in the strict sense. make it so, at least the objects. Choose representatives for the proj. f.t. R-modules; form a set ∇. then consider

$$\coprod_{P\in\nabla} B\,\mathrm{Aut}(P)$$

and form category of exact sequences

~~Classifying space of the category of exact sequences.~~

analogy is this: think of k as $H_0(L.)$ where $L.$ is a chain complex in an abelian category. Think of $\underline{k}$ as $L.$ itself and k_1 as $H_1(L.)$. Then I want to think of B as a complex $B.$ and what I think I should gain by working with 2-cats, etc. is embodied by the difference between

$$\mathrm{Hom}_D(L., B.) \quad \text{and} \quad \mathrm{Hom}(H_0(L.), H_0(B.)).$$

keep up the analogy:

Go on to what you need to finish off your theorem. Thus you have constructed a natural transformation $k \longrightarrow [\,, E\Psi^8]$ but you need more, namely a functor

$$\underline{k} \longrightarrow \underline{\underline{E\Psi^8}}$$

Suppose I have two cartesian functors

$$\theta_i : \underline{\underline{k}} \longrightarrow \underline{\underline{X}},$$

such that $f\theta_1$ is isomorphic to $f\theta_2$. What θ_1 does is to for each space S associate a functor

$$\underline{\underline{k}}(S) \longrightarrow \mathrm{Hom}(S, X)$$

plus functorality for all maps in $\underline{\underline{k}}$. Thus if I am given a map $u: S' \longrightarrow S$ and a map of virtual bundles $v: x' \longrightarrow x$ over u, then I want to have a homotopy $\theta_i(v)$ making

$$\begin{tikzcd} S' \arrow[rr, "u"] \arrow[dr, "\theta_i(x')"'] & \arrow[d, phantom, "\overset{\theta_i(v)}{\Rightarrow}"] & S \arrow[dl, "\theta_i(x)"] \\ & X & \end{tikzcd}$$

commute, plus transitivity, etc. Now suppose ξ is an isomorphism from $f\theta_1$ to $f\theta_2$, so for each pair (S, σ) I have a homotopy $\xi(S,\sigma)$

$$\begin{tikzcd} S \arrow[r, shift left, "f\theta_1(S,\sigma)"] \arrow[r, shift right, "f\theta_2(S,\sigma)"'] & Y \end{tikzcd}.$$

Then let $\beta(S,\sigma) = \left(\theta_1(S,\sigma), \xi(S,\sigma), \theta_2(S,\sigma)\right) : S \longrightarrow X \times_Y Y^I \times_Y X$

~~side what I probably need is to express the~~

I form a fibred category over the homotopy category, associating to each X the Picard category $\underline{\underline{k}}(X)$ generated by R-vector bundles over X. An object of $\underline{\underline{k}}(X)$ might be defined to be an element of the free group generated by the set of R-vector bundles over X. On the other hand given a space B I can form the fibered category $\underline{\underline{B}}$ over the homotopy category where $\underline{\underline{B}}(X) = \mathrm{Hom}(X, B)$ and maps are homotopy classes of homotopies. Note that a map $B_1 \longrightarrow B_2$ induces a functor $\underline{\underline{B}}_1 \longrightarrow \underline{\underline{B}}_2$ over $\mathcal{H}o$ and that homotopic maps give rise to isomorphic functors.

The basic question is now whether there exists a morphism of fibered categories $\underline{\underline{k}} \longrightarrow \underline{\underline{B}}$ which is universal, i.e. given $\underline{\underline{k}} \longrightarrow \underline{\underline{B}}'$ there is a map $f : B \longrightarrow B'$ and an isomorphism θ rendering

$$\begin{tikzcd}
\underline{\underline{k}} \arrow[r] \arrow[dr] & \underline{\underline{B}} \arrow[d, "f"] \arrow[dl, phantom, "\Rightarrow\theta"] \\
 & \underline{\underline{B}}'
\end{tikzcd}$$

and the pair (θ, f) is unique in some sense.

The semi-simplicial approach is wrong! Only a universal approach is categorically acceptable.

So ~~suppose~~ before giving up the simplicial stuff observe that if $Q(X)$ is the simplicial set constructed out of R-vector bundles over X, then there is a map

$$Q(X) \longrightarrow \underline{\mathrm{Hom}}(X, Q)$$

or equivalently a map

$$X \times Q(X) \longrightarrow Q.$$

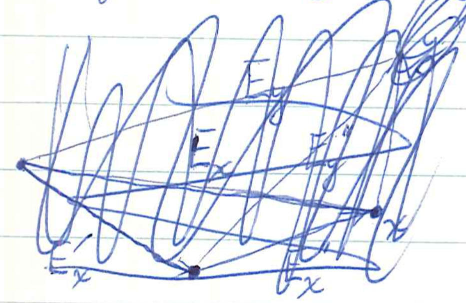
Indeed ~~[illegible]~~ a map $Y \longrightarrow X \times Q(X)$ is the same thing as a pair consisting of a map $u: Y \longrightarrow X$ and a 1-cocycle f on Y with values in R-vector bundles over X. So if y is a 1-simplex in Y, then associate to y the R-module $f'(y) = f(y)_{u(y)}$ i.e. the fiber of $f(y)$ over the 1-simplex $u(y)$. ~~[illegible]~~ Then ~~[illegible]~~ f' is a 1-cocycle on Y with values in R-modules, so we have a map $Y \longrightarrow Q$.

Ideas: to $Q(X)$ I associate the Picard category $\underline{k}(X)$, say defined as the Postnikov part of $G(Q(X))$. Then what I ~~[illegible]~~ want to produce is a map of Picard categories from

$$\underline{k}(X) \longrightarrow \underline{\Pi}(X, E\Phi^{\delta})$$

when I have a finite field. ~~But from the theoretical~~

~~bundle over X. If $0 \to E' \to E \to E'' \to 0$ is~~
~~an exact sequence then you form a simplicial set $\mathcal{E}^2 X$~~



Perhaps it is slightly better to use the suspension $\Sigma X = \Delta(1) \times X / \dot{\Delta}(1) \times X$ and to associate the 1-cocycle which over the product of $\Delta(1)$ and a 1-simplex u looks like

$$\begin{tikzcd} \bullet \arrow[r, "E_y"] & y \\ \bullet \arrow[u, "O"] \arrow[r, "E_x"'] \arrow[ur, "E_u"] & x \arrow[u, "O"'] \end{tikzcd}$$

The reason is that if $0 \to E' \to E \to E'' \to 0$ is an exact sequence of R-bundles over X, then we get a map

$$\Delta(2) \times X / \dot{\Delta}(2) \times X \longrightarrow Q$$

whose faces are the maps associated to E', E, and E''. This shows that we get a map

$$k(X) \longrightarrow \pi_1 \, \underline{\mathrm{Hom}}(X, Q)$$

for any "space" X (note that the Grothendieck group as a group defined by generators + relations is necessarily abelian since $E' \oplus E''$ fits into two exact sequences.)

Situation:

For each $n \geq 0$ I have a simplicial set (n-reduced) $Q(n)$ and probably maps $\Sigma Q(n) \longrightarrow Q(n+1)$ which come from the fact an exact sequence

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots\cdots \longrightarrow V_{n+1} \longrightarrow 0$$

can be extended one more step by adding 0 at the end (or beginning depending on how $\Sigma Q(n)$ is defined).

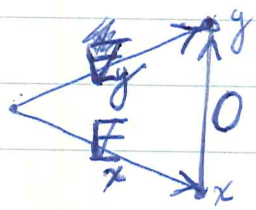
Problem 1: Is induced map $\pi_i Q(n) \longrightarrow \pi_{i+1} Q(n+1)$ an isomorphism?

If so then the cup product maps

$$Q_R(p) \times Q_S(q) \longrightarrow Q_{R \otimes S}(p+q)$$

will give the appropriate product structures on K-groups. (How about λ-operations? Seem OKAY in char 0!)

Suppose E is a (R-) vector bundle over X. Then over each vertex you get a vertex so over ΣX you get for each 1 simplex a vector space



this is a typical 2-simplex in the cone on X.

This shows that there is a map

$$\Sigma X \longrightarrow Q = Q(1)$$

associated to an R-vector bundle over X.

part of paper on algebraic K-theory

~~[illegible]~~ theorems:

I. $BGL^+ = \Omega B\left(\coprod_n BG_n\right)_0$

II. $K_i'A \simeq K_iA$.

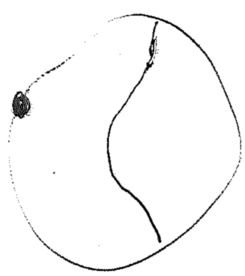
---

stability: manifold X compact with basepoint then get a framed codim 2 variety Z & must be done in a certain order.

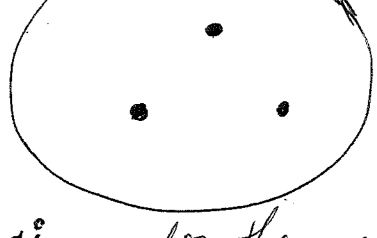
$$\begin{tikzcd}
0 \arrow[r] & E' \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \otimes_{\mathbb{Z}} E' \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & E'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \otimes_{\mathbb{Z}} E' \arrow[r] & A \otimes_{\mathbb{Z}} E \arrow[r] & A \otimes_{\mathbb{Z}} E'' \arrow[r] & 0
\end{tikzcd}$$

don't

and a representation of $\pi_1(X - Z)$ except that certain ramification behavior is to take place.
~~each of the latter ramification pieces is clear & refined.~~



so over the 2-sphere I get so the fundamental group is free group on generators



cobordism is a bit amusing for then singularities come in.

Problems: Suppose $K(X)$ is a representable contravariant functor from finite complexes to $\mathcal{Ab}$ endowed with traces for finite coverings. Does K extend to a generalized cohomology theory? (Segal needs $K(X)$ defined for all CW complexes X, I think.)

Suppose $K(X)$ has a natural ring structure such that the projection formula holds. Then does the cohomology theory $K^*(X)$ admit products?

In the case of the connected theory $k^*(X,A)$ does Ψ^{ℓ} extend to a stable cohomology operation when denominators are introduced? And how about the filtration properties signalled by Milgram as a possible method of attack on the stability problem? Any relation between the two kinds of stability?

If $\mathcal{Q}$ is a small abelian category and we form $\Omega|C.|$ is its homology the same as what one gets from characteristic classes?

Connection of algebraic K-theory and the theory of motives: Fix some groundfield k and consider algebraic schemes over k. Assume that k is small, i.e. of finite type whence one expects that absolute etale cohomology is a good functor in the sense that it detects lots of stuff. To each scheme X, let $K_\bullet(X)$ be the rational K-groups. Then $K_\bullet(X)$ should satisfy the rpojective bundle theorem, and this suggests stronly that it has a Gysin homomorphism; assume so. Wait. Let's review the proj. bundle thm. Thus if E is a vector bundle over X, one knows that $R(G,\mathbb{P}E)$ is a free module over $R(G,X)$ with standard generators, and this isomorphism is compatible with changes in G. Thus the universal map

$$\underline{o}(1) \qquad \underline{X}\,R(G,\mathbb{P}E) \qquad H^o(G,K_\bullet(\mathbb{P}E))$$

Thus $\underline{o}(1)$ defines an element $K_o(\mathbb{P}E)$ and $K_{\frac{1}{2}}(\mathbb{P}E)$ is a free module over $K_{\frac{1}{2}}(X)$ with usual basis and relation. So what we're getting is the various difference in weights already in K_o. Thus if there is a Gysin morphism it is of degree 0 for the grading but changes the weights in the appropriate way.

Conjecture which should be answerable: Show that the Adams operations on $K_\bullet$ admit eigenspaces of the standard sort.

What's missing is a typr of periodicity result which would connect up K-groups of different dimensions.

Grand conjecture would run like this: Take the groups $K_\bullet$ and rearrange the grading so as to be by weights Vaguely the K-groups should measure the deficiency between motives and cohomology. Nonsense.

~~The reason that $BGL_\infty(A)^+$ is an H-space should follow exactly the way it does for BU once you work with the spaces $BGL_n(A)^+$ and the fact that these pointed spaces have the homotopies that you need. Stability theorem should be absolutely formal due to the fact that the cells you a mixing with are of dimensions 2 adn 3. Thus an honest bundle has the property that~~

$$X^n \qquad BGL^+ n = BGL_n \cup e_2 \cup e_3$$

~~Suppose that X^n is an n-manifold. Then a map to BGL_n^+ should be first of all a~~

Program: Characterisitic classes with field coefficients. Formula for the rational K groups.

Universal property of $K_*(A)_{\mathbb{Q}}$: There is a morphism of functors from $\underline{C}$ to $\underline{Ab}$

$$R_A(G) \longrightarrow H^0(G, K_*(A)_{\mathbb{Q}})$$

which is a universal additive transformation from R_A to cohomology with coefficients in a vector space over $\mathbb{Q}$.

Corollary: If A is commutative, then there is a unique ring structure on $K_*(A)_{\mathbb{Q}}$ such that the above is a ring homomorphism.

What about the ψ^k ? They must act on $K_*(A)_{\mathbb{Q}}$; is it reasonable to expect that they give a weight decomposition of this space, i.e. does it split up as a direct sum of eigenspaces for the various characters of $\mathbb{G}_m$. For example there is the homo

$$ch \; ; \; K_*(A)_{\mathbb{Q}} \longrightarrow \bigoplus_i H^{2i-*}(\mathrm{Spec}\, A, T_\ell^{\otimes i})_{\mathbb{Q}}$$

and one knows how the ψ^k act on the ch_i . Thus if the homo above is something like an isomorphism, then

$$c_i \quad K_a(X)_{\mathbb{Q}} \longrightarrow H^{2i-a}(\mathrm{Spec}\, A, T^{\otimes i})$$

should be the projection on the part of weight i .

~~Connection with the theory of motives is becoming pretty clear now.~~ MQ