February 1, 1968 Summary:

Problems:

- Irreducibility of $I(\xi)$ + isomorphism
  $\mathrm{Hom}_{\mathfrak{g}}(I(\xi_1), I(\xi_2))$ related to Weyl group.
  $H_*(\mathfrak{n}, I(\xi))$ (Kostant's BW thm.)
  $H_0(\mathfrak{n}, V) = 0$, V f.f. $\mathfrak{g}, k$ mod $\Rightarrow V = 0$.
  Structure of Ω_Λ. Injectivity of $\Omega_\Lambda \to U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda)$.

Ideas:

principal series - induced + coinduced

relation of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$.

had trouble forming $N \otimes_{\Omega_\Lambda} (U(\mathfrak{g}) \otimes \Lambda)$, reexamine

Calculate $H_*(\mathfrak{n}, I(\xi))$ carefully for $\mathfrak{sl}(2, \mathbb{R})$.

Conjecture: $I(\xi)$ finite length + every irred occurs within.

Borel-Weil problem: Construct a canonical maximal left ideal for each ~~[illegible]~~ orbit closure in $\mathfrak{g}'$.

~~Klem~~ Nakayama's lemma.

homological alg.
$$\begin{tikzcd}(g,k) \arrow[r, shift left=2, "1 \otimes_{\mathfrak{n}}"] \arrow[r, shift right=2, "\mathrm{Hom}(J, \cdot)"'] & (b, \mathfrak{n}) \arrow[l, "I"'] \end{tikzcd} \text{ mod.}$$

(duality thm.?)

I exact $\therefore \simeq J \otimes_b \cdots$

$$\mathrm{Ext}^*_{\mathfrak{g}}(I(\xi_1), I(\xi_2)) = \mathrm{Ext}^*_{b \times b^-}(J, \mathrm{Hom}(\xi_1, \xi_2)).$$

Bruhat 2

February 5, 1968. Summary

Problem: Understand how Bruhat calculated intertwining morphisms.

The case of finite groups:

$j : B \to G$ inclusion of a subgroup. Have

$$j_! \mathcal{J} = G \times_B \mathcal{J} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Hom}_B(G, \mathcal{J}) = j_*(\mathcal{J})$$

$$\Phi(g, \mathcal{J}) = (g_1 \longmapsto \chi_B(g_1 g) g_1 g \mathcal{J})$$

$$\Psi(f) = \sum_i (g_i, f(g_i^{-1})) \qquad \text{where } G = \coprod_i g_i B$$

So

$$\mathrm{Hom}_G(j_* \mathcal{J}_1, j_* \mathcal{J}_2) \simeq \mathrm{Hom}_G(j_! \mathcal{J}_1, j_* \mathcal{J}_2) \simeq \mathrm{Hom}_{B\times B}(G, \mathrm{Hom}(\mathcal{J}_1, \mathcal{J}_2))$$

$$\varphi^\# \longleftarrow\!\!\!\!\!\mapsto \varphi$$

$$(\varphi^\# f)(g_\#) = \sum_i \varphi(g g_i) f(g_i^{-1}) \qquad \varphi(b_1 g b_2^{-1}) = \mathcal{J}_2(b_1)\varphi(g)\mathcal{J}_1(b_2)^{-1}$$

Then $\varphi^\# \circ \psi^\# = (\varphi * \psi)^\#$ where

$$(\varphi * \psi)(g_1) = \sum_i \varphi(g_1 g_i) \psi(g_i^{-1})$$

Now let $\coprod_{u \in W} B\alpha_u B = G$

$$\mathrm{Hom}_{B\times B}(G, \mathrm{Hom}(\xi_1, \xi_2)) \xrightarrow{a} \prod_{u\in W} \mathrm{Hom}_T(\xi_1^{\alpha_u}, \xi_2)$$

$$\varphi \longmapsto (\varphi(\alpha_u))$$

where $\qquad \xi_1^{\alpha_u}(t) = \xi_1(\alpha_u^{-1} t \alpha_u)$.

Here we are assuming that $\quad B = T \times_{\sigma} N,$ that

$$B \cap \alpha_u B \alpha_u^{-1} = T \times (N \cap \alpha_u N \alpha_u^{-1})$$

and that ξ_1, ξ_2 are trivial on N. *Typical Bruhat situation!*

We now ~~wish to calculate~~ define

$$\Gamma(u, \xi)$$

to be that element of $\mathrm{Hom}_{B\times B}(G, \mathrm{Hom}(\xi, \xi^{\alpha_u}))$ such that

$$\Gamma(u,\xi)(\alpha_v) = \begin{cases} 0 & v \neq u \\ id_{\xi^{\alpha_u}} & v = u \end{cases}$$

Thus

$$\Gamma(u,\xi)(g) = \begin{cases} 0 & \text{if } g \notin B\alpha_u B \\ \xi^{\alpha_u}(b_1)\,\xi(b_2) & \text{if } g = b_1 \alpha_u b_2 \end{cases}$$

<u>Problem:</u> Relate

$$\Gamma(v, \xi^{\alpha_u}) \circ \Gamma(u, \xi) \qquad \text{and} \qquad \Gamma(vu, \xi).$$

Special case: $u = s = v$ is a reflection.

$$[\Gamma(s,\rho^{\alpha_s}) * \Gamma(s,\rho)](\alpha_w) = \sum_i \underbrace{\Gamma(s,\rho^{\alpha_s})(\alpha_w g_i)}_{\substack{\neq \\ 0 \\ \Downarrow \\ \alpha_w g_i \in B\alpha_s B}} \underbrace{\Gamma(s,\rho)(g_i^{-1})}_{\substack{\neq \\ 0 \\ \Downarrow \\ g_i^{-1} \in B\alpha_s B}}$$

$$\Downarrow$$

$$\alpha_w \in B\alpha_s B \cdot B\alpha_s B$$

But as s is a reflection $B\alpha_s B \cup B$ is a subgroup of G so either $w = s$ or $w = e$.

Recall that

$$N/N \cap \alpha_s N \alpha_s^{-1} \xrightarrow{\sim} B\alpha_s^{-1} B/B = B\alpha_s B/B.$$

and we can write $N = J \times_s \alpha_s N \alpha_s^{-1}$ J abelian. Thus can take $g_j = j\alpha_s^{-1}$ $j \in J$ for coset representatives.

Thus we want $\alpha_w j \alpha_s^{-1} \in B\alpha_s B$.

I would like to rule out the possibility $w = s$, so for the moment assume that

$$\sum_j \Gamma(s,\rho^{\alpha_s})(\alpha_s j \alpha_s^{-1}) \underbrace{\Gamma(s,\rho)(\alpha_s j^{-1})}_{\rho(\alpha_s)} = 0$$

This will certainly be the case if $\mathrm{Hom}_T(\sigma^{\alpha_s}, \sigma) = 0$.

Conclusion: want

$$\sum_j \Gamma(s, \sigma^{\alpha_s})(j\alpha_s^{-1})\, \Gamma(s, \sigma)(\alpha_s j^{-1})$$

$$= \sum_j \quad ? \qquad \cancel{\sigma(\alpha_s)}\, \mathrm{id}_{\sigma^{\alpha_s}}$$

write $\alpha_s^{-1} = t_s \alpha_s$ $\qquad t_s \in M$.

$$\Gamma(s, \sigma^{\alpha_s})(j t_s \alpha_s) = \begin{cases} \cancel{0 \qquad j t_s \alpha_s \notin B\alpha_s B} \\ (\sigma^{\alpha_s})^{\alpha_s}(j t_s) \end{cases}$$

$$\|$$

$$(\sigma^{\alpha_s})^{\alpha_s}(t_s) = \sigma^{\alpha_s}(\alpha_s^{-1} t_s \alpha_s)$$

$$= \sigma(\alpha_s^{-1}\alpha_s^{-1} t_s \alpha_s \alpha_s)$$

$$= \sigma(t_s) \qquad \text{indep of } j.$$

Therefore

$$\boxed{\Gamma(s, \sigma^{\alpha_s}) * \Gamma(s, \sigma) = |J| \cdot \sigma(t_s) \cdot \Gamma(\underset{\substack{\| \\ \mathrm{id}\ I(\sigma)}}{1}, \sigma).}$$

doesn't look good in char p.

What is $\sigma(t_s)$ in $sl(2, \mathbb{R})$?

$$\alpha_s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad t_s = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M\,A \bumpeq Z \times e^{tA}$$

$$\varphi\left(e^{2\pi i H \cdot n}\, e^{tA}\right) = \nu^{n} e^{t\lambda}$$

$$\alpha_s = e^{\pi i H}$$

$$\sum_{i} \Gamma(s, \varphi^{\alpha_s})(g_i)\, \Gamma(s, \varphi)(g_i^{-1}) \qquad g_i = j\alpha_s$$

$$\sum_{j} \underbrace{\Gamma(s, \varphi^{\alpha_s})(\underline{j}\alpha_s)}_{} \quad \underbrace{\Gamma(s, \varphi)(\alpha_s^{-1} j)}_{}$$

$$\begin{array}{cc} \parallel & \parallel \\ (\varphi^{\alpha_s})^{\alpha_s}(j) & \Gamma(s, \varphi)(t_s \alpha_s j) \\ \parallel & \parallel \\ 1 & \varphi^{\alpha_s}(t_s)\, \varphi(j) \\ & \parallel \\ & \varphi^{\alpha_s}(t_s) \\ & \parallel \\ & \varphi(\alpha_s^{-1} t_s \alpha_s) = \underline{\varphi(t_s)} \end{array}$$

$$\therefore \quad \sum_{j} \varphi(t_s) = |J| \cdot \varphi(t_s) \cdot \mathrm{id}_J$$

In the case of char. p this is very discouraging!

~~p is a local homeomorphism.~~

Nakyama's lemma: $H_0(\mathfrak{n}, V) = 0 \Rightarrow V = 0.$

V finitely generated over $U(\mathfrak{n})$ since we know its finitely generated over $U(\mathfrak{n}) \otimes Z$.

$$\begin{tikzcd}
(U(\mathfrak{g}) \otimes_k \Lambda) \otimes_{\Omega_\Lambda} \cancel{\xi}\, \underbrace{\mathrm{Hom}_M(\Lambda, \xi)}_{} \arrow[r] \arrow[d] & I(\xi) \\
V &
\end{tikzcd}$$

There are many possible ways of extending the ~~action of the Weyl~~ ~~g~~ Ω_Λ action on $\mathrm{Hom}_M(\Lambda, \xi)$ to an $U(\mathfrak{g}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda)$ action and each gives maps

$$\begin{tikzcd}
(U(\mathfrak{g}) \otimes_k \Lambda) \otimes_{\Omega_\Lambda} \xi \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & I(\xi) \arrow[dl, dash] \\
& I(\xi) &
\end{tikzcd}$$

Suppose knew these were isomorphism. ~~is induced of principal series.~~ But we might be able to show that $I(\xi)$ is free / $\mathfrak{n}$.

Take an irreducible Ω_Λ module $\xi = \mathrm{Hom}_k(\Lambda, V)$. Now have a map

$$\Omega_\Lambda \xrightarrow{\gamma} U(\mathfrak{a})^W \otimes \mathrm{Hom}_M(\Lambda, \Lambda).$$

which you get by passing the mod $\underline{\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{k})\delta_\Lambda}$ ~~then adding~~. to get in $U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda)$

then map $U(\mathfrak{a}) \xrightarrow{'} U(\mathfrak{a})$

$$\underline{A \longmapsto A - \rho(A)}.$$

and you land in $U(\mathfrak{a})^W \otimes \mathrm{Hom}_M(\Lambda, \Lambda)$.

~~we have to take~~ ξ irred. over Ω_Λ of the form

$$\square \otimes \mathrm{Hom}_M(\Lambda, \nu)$$

where $\square$ is an irreducible $\underline{U(\mathfrak{a})^W}$ module given by essentially

send $\boxed{A \mapsto A - \rho(A).}$

---

extends to

~~[illegible]~~

Take $\underline{P(A)}$ Weyl symmetric

$P(A), e^{\lambda}$ A acts

~~[illegible]~~ $\to$ A

$\xi = \mathrm{Hom}_M(\Lambda, \nu)$

with A action given by

have a

$\underline{U(\mathfrak{a})^W} \otimes \mathrm{Hom}_M(\Lambda, \Lambda)$ action

extend to a $U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda)$

Thus $I(\xi)$ and $I(\xi')$ isom if

$(\nu, \lambda-\rho) \overset{\text{conj}}{\sim}$ to $(\nu', \lambda'-\rho)$.

Casimir operator

$$\sum_i H_i K_i + \sum_\alpha e_\alpha e_{-\alpha}$$

$\underline{A^2 - A}$ $\underline{\lambda^2 - \lambda}$

$|\lambda - \rho|^2 - |\rho|^2$.

$A - \rho(A)$

$\lambda(A) - \rho(A)$

$\boxed{\lambda - \rho}$

Therefore suppose that

~~[illegible]~~

$$\begin{tikzcd}
\boxed{(U(\mathfrak{g})\otimes_k\Lambda)\otimes_{\Omega_\Lambda}\mathrm{Hom}_M(\Lambda,\xi)} \arrow[r] \arrow[d, "1\otimes_{\mathfrak{n}}"] & I \\
U(\mathfrak{a})\Lambda \otimes_{U(\mathfrak{a})^W\otimes\mathrm{Hom}_M(\Lambda,\Lambda)} \mathrm{Hom}_M(\Lambda,\xi) \arrow[d, phantom, "\cong"] & \\
\underline{U(\mathfrak{a})\otimes_{U(\mathfrak{a})^W}\mathrm{Hom}_M(\Lambda,\xi).} &
\end{tikzcd}$$

$$\underline{\Omega_\Lambda \xrightarrow{\text{Fun}} U(\mathfrak{a})^{[illegible]}\otimes\mathrm{Hom}_M(\Lambda,\Lambda)}$$

$$U(\mathfrak{a})\otimes_{\mathfrak{a}+\mathfrak{n}}\Lambda$$

Morse inequalities should lead to a refinement of Riemann-Roch

Review the integral formulas.

Let $\varphi : G \longrightarrow \mathrm{Hom}(\mathfrak{I}_1, \mathfrak{I}_2)$ be a "function"

such that $\varphi(b_1 g b_2) = \mathfrak{I}_2(b_1)\varphi(g)\mathfrak{I}_1(b_2)\rho(b_2^{-1})$. Then

if $f \in I(\mathfrak{I}_1)$ $f(bg) = \mathfrak{I}_1(b)f(g)$

$$\theta(g_2^{-1}) = \varphi(g_1 g_2) f(g_2^{-1})$$

~~$\theta((g_2 b)^{-1})$~~ ~~$\theta((bg_2)^{-1}) = \varphi(g_1 g_2^{-1} b^{-1}) f(bg_2^{-1} b^{-1}$~~

~~$\theta((bg_2)^{-1})$~~ $\theta(x) = \varphi(g_1 x^{-1}) f(x)$

~~$\theta(bg_2^{-1}) = \varphi\,\theta b$~~

$$\theta(bx) = \varphi(g_1 x^{-1})\,\mathfrak{I}_1(b^{-1})\,\rho(b)\,\mathfrak{I}_1(b) f(x)$$

$= \rho(b)\theta(x)$ so defines a section of $\Lambda^r T^*(G/B)$.

Now I am going to need to calculate φ as a distribution. So take the Bruhat cell belonging to a reflection s ie

$$B\alpha_s B = \boxed{N\alpha_s B}$$

now we know that this is a manifold of dimension 1 in fact the orbit of a 1-parameter subgroup J of N

$J = \exp te_\alpha$ where $s = s_\alpha$.

Then in fact $B\alpha_s B \cup B =$ ~~[illegible]~~ $\boxed{BJB}$

Recall old formula

$$\alpha_s B \alpha_s^{-1} = T \times \alpha_s N \alpha_s^{-1}$$

$$B = (T \times N)$$

$$\Gamma(s, \xi^{\alpha_s})\, \Gamma(s, \xi) = |J| \cdot \xi(t_s) \cdot \mathrm{id}_{I(s)}$$

$$\boxed{B \cap \alpha_s B \alpha_s^{-1} = T \times N \cap \alpha_s N \alpha_s^{-1}}$$

$$B\alpha_s B / B \cong J \qquad N / N \cap \alpha_s N \alpha_s^{-1}.$$

Problem: Define $\Gamma(\alpha_s, \xi)$ where $\alpha_s \in (MA)^\wedge$

Check that

$$B \cap \alpha_s B \alpha_s^{-1} \ni t \cdot n$$

$$x \in B \cap \alpha_s B \alpha_s^{-1}$$

$$\Longrightarrow \quad x = t_1 n_1$$

$$\alpha_s^{-1} x \alpha_s = t_2 n_2$$

$$\Longrightarrow \quad x = \alpha_s t_2 \alpha_s^{-1} \cdot \alpha_s n_2 \alpha_s^{-1} = \tilde{t}_2 \cdot \underline{\alpha_s n_2 \alpha_s^{-1}} = t_1 n_1$$

But $\quad n_2 = \exp \log n_2$

so $\quad \alpha_s n_2 \alpha_s^{-1} = \exp \mathrm{Ad}\alpha_s (\log n_2).$

~~But Ad $\alpha_s(\log n_2)$~~

$$\log n_2 = \sum{}' a_\alpha e_\alpha$$

$$\mathrm{Ad}\alpha_s(\log n_2) = \underline{\sum{}' a_\alpha e_{s\alpha}}$$

But we know that

$$\underline{\exp \mathrm{Ad}\alpha_s(\log n_2)} \in B. \quad \text{and}$$

If $x \in B \cap \alpha_s B \alpha_s^{-1}$ write $x = t_1 n_1$

$$\alpha_s^{-1} x \alpha_s = \alpha_s^{-1} t_1 \alpha_s \cdot \alpha_s^{-1} n_1 \alpha_s = \tilde{t}_1 \cdot \alpha_s^{-1} n_1 \alpha_s \in B$$

$$\therefore \alpha_s^{-1} n_1 \alpha_s \in B$$

But $\underline{\alpha_s^{-1} n_1 \alpha_s}$ is clearly unipotent $\therefore \alpha_s^{-1} n_1 \alpha_s \in N$

~~$N \to B \to T$~~ $\therefore n_1 \in N \cap \alpha_s N \alpha_s^{-1}$.

Thus N

$$\left(\begin{array}{cc|c} \cdot & \cdot & \\ +1 & 0 & \\ \hline & & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & t & & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & \\ \hline & & 1 \end{array}\right) = \left(\begin{array}{cc|ccc} 1 & 0 & & & \\ -t & 1 & & & \\ \hline & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{array}\right)$$

$$\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$$

∴ calculation OKAY. $\therefore$ $$\boxed{N = J \times (N \cap \alpha_s N \alpha_s^{-1}).}$$

$$B\alpha_s B/B \longleftarrow N$$

$$n\alpha_s B \longmapsfrom n$$

if $n\alpha_s B = \alpha_s B$

$\iff \alpha_s^{-1} n \alpha_s \in B \iff n \in N \cap \alpha_s N \alpha_s^{-1}$

$$J \simeq N/N \cap \alpha_s N \alpha_s^{-1} \simeq B\alpha_s B/B$$

so again $j\alpha_s B$ form a complete set of coset reps.

Now want to calculate a function supposted on the Shubert cell. $B\alpha_s B$

Can you calculate the Bruhat intertwining operator?

Recall map in finite case

$$\begin{tikzcd}
\operatorname{Hom}_{B\times B}(G, \operatorname{Hom}(\rho_1,\rho_2)) \arrow[r] \arrow[dr, bend right, "a"'] & \prod_u \operatorname{Hom}_{B\times B}(B\alpha_u B, \operatorname{Hom}(\rho_1,\rho_2)) \\
 & \cong \prod_u \operatorname{Hom}_T(\rho_1^{\alpha_u}, \rho_2)
\end{tikzcd}$$

$\varphi(t\alpha_u) = \rho_2(t)\,\varphi(\alpha_u)$

$\varphi(\alpha_u\alpha_u^{-1}t\alpha_u) = \varphi(\alpha_u)\,\rho^{\alpha_u}(t).$ ✓

where $(a\varphi)(u) = \varphi(\alpha_u) \in \operatorname{Hom}_T(\rho_1^{\alpha_u}, \rho_2).$

$$B\alpha_u B/B \simeq B/B\cap\alpha_u B\alpha_u^{-1}$$

Thus in fact if $\Gamma(\alpha_u,\rho)$ is $\ni$ $\Gamma(\alpha_u,\rho)(\alpha_v) = 0$ $v\neq 0$

$(\alpha_u) = id_{\rho^{\alpha_u}}$

then we have

$$\Gamma(\alpha_u,\rho)(g) = \begin{cases} 0 & \text{if } g\notin B\alpha_u B \\ \rho^{\alpha_u}(b_1)\,\rho(b_2) & \text{if } g = b_1\alpha_u b_2 \end{cases}$$

and finally as an integral operator we have

$$\left[\Gamma(\alpha_u,\rho)^{\#} f\right](g) = \sum_i \Gamma(\alpha_u,\rho)(g g_i)\, f(g_i^{-1}) \qquad G = \coprod g_i B$$

$$= \sum_i \Gamma(\alpha_u,\rho)(g_i)\, f(g_i^{-1}g).$$

Recall that this sum may be taken over $g_j = \underline{j\alpha_s}$

so

$$[\Gamma(\alpha_s, \rho)^{\#} f](g) = \sum_{j\in J} \underbrace{\rho^{\alpha_u}(j)}_{\overset{\shortparallel}{1}} f(\alpha_s^{-1} j^{-1} g)$$

$$= \sum_{j\in J} f(\alpha_s^{-1} j^{-1} g)$$

why is this in $I(\rho^s)$?

$$\sum_{j\in J} f(\alpha_s^{-1} \underline{j^{-1} b} g)$$

~~$\alpha_s^{-1}(j^{-1}nj)j^{-1}g$~~

write $b = t j_0 n$ where $n \in N \cap \alpha_s N \alpha_s^{-1}$ is norm. by J

Then $\displaystyle\sum_{j} f(\alpha_s^{-1} j^{-1} t j_0 ng))$ n

$$\sum \rho(\alpha_s^{-1} t \alpha_s) f(\alpha_s^{-1} \underline{j^{-1}} ng))$$

$$f(\alpha_s^{-1} \underbrace{j^{-1}nj}_{n_j} j^{-1} g.$$

✓

Thus we now have to develop ~~a~~ continuous analogue of this!
First ~~attempt~~

$$\left[\Gamma(\alpha_s, \xi)^{\#} f\right](g) = \int_{J} f(\alpha_s^{-1} j g).$$

This integral as it stands is incorrect because we have not accounted for the volume factor.

~~[illegible] $\psi \in Hom_T(\xi_1^{\alpha_u}, \xi_2)$~~

~~First problem is to determine the image representation~~.

~~$[\Gamma(\alpha_s, \xi)^{\#} f](bg) =$~~

$g = \frac{1}{2}\rho$
$\rho = \sum$ pos. roots.

Work generally: **Take** $\psi \in Hom_T(\xi_1^{\alpha_u}, \xi_2)$ and consider the function φ on G given by

$$\varphi(g) = \begin{cases} 0 & g \notin B\alpha_u B \\ \xi_2(b_1)\,\psi\,\xi_1^{\alpha_u}(b_2) & \text{if } g = b_1 \alpha_u b_2 \end{cases}$$

This function is supported on $B\alpha_u B$ and will give rise to an integral operator of $B\alpha_s B/B$. Therefore you should try to write down an integral operator which keeps the volume factors straight

Old Fmla

$$Ff(g_1) = \int_{g_2 B \in G/B} \varphi(g_2)\, f(g_2^{-1} g_1)$$

Now I want

$$Ff(g_1) = \int_{g_2 B \in B\alpha_u B/B} \varphi(g_2)\, f(g_2^{-1} g_1)$$

$$B\alpha_u B/B \simeq B/B \cap \alpha_u B \alpha_u^{-1}$$

Define $\quad [\Gamma(\alpha_s, J) f](g) = \int_J \psi f(\alpha_s^{-1} j g)$ (Haar measure on J)

where ψ may but probably doesn't depend on j. Want to determine image representation. So proceed as follows:
write $b = t j_0 n$

$$\int \psi\, f(\alpha_s^{-1} j\, bng) = \int_J \psi\, f(\alpha_s^{-1} j\, \overset{\shortparallel}{t} j_0 n g).\, dj$$

$$= \int_J \psi\, f(\alpha_s^{-1} t\, \underbrace{t^{-1} j t}\, j_0 n g)\, dj$$

Be more careful $t^{-1} j t = n_t j_t$ if $t \in M$ not in Cartan.

$$\int_J \psi\, f(\alpha_s^{-1} t j n g)\, d(tjt^{-1}) = \int_J \psi\, f(\alpha_s^{-1} t n n^{-1} j n g)\, d(tjt^{-1})$$

$$= \int_\psi \psi\, \xi(\alpha_s^{-1} t \alpha_s)\, \underbrace{\xi(\alpha_s^{-1} \overset{\shortparallel}{n} \alpha_s)}\, f(\alpha_s^{-1} j g)\, d(tnjn^{-1}t^{-1}).$$

note as before that $n \in N \cap \alpha_s N \alpha_s^{-1}$.

Suppose that $d(tn_j n^{-1} t^{-1}) = \rho_J(t)\, dj.$

$\therefore$ If $\psi \in \mathrm{Hom}_T(\sigma^{\alpha_s}, \sigma_2 \otimes \rho_J^{-1})$ we find that

above $= \quad \sigma_2(t) \int_J \psi f(\alpha_s^{-1} j g)\, dj = \sigma_2(b)[\Gamma(\alpha_s, \sigma)f](b)$

<u>Conclusion</u>: If

$$\psi \in \mathrm{Hom}_T(\sigma_1^{\alpha_s}, \sigma_2 \otimes \rho_J^{-1})$$

get a map $\psi^{\#}: I(\sigma_1)$ to $I(\sigma_2)$ given by

$$[\psi^{\#}(f)](g) = \int_J \psi f(\alpha_s^{-1} j g)\, dj$$

~~Here we have assumed that J~~

Recall $J = \exp \mathbb{C} e_\alpha$ where $s = s_\alpha$ so if $j = \exp z e_\alpha$

$$tjt^{-1} = \exp \mathrm{Ad}\, t\; z e_\alpha$$
$$= \exp \underline{\alpha(t) z e_\alpha}$$

may assume $t \in$ Cartan.

$\therefore d(tjt^{-1}) = \alpha(t)\, dj.$

$\therefore \rho_J(t) = \alpha(t)$

~~Probably OKAY even if t not in Cartan~~

There seems to a problem where M does not normalizes the e_α space. So assume the cx. case!!

$$\mathrm{Hom}_T(\varrho_1^{\alpha_s}, \varrho_2 \otimes \alpha^{-1})$$

$$\parallel$$

$$\mathrm{Hom}_T(\varrho_1^{\alpha_s} \otimes \alpha, \varrho_2)$$

$$\parallel$$

$$\mathrm{Hom}_T((\varrho_1 \otimes \alpha^{-1})^{\alpha_s}, \varrho_2)$$

Let g be such that $g^{\alpha_g} = g \otimes \alpha^{-1}$

$$= \mathrm{Hom}_T(\varrho_1^{\alpha_s} \otimes (g^{-1})^{\alpha_s}, \varrho_2 \otimes \cancel{g}\, \alpha^{-1} \otimes g^{-1} \otimes \alpha)$$

$$\parallel$$

$$= \mathrm{Hom}_T((\varrho_1 \otimes g^{-1})^{\alpha_s}, (\varrho_2 \otimes g^{-1}))$$

$$-\lambda_1 + \frac{1}{2} = \lambda_2 - \frac{1}{2}$$

$$1 - \lambda_1 = \lambda_2 \quad \checkmark$$

<u>Conclusion:</u> (at least when M normalizes e_α). If

$$\psi \in \mathrm{Hom}_T((\varrho_1 \otimes g^{-1})^{\alpha_s}, (\varrho_2 \otimes g^{-1})) \quad \text{then}$$

$$F(f)(g) = \int_J \psi\, f(\alpha_s^{-1} j g)\, dj$$

defines a map F: $I(\varrho_1)$ to $I(\varrho_2)$.

C

Definition of the principal series.

Let ν be an irreducible finite rep of M, let $\lambda \in \mathfrak{a}'$ and let W be the underlying vector space of ν~~[illegible]~~. We shall denote by ξ the vector space W with ~~its~~ the following ~~[illegible]~~ module structures

over M ~~[illegible]~~ as ν.

over $\mathfrak{a}$ $\quad X \cdot w = \lambda(X) w \quad X \in \mathfrak{a}, \ w \in W$

over $\mathfrak{n}$ $\quad Y w = 0 \quad Y \in \mathfrak{n}, \ w \in W$.

Clearly ξ is an irreducible finite $B = MAN$ module and every finite B module is obtained in this way. We shall ~~let~~ write $\xi = (\lambda, \nu)$ to indicate the relation of ξ and ν.

Let $\mathrm{Hom}_B(U(\mathfrak{g}), \xi)$ be defined as follows (note B does not left act on $U(\mathfrak{g})$ so one must be careful). ~~Let $f \in \mathrm{Hom}_{\mathfrak{a}+\mathfrak{n}}(U(\mathfrak{g}), \xi)$ and let $m \in M$. Then $\mathrm{Ad}\, m$ is a well defined autom. of $U(\mathfrak{g})$ so we may define~~

$$\cancel{(m \cdot f)(D) = m f(D) \quad f(\mathrm{Ad}\, m\, (D))}$$

~~As $\mathrm{Ad}\, m$ normalizes $\mathfrak{a}+\mathfrak{n}$, $m f \in \mathrm{Hom}_{\mathfrak{a}+\mathfrak{n}}(U(\mathfrak{g}), \xi)$. We let $\mathrm{Hom}_B(U(\mathfrak{g}), \xi)$ be the space of those f such that $m \cdot f = f$.~~

$$\mathrm{Hom}_{\mathfrak{a}+\mathfrak{n}}(U(\mathfrak{g}), \xi) \xrightarrow[k]{\sim} \mathrm{Hom}(U(\mathfrak{k}), \xi)$$

$$f$$

~~want~~ $\cancel{f(m \cdot D) = m f(D)}$ M acts on the right by

$$(m f)(d) =$$

If ~~th~~ $\underline{m} \in \mathfrak{m}$, then want

$$\underline{f(\underline{m}}D) = \underline{m} f(D)$$ ✓

~~$(\underline{m}f)(D) = \underline{m}f(D) = f(\underline{m}D) =$~~

$$\underline{m} f(D) \underset{\text{want}}{=} f(\underline{m} D) = \underset{\substack{\| \\ (\underline{m}f)(D)}}{f(D\underline{m})} + f([\underline{m}, D]).$$

Thus you want f's such that

$$\underline{m} f(D) = (\underline{m} f)(D) + f([\underline{m}, D]).$$

And so integrating we want

$$m f(D) = \underset{?}{\underline{(mf)}} D + f(\mathrm{Ad}\, m\, (D)).$$

$$I_\xi = \{ f : U(\mathfrak{g}) \longrightarrow \xi \mid \text{(i)} \; f(XD) = Xf(D) \quad X \in \mathfrak{a}+\mathfrak{n}$$

(ii) f k finite, i.e. ~~$f(D) = 0$ if $D \in U(\mathfrak{g})\mathfrak{a}$~~

~~[illegible] ideal in $U(k)$~~

kf semi-simple finite k module

(iii) ~~[illegible]~~

$$m \circ f = mf + f \circ \mathrm{Ad}\, m \qquad m \in M$$

Note that mf makes sense by (ii).

Definition: I_ξ is the $\mathfrak{g}, k$ module ~~belonging to~~ of the principal series associated to ξ.

Problem: When is I_ξ irreducible?

Conjecture: I_ξ ~~has a~~ is of finite ~~number of J-H~~ length and all irreducible $(\mathfrak{g}, k)$ modules occur as J-H components of exactly one I_ξ.

Condition for irreducibility for $sl(2,R)$ is calculate in the α, λ_0 situation when

$$\underbrace{\mathrm{Hom}_B(V, \xi) \neq 0}_{1 \text{ dimensional}}$$

$\boxed{V_{\lambda_0,\alpha}}$ $\alpha =$ eigenvalue of XY in λ_0.

$f : V \longrightarrow \mathfrak{J}$ · has gen. 1.

$\underline{f \in \mathfrak{J} \otimes V'}$ $f = \sum' \mathfrak{J} \otimes m_k e_k^*$

$\ni$ $\mathfrak{a} f = 0$

$\mathfrak{n} f = 0$.

$\underline{\mathfrak{J} \otimes V'_{\lambda_0,\alpha}} \simeq \mathfrak{J} \otimes \hat{V}_{-\lambda_0,\alpha-\lambda_0}$

$$H e_{\lambda_0}^* = -\lambda_0(H) e_{\lambda_0}^* .$$

$$\begin{aligned} \langle e_{\lambda_0}, XY e_{\lambda_0}^* \rangle &= \langle YX e_{\lambda_0}, e_{\lambda_0}^* \rangle \\ &= \langle (XY-H) e_{\lambda_0}, e_{\lambda_0}^* \rangle \\ &= \alpha - \lambda_0 \end{aligned}$$

$$XY e_{\lambda_0}^* = (\alpha - \lambda_0) e_{\lambda_0}^*$$

Assume have an eigenvector for $\mathfrak{a}+\mathfrak{n}$ $v = \sum' m_k \in \hat{V}_{-\lambda_0,\alpha-\lambda_0}$

~~$Av = \beta v$ $Nv = 0$.~~

then

$$\therefore X m_{k-1} = \frac{\beta + k}{\sqrt{2}} m_k \qquad A 1_{\mathfrak{J}} = -\beta 1_{\mathfrak{J}}$$

$$Y m_{k+1} = \frac{\beta - k}{\sqrt{2}} m_k \qquad \therefore -\beta = \lambda$$

~~so taking~~

$$X m_{k-1} = \frac{-\lambda + k}{\sqrt{2}} m_k$$

$$Y m_{k+1} = \frac{-\lambda - k}{\sqrt{2}} m_k$$

$$\therefore X Y m_k = X \left(\frac{-\lambda - k + 1}{\sqrt{2}} \right) m_{k-1}$$

$$= \frac{(-\lambda - k + 1)(-\lambda + k)}{\sqrt{2} \cdot \sqrt{2}} m_k$$

Now suppose $m \in M = \{ e^{2\pi i n H} \mid n \in \mathbb{Z} \}$

$$e^{2\pi i H} 1_\gamma = \nu 1_\gamma$$

$$e^{2\pi i H} \left(1_\gamma \otimes \sum{}' m_k \right) = \nu e^{2\pi i (-\lambda_0)} = 1$$

$$\therefore \quad e^{2\pi i \lambda_0} = \nu .$$

Set $k = -\lambda_0$ so that

$$\boxed{\begin{aligned} \frac{(-\lambda \overset{+\lambda_0}{\cancel{-k}} + 1)(-\lambda \overset{-\lambda_0}{\cancel{+k}})}{2} &= \alpha - \lambda_0 \\ \nu &= e^{2\pi i \lambda_0} \end{aligned}}$$

← These are the equations relating λ, ν to λ_0, α.

The next calculation is what irreducibility means in the λ, ν.

i.e.

$$\nu = e^{2\pi i \lambda_0}$$

$$\frac{(\lambda+\lambda_0)(\lambda-\lambda_0-1)}{2} = \alpha - \lambda_0$$

$$\parallel$$

$$\frac{\lambda^2-\lambda_0^2-\lambda+\lambda_0}{2} = \alpha$$ ~~$-\lambda_0$~~

$$\lambda^2-\lambda_0^2-\lambda+\lambda_0 = 2\alpha$$

~~$(\lambda^2-\lambda) = (\lambda_0^2+\lambda_0)$~~

$$\boxed{\begin{array}{c} \lambda^2-\lambda = \lambda_0^2-\lambda_0+2\alpha \\ \nu = e^{2\pi i \lambda_0} \end{array}}$$

Corollary is that
one gets isomorphic W
ν, k reps. from $\lambda-\frac{1}{2}$.
conjugate

want $\quad 2\alpha \notin \{2k\lambda_0 + k(k-1) \mid k \in \mathbb{Z}\}$.

To simplify things I set

$$\mu = \lambda - \tfrac{1}{2}$$

$$\mu_0 = \lambda_0 - \tfrac{1}{2}$$

$$2k(\mu_0+\tfrac{1}{2}) + k(k-1) = 2k\mu_0 + k^2$$

then

$$\boxed{\begin{aligned} \mu^2 &= \mu_0^2 + 2\alpha \\ \nu &= -e^{2\pi i \mu_0} \end{aligned}}$$

$$2\alpha \notin \{2k\mu_0 + k^2 \mid k \in \mathbb{Z}\}.$$

$$\Updownarrow$$

$$\mu^2 \notin \{(\mu_0 + k)^2 \mid k \in \mathbb{Z}\}.$$

$$\Updownarrow$$

$$\mu \neq \pm(\mu_0 + k) \qquad k \in \mathbb{Z}.$$

$$e^{2\pi i \mu} \neq -\nu, -\nu^{-1}$$

$$(e^{2\pi i \mu} + \nu)(e^{2\pi i \mu} + \nu^{-1}) \neq 0.$$

$$\boxed{(e^{2\pi i \lambda} - \nu)(e^{2\pi i \lambda} - \nu^{-1}) \neq 0}$$

$$(e^{2\pi i \lambda} - \nu)(e^{-2\pi i \lambda} - \nu) \neq 0.$$

More $\mathfrak{sl}(2)$ calculations

__Old idea:__ Had an abelian category with exact ind. limits and a set $(U(\mathfrak{g}) \otimes_k \Lambda)$ of __small__ projective generators, hence abelian category is a diagram category.

Alternatively we have a functor

$$\mathrm{Res} : \mathcal{M}(\mathfrak{g}, k) \longrightarrow \mathcal{M}_k^{ss}$$

which is exact and faithful and we want to set this up as a descent theory.

$$\mathcal{M}(\mathfrak{g}, k) \longrightarrow \mathcal{M}_k^{ss} \rightrightarrows$$

__New idea:__ Consider functor

$$\begin{tikzcd} F : \mathcal{M}(\mathfrak{g}, k) \arrow[r, shift left] & \mathcal{M}(M \times \alpha) \arrow[l, shift left] \\ V \arrow[r, mapsto] & V/\mathfrak{n}V \end{tikzcd}$$

F is clearly right exact and has a left adjoint

$$(\Gamma_k \mathrm{Hom})_B (U(\mathfrak{g}), J) \longleftarrow\!\!\shortmid J$$

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{g})\otimes_k\Lambda & \Lambda \arrow[l, mapsto] \\
\mathcal{M}(\mathfrak{g},k) \arrow[r, "res"] & \mathcal{M}_k^{ss} \arrow[l, shift right=2] \arrow[l, shift left=2] \\
\Gamma_k \operatorname{Hom}_k(\mathcal{U}(\mathfrak{g}), \Lambda) & \Lambda \arrow[l, mapsto]
\end{tikzcd}$$

note that these left and right adjoint functors are ~~canonically~~ isomorphic i.e. $\exists$ isom.

$$\#: \mathcal{U}(\mathfrak{g})\otimes_k \Lambda \xrightarrow{\sim} \Gamma_k \operatorname{Hom}_k(\mathcal{U}(\mathfrak{g}), \Lambda)$$?

defined how? ~~two possibilities~~ Take

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[dr] & \operatorname{Hom}_k(\mathcal{U}(\mathfrak{g}), \Lambda) \\
& \operatorname{Hom}(S(\mathfrak{p}), \Lambda) \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}$$

kills positive degree stuff.

so $\#(D_1\otimes\lambda)(D_2) = \#(1\otimes\lambda)(D_2D_1)$

write $D_2D_1 = \sum \alpha_i\otimes\beta_i$ $\qquad \alpha_i\in\mathcal{U}(k) \qquad \beta_j\in S(\mathfrak{p})$.

then $\#(1\otimes\lambda)(D_2\otimes D_1) = \sum' \epsilon(\beta_i)\alpha_i\lambda$.

why is it clear this is an isomorphism ?

$$S(\mathfrak{p})\otimes\Lambda \longrightarrow \operatorname{Hom}(S(\mathfrak{p}), \Lambda).$$?

should eventually return to this.

I would like to improve F by working in Weyl group so that it gives an equivalence of categories.

Example: Replace $\mathfrak{g}$ by $\tilde{\mathfrak{g}} = \mathfrak{p} \times_0 k$ and let $\mathfrak{n}$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. Then $\bar{W}$ normalizes $\mathfrak{n}$, hence we may define a $\bar{W}$ module structure on $V/\mathfrak{n}V$ thereby obtaining an equivalence

$$\mathcal{M}(\mathfrak{g}, k) \longrightarrow \mathcal{M}($$

Careful: Given $\overbrace{\text{a } \tilde{\mathfrak{g}}^{k}}^{\text{finite type}}$ module, it's finite type over $\mathfrak{p}$ and has support stable under k. Support $\subset \mathfrak{p}'$ and we can concentrate on its intersection with $\mathfrak{a}' = \mathfrak{n}^{\perp}$ i.e. can look at $V/\mathfrak{n}V$ as a $S(\mathfrak{a})$ module. In this case we have a $N_{\mathfrak{a}} \times \mathfrak{a}$ module. Seems that

$$\mathrm{Hom}_{\tilde{\mathfrak{g}}}(U(\tilde{\mathfrak{g}}) \otimes_k \Lambda_1, U(\tilde{\mathfrak{g}}) \otimes_k \Lambda_2) \longrightarrow \left[U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2)\right]^W$$

should be an isomorphism

$$\| $$

$$\mathrm{Hom}_{N_{\mathfrak{a}} \times \mathfrak{a}}(U(\mathfrak{a}) \otimes \Lambda_1, U(\mathfrak{a}) \otimes \Lambda_2).$$

More reasonable to select a cross-section of the closed orbits. Take linear space v of companion matrices $v' \to \mathfrak{p} \twoheadrightarrow v$, then form

$$\begin{tikzcd} S(\mathfrak{p}) \arrow[r, two heads] & S(v) \\ S(\mathfrak{p})^k \arrow[u, hook] \arrow[ur, "\sim"'] & \end{tikzcd}$$

$V/v'V$ is an $S(\mathfrak{p})^k$ module

Should check details here.

$\widetilde{sl(2,\mathbb{R})} = (H, X, Y)$

$$\begin{cases} [H,X] = X \\ [H,Y] = -Y \\ [X,Y] = 0 \end{cases} \qquad [H, XY] = XY - XY = 0.$$

Classify irreducible $\tilde{\mathfrak{g}}, k$ modules. Decompose under k get $V = \bigoplus_{n\in\mathbb{Z}} V_{\lambda_0+n}$ where $\underbrace{\dim V_{\lambda_0+n} \leq 1,}$ and set of $n \ni V_{\lambda_0+n} \neq 0$ forms an interval in $\mathbb{Z}$. First invariant is

$$\nu = e^{2\pi i \lambda_0}.$$

Another invariant is eigenvalue of $XY = \alpha$.

If $\alpha \neq 0$, then $\dim V_{\lambda_0+n} = 1$ all $n \in \mathbb{Z}$.

If $XY \equiv 0$, then

$$\begin{tikzcd} V_{\lambda_0} \arrow[r, shift left, "X"] & V_{\lambda_0+1} \arrow[l, shift left, "Y"] \end{tikzcd}$$

Case 1: $X \neq 0$, $Y = 0$. Follows that $\bigoplus_{n\geq 1} V_{\lambda_0+n}$ is a proper submodule so must be zero. Thus $V_{\lambda+n} = 0$ if $V_\lambda \neq 0$ $\forall n \geq 1$ $\Longrightarrow$ only one $\lambda \ni V_\lambda \neq 0$ say λ_0. ~~Thus~~

Case 2: $X = 0$, $Y \neq 0$ similar

Answer: $\alpha \neq 0$ get infinite series with invariant $\nu = e^{2\pi i \lambda_0}$

$\alpha = 0$, get 1 dimensional repn with eigenvalue λ_0.

Can you determine the category of $\tilde{\mathfrak{g}}, k$ modules. Any chance that it is equivalent to $\mathfrak{a}\, N_\alpha$ modules?

Is $\underline{V \longmapsto V/\mathfrak{n}V}$ exact?

Conjecture true in generic cases i.e. if XY is invertible.

$$\mathfrak{n} = (N) \qquad N = (X - Y)\tfrac{1}{\sqrt{2}}$$

$$\mathfrak{a} = (A) \qquad A = (X + Y)\tfrac{1}{\sqrt{2}}$$

$$\langle A, N \rangle = 2 - 2 = 0.$$

$$\langle A, A \rangle = 2$$

$$\langle N, N \rangle = -2.$$

$$XY = \tfrac{1}{2}(A^2 - N^2)$$

To calculate $H_*(\mathfrak{n}, V)$ use

$$0 \to H_1(\mathfrak{n}, V) \longrightarrow V \xrightarrow{N} V \longrightarrow H_0(\mathfrak{n}, V) \to 0$$

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda .$$

since it's a $\tilde{\mathfrak{g}}, k$ module.
may assume $V_\lambda \neq 0 \Rightarrow e^{2\pi i \lambda} = \nu$.

$v = \mathrm{Ker}\, N \qquad v = \sum' v_\lambda$

$$Nv = 0 \quad \text{ie} \quad \sum X v_\lambda - Y v_\lambda = 0$$

$$\text{ie} \quad X v_{\lambda-1} = Y v_{\lambda+1} \quad \text{all } \lambda.$$

So let λ be least $\ni$. $v_\lambda \neq 0 \Rightarrow Yv_\lambda = 0$.
$\Rightarrow XY$ not invertible.

?

Prop: Let V be a $\tilde{\mathfrak{g}}, k$ module such that δ is invertible on V where $\delta = 0$ is the locus of non-regular elements of $\mathfrak{p}$. Then $H_+(\mathfrak{n}, V) = 0$ where $\mathfrak{n}$ is the orth. comp. of $\mathfrak{a}$ in $\mathfrak{p}$.

Proof: ~~$\mathfrak{n}$ is stable under k~~

Idea is that $H_+(\mathfrak{n}, V)$ as an $\mathfrak{a}$ module should have its support in bad set of $\mathfrak{a}'$.

In the simple case $\mathrm{Ker}\ N = \mathrm{Ker}\ \underline{X - Y}$.
think of as a module over $X + Y \sim 2X$. ?

Q: Does $\exists$ a unique y-irred ~~quotient~~ submodule of I_y mapping onto y?

$$\begin{tikzcd}
I_y \arrow[rr, "B"'] & & y \\
& J_y \arrow[ul, hook] \arrow[ur, "\text{onto}"'] &
\end{tikzcd} \Bigg\} \exists ?$$

Suppose V irred., can you show $\overset{\text{f.t. over } \mathfrak{a}}{\overset{\downarrow}{V/\mathfrak{r}V}}$ irred over $M_\mathfrak{a}$? i.e. suppose $\exists$ non-trivial quotient mod y.

$$V/\mathfrak{r}V \longrightarrow y \longrightarrow 0$$

This gives rise to $V \hookrightarrow I_y$ which must be injective by irreducibility. so get map

$$V/\mathfrak{r}V \longrightarrow I_y/\mathfrak{r}I_y$$

Suppose I knew this were injective (if $H_1(\mathfrak{r}, I_y/V) = 0$ in particular. Then if I were to know that

$$\begin{tikzcd}
V/\mathfrak{r}V \arrow[r, hook] \arrow[dr] & I_y/\mathfrak{r}I_y \arrow[d, "\wr\|"] & \arrow[l] \text{probably false} \\
& y &
\end{tikzcd}$$

I would be able to conclude that that $V/\mathfrak{r}V$ is irreducible.

The problem of course is that $V/\mathfrak{n}V$ is not Moc irred. but only when W is taken into account.

Imperative calculation: $H_*(\mathfrak{n}, I_\gamma)$

Do for $sl(2,\mathbb{R})$. Let $\gamma = (\lambda, \nu)$ be an irred. repn. of $sl(2,\mathbb{R})$, ~~so that~~ given by

$$\boxed{A\,1_\gamma = \lambda\,1_\gamma}$$

$$e^{2\pi i H}$$

$$K = \text{s.c. covering of } k$$
$$= \{\exp zH \mid z \in \mathbb{C}\}.$$

$$M = \{\exp zH \mid \exp zH(A) = A$$
$$e^{z}X + e^{-z}Y = X + Y\}.$$
$$= \{\exp 2\pi i n H \mid n \in \mathbb{Z}\}.$$

$$\boxed{\exp(2\pi i n H)\cdot 1_\gamma = \nu^{n}\cdot 1_\gamma.}$$

Now need a formula for I_γ. So want to look at functions $f: U(\mathfrak{g}) \longrightarrow \gamma$ such that $f(\mathfrak{n}d) = 0$.
$f(Ad) = \lambda f(d)$.

So want to look at $f: U(\mathfrak{k}) \longrightarrow \mathcal{J}$ which vanish on a cosemisimple right ideal and finally f should be M invariant

Here $U(\mathfrak{k}) = \mathbb{C}[H]$, so

$$f(P(H)) = \sum_{\lambda \in \mathbb{C}} a_\lambda \langle P, e^\lambda \rangle 1_{\mathcal{J}}$$

ie $$f(P) = \sum_{\lambda \in \mathbb{C}} a_\lambda P(\lambda) 1_{\mathcal{J}}$$ $a_\lambda \in \mathbb{C}$. all $P \in \mathbb{C}[H]$.

What does M invariance mean? ~

~~$\exp(2\pi i H)(f(P)) = f(\mathrm{Ad}\exp 2\pi i H\,(P))$~~

If $f \in \mathrm{Hom}(U(\mathfrak{g}), \mathcal{J})$ is $\mathfrak{k}$-finite, then we know how to define mf for any $m \in K$, in this case

~~$\exp(2\pi i H) f$~~

$$\left[\frac{d}{d\alpha}\left(\exp(\alpha H)\cdot f\right)\right](P) = \left(H \cdot \exp(\alpha H) f\right)(P)$$

$$= \left(\exp(\alpha H) f\right)(P \cdot H)$$

try $$g_\alpha(P) = \sum a_\lambda P(\lambda) e^{\alpha\lambda} 1_{\mathcal{J}}.$$

$$\left(\frac{d}{d\alpha} g_\alpha\right)(P) = \sum a_\lambda P(\lambda) e^{\alpha\lambda} 1_{\mathcal{J}} = g_\alpha(PH).$$

If $$f(P) = \sum a_\lambda P(\lambda) 1_\gamma$$

$$\therefore \quad \left[(\exp \alpha H) f\right](P) = \sum a_\lambda P(\lambda) e^{\alpha \lambda} 1_\gamma$$

Given a k finite f want

$$\underline{f \circ L_m} = m \circ f$$

$$\|$$

$$\underline{f \circ (L_m R_{m^{-1}})} \circ R_m$$

$$\|$$

$$m \cdot (f \circ Ad\, m).$$

Thus

M invariance means

$$m \circ f = m \cdot (f \circ Ad\, m)$$

in $sl(2, \mathbb{R})$ $Ad\, m = 1$ on $U(k)$. so

$$\left[(\exp 2\pi i H) \circ f\right](P) = \sum a_\lambda P(\lambda) \nu 1_\gamma = \nu \cdot f(P)$$

$$\left[(\exp 2\pi i H) f\right](P) = \sum a_\lambda P(\lambda) e^{2\pi i \lambda} 1_\gamma$$

$$= \cancel{e^{2\pi i \lambda} (a_\lambda P(\lambda) 1_\gamma f(P)}.$$

$$\therefore \quad e^{2\pi i \lambda} = \nu \quad \text{for all } \lambda \text{ appearing.}$$

Thus

$$f(P) = \sum_{n\in\mathbb{Z}}' a_n P(\lambda_0+n) \underset{\smile}{1_j}$$

which means that as k modules

$$I_j \simeq \sum_{n\in\mathbb{Z}}' \mathbb{C}_{\lambda_0+n} \qquad e^{2\pi i\lambda_0} = \nu$$

Let $\delta_{\lambda_0+n}(P) = P(\lambda_0+n)1_j$.

$$HX = XH + X$$
$$= X(H+1)$$

Then

$$\begin{aligned}(X\delta_{\lambda_0+n})(P) &= \delta_{\lambda_0+n}(PX) \\ &= \delta_{\lambda_0+n}(XP(H+1)) \\ &= \delta_{\lambda_0+n}\left(\tfrac{1}{\sqrt{2}}(A-N+H)P(H+1)\right) \\ &= \tfrac{1}{\sqrt{2}}\left[\lambda\,\delta_{\lambda_0+n}(P(H+1)) + \delta_{\lambda_0+n}(HP(H+1))\right] \\ &= \tfrac{1}{\sqrt{2}}\left[\lambda P(\lambda_0+n+1) + (\lambda_0+n)P(\lambda_0+n+1)\right]1_j \\ &= \tfrac{1}{\sqrt{2}}(\lambda+\lambda_0+n)\,\delta_{\lambda_0+n+1}(P).\end{aligned}$$

$$\begin{aligned}(Y\delta_{\lambda_0+n})(P) &= \delta_{\lambda_0+n}(PY)\\ &= \delta_{\lambda_0+n}(Y P(H-1))\\ &= \delta_{\lambda_0+n}\left(\frac{1}{\sqrt{2}}(N-H+A)P(H-1)\right)\\ &= \frac{1}{\sqrt{2}}\left[-(\lambda_0+n)P(\lambda_0+n-1)+\lambda P(\lambda_0+n-1)\right]\\ &= \frac{1}{\sqrt{2}}(\lambda-\lambda_0-n)\,\delta_{\lambda_0+n-1}(P).\end{aligned}$$

Thus ~~as a~~ we get

$$\boxed{\begin{aligned}X\delta_{\lambda_0} &= \frac{1}{\sqrt{2}}(\lambda+\lambda_0)\,\delta_{\lambda_0+1}\\ Y\delta_{\lambda_0} &= \frac{1}{\sqrt{2}}(\lambda-\lambda_0)\,\delta_{\lambda_0-1}\end{aligned}}$$

Check: $XY\delta_{\lambda_0} = \frac{1}{\sqrt{2}}(\lambda-\lambda_0)\frac{1}{\sqrt{2}}(\lambda+\lambda_0-1)\,\delta_{\lambda_0}$

So $\alpha = \frac{1}{\sqrt{2}}(\lambda-\lambda_0)(\lambda+\lambda_0-1)$

$$= \frac{1}{2}\left[\left(\lambda-\frac{1}{2}\right)^2-\left(\lambda_0-\frac{1}{2}\right)^2\right]$$

So we get

$$\boxed{\begin{aligned}\left(\lambda-\frac{1}{2}\right)^2 &= \left(\lambda_0-\frac{1}{2}\right)^2+2\alpha \quad = \text{eigenvalue of } 2C+\frac{1}{4}\\ \nu &= e^{2\pi i\lambda_0}\end{aligned}}$$

irreducibility means that for all $n \in \mathbb{Z}$

$$\left(\lambda - \tfrac{1}{2}\right)^2 \neq \left((\lambda_0 + n) - \tfrac{1}{2}\right)^2$$

$$\pm\left(\lambda - \tfrac{1}{2}\right) \neq \lambda_0 - \tfrac{1}{2} + n$$

$$e^{\pm 2\pi i\left(\lambda - \frac{1}{2}\right)} \neq e^{2\pi i\left(\lambda_0 - \frac{1}{2}\right)}$$

$$e^{\pm 2\pi i\left(\lambda - \frac{1}{2}\right)} \neq -\nu .$$

---

get isomorphic representations if $\left(\lambda' - \tfrac{1}{2}\right) = \left(\lambda - \tfrac{1}{2}\right)^s$ $s \in \mathbb{N}$

$$\nu' = \nu$$

~~Note that in this case [illegible] the action of W on M is trivial~~

$$W = \left\{ \exp zH \;\middle|\; e^{z}X + e^{-z}Y \sim X + Y \right.$$

$$\Updownarrow$$

$$e^{2z} = 1$$

$$\Updownarrow$$

$$\left. z = \pi i n . \quad n \in \mathbb{Z} \right\}.$$

W acts trivially on M in this case because K is abelian.

Now that we know what I_ν is, calculate $H_*(\mathfrak{n}, I_\nu)$.

$$I_\nu \xrightarrow{N} I_\nu$$

$$N = H - \frac{1}{\sqrt{2}}(X - Y).$$

$$f = \sum_\sigma{}' a_\sigma \delta_\sigma$$

$$\underline{\delta_\sigma(P) = P(\sigma) 1_\nu}$$

Suppose $Nf = 0.$

$$(H\delta_\sigma)(P) = \delta_\sigma(PH)$$
$$= \sigma\delta_\sigma(P).$$

$$Hf = \sum_\sigma{}' a_\sigma H\delta_\sigma$$
$$= \sum_\sigma{}' a_\sigma \sigma \delta_\sigma$$

$$e^{2\pi i\sigma} = \nu$$
$$\sigma = \lambda_0 + n \qquad n \in \mathbb{Z}.$$

$$Xf = \frac{1}{\sqrt{2}} \sum{}' (\lambda + \sigma) a_\sigma \delta_{\sigma+1}$$

$$= \frac{1}{\sqrt{2}} \sum (\lambda + \sigma - 1) a_{\sigma-1} \delta_\sigma$$

$$Yf = \frac{1}{\sqrt{2}} \sum{}' (\lambda - \sigma) a_\sigma \delta_{\sigma-1}$$

$$= \frac{1}{\sqrt{2}} \sum (\lambda - \sigma - 1) a_{\sigma+1} \delta_\sigma$$

$$Nf = 0 \Longrightarrow \sum_\sigma \left(\sigma a_\sigma - \frac{1}{2}(\lambda + \sigma - 1) a_{\sigma-1} + \frac{1}{2}(\lambda - \sigma - 1) a_{\sigma+1}\right) \delta_\sigma = 0$$

$$-2HN + \underline{2NH} = +2A$$

$$2C = A^2 - A \cancel{+} - N^2 + 2NH$$

$$= \underbrace{A^2 + A} - N^2 + 2HN$$

$$\therefore \; Nf = 0 \implies (A^2 + A)f = 2Cf = (\lambda^2 - \lambda)f.$$

$$\operatorname{Ker} N = N_1 + N_2 \qquad \left[(A+\tfrac{1}{2})^2 - (\lambda - \tfrac{1}{2})^2\right] f = 0$$

$$g \in N_1 \iff Ag = -\lambda g \;\Big|\; g \in N_2 \iff Ag = (\lambda - 1)g.$$

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & I_\chi \arrow[r, "N"] & I_\chi \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & K' \arrow[r] & I'_\chi & I'_\chi \arrow[l, "N^t"'] & C' \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Idea: Identify

$$I'_\chi \cong \hat{I}_{-\chi}$$

$$A = \frac{1}{\sqrt{2}}(X + Y)$$

---

how many solutions of $Nf = 0$ are there i.e.

$$\sigma a_\sigma - \frac{1}{2}(\lambda + \sigma - 1) a_{\sigma-1} + \frac{1}{2}(\lambda - \sigma - 1) a_{\sigma+1} = 0$$

generically only 2 dimensions i.e. choose two consecutive a_σ and then done. Therefore

---

Question: Any chance that the full dual of I_χ should be an unrestricted $\operatorname{Hom}_b(\mathfrak{g}, -)$?

Assume M conn.

$$I_\xi = \Gamma_k \underbrace{\mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), \xi)}_{\cong\, k}$$

$$\Gamma_k \underline{\mathrm{Hom}_{\mathfrak{m}}(U(\mathfrak{k}), \xi)}.$$

$$\begin{tikzcd}
U(\mathfrak{g}) \otimes_{\mathfrak{b}_-} \xi \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{Hom}_{\cancel{\mathfrak{m}}\mathfrak{b}}(U(\mathfrak{g}), \xi) \arrow[d, phantom, "\cong"] \\
U(\mathfrak{k}) \otimes_{\mathfrak{m}} \xi \arrow[r] & \mathrm{Hom}_{\mathfrak{m}}(U(\mathfrak{k}), \xi)
\end{tikzcd}$$

?

When are I_ξ and $I_{\xi'}$ isom.

necessary conditions $\quad (\lambda - \frac{1}{2}) = -(\lambda' - \frac{1}{2})$

$$\nu = \nu'.$$

Conversely suppose this holds and try to define a map $\quad \varphi : I_\xi \longrightarrow I_{\xi'}$ by

$$\varphi(\delta_\sigma) = \varphi_\sigma \, \delta'_\sigma$$

$$\varphi(X\delta_\sigma) = \frac{1}{\sqrt{2}}(\lambda + \sigma)\, \varphi(\delta_{\sigma+1}) = \frac{1}{\sqrt{2}}(\lambda+\sigma)\, \varphi_{\sigma+1}\, \delta'_{\sigma+1}$$

$$X(\varphi_\sigma \delta'_\sigma) = \varphi_\sigma \frac{1}{\sqrt{2}}(\lambda' + \sigma)\, \delta'_{\sigma+1}$$

$$\boxed{\varphi_\sigma(\lambda' + \sigma) = (\lambda+\sigma)\, \varphi_{\sigma+1}.}$$

$$\varphi(Y\delta_\sigma) = \frac{1}{\sqrt{2}}(\lambda-\sigma)\varphi(\delta_{\sigma-1}) = \frac{1}{\sqrt{2}}(\lambda-\sigma)\varphi_{\sigma-1}\delta'_{\sigma-1}$$

$$Y(\varphi_\sigma\delta'_\sigma) = \varphi_\sigma\frac{1}{\sqrt{2}}(\lambda'-\sigma)\delta'_{\sigma-1}$$

$$\therefore \quad \boxed{\varphi_\sigma(\lambda'-\sigma) = (\lambda-\sigma)\varphi_{\sigma-1}}$$

$$\begin{cases} \varphi_{\sigma+1}(\lambda'-\sigma-1) = (\lambda-\sigma-1)\varphi_\sigma \\ \varphi_{\sigma+1}(\lambda+\sigma) = (\lambda'+\sigma)\varphi_\sigma \end{cases}$$

put in $\lambda' = \frac{1}{2}-(\lambda-\frac{1}{2}) = 1-\lambda$

get
$$\begin{cases} \varphi_{\sigma+1}(-\lambda-\sigma) = (\lambda-\sigma-1)\varphi_\sigma \\ \boxed{\varphi_{\sigma+1}(\lambda+\sigma) = (1-\lambda+\sigma).\varphi_\sigma} \end{cases}$$

Same equations. Now ~~$\varphi_{\sigma+n}$~~ φ_σ φ_{σ} never 0

So when can we choose ~~$\varphi_{\lambda_0+n} \neq 0 \quad n \in \mathbb{Z}$~~

$\ni$

~~$\varphi_{\lambda_0+n+1}(\lambda+\lambda_0+n) = (1-\lambda-\lambda_0-n)\varphi_{\lambda_0+n}$~~

~~Now Problem if $\lambda+\sigma \neq 0$ or 1. ie if~~

~~$\lambda+\lambda_0$ not an integer ie~~

~~$e^{2\pi i\lambda} \neq 1$~~

~~If $\lambda+\lambda_0 \in \mathbb{Z}$~~ If isom, then

~~$\lambda+\lambda_0 \in \mathbb{Z}$~~

$$\underbrace{\lambda+\lambda_0 = n \iff 1-\lambda+\lambda_0-n = 0.}$$

$$\Downarrow$$

$$1-\lambda+\cancel{\lambda_0} = \lambda+\cancel{\lambda_0}$$

$$\Downarrow$$

$\lambda = \frac{1}{2}$ in which case $\lambda = \lambda'$

So if $\lambda \neq \frac{1}{2}$, we have an isomorphism when and only when

$$e^{2\pi i\lambda}\nu \neq 1 \quad \text{and} \quad e^{-2\pi i\lambda}\nu \neq 1.$$

Conclusion: If $\lambda \neq \frac{1}{2}$, then $I_{\lambda,\nu} \simeq I_{1-\lambda,\nu}$

$\iff (e^{2\pi i\lambda}\nu-1)(e^{-2\pi i\lambda}\nu - 1) \neq 0 \iff I_{\lambda,\nu}$ irreducible

Short paper on max ideals in $U(\mathfrak{g})$.

a. Schur's lemma

b. PRV.

Given $\xi = \lambda, \nu$ when is I_ξ irreducible?

$$\boxed{(e^{2\pi i\lambda}\nu - 1)(e^{-2\pi i\lambda}\nu - 1) \neq 0.}$$

make sense out of this.

PRV for $sl(2,\mathbb{C})$, $\nu = 0$.

The center is $\mathbb{C}[\Delta]$ where Δ = Casimir op.

and $\boxed{\Delta - \alpha}$ generates a maximal ideal $\Leftrightarrow$

α not eigenvalue of a f.d. repn. i.e. of the form

$$\left(\lambda_0 + \frac{1}{2}\right)^2 - \frac{1}{4} = \lambda_0^2 + \lambda_0$$

where λ_0 not a half integer ≥ 0.

dominant wgt.

$$\alpha \notin \left\{ \frac{l^2}{4} + \frac{l}{2} \;\middle|\; l \in \mathbb{Z} \geq 0 \right\} \Leftrightarrow \text{irreducible.}$$

$$\Updownarrow$$

$$4\alpha + 1 \notin \left\{ (l+1)^2 \mid l \in \mathbb{Z} \geq 0 \right\}$$

$$\Updownarrow$$

$$4\alpha + 1 \notin$$

Fundamental problem: Define a map

$$I_\xi \longrightarrow I_{\xi^s} \qquad s \in W$$

I_ξ = functions on ~~$G/B \to K/M$ which are~~ G which are constant on right N cosets, with values in ξ, and which transform under MA as ξ. and furthermore which transform as a finite diml rep. under K.

$f : G \longrightarrow \xi$ be such. Now suppose that $s \in W$ s rep by is an element of K which normalizes M and A but not N.

First do it by Hilbert space methods!

$$\operatorname{Hom}_G(j_* \xi, j_* \xi) = \prod_{\substack{\text{double cosets} \\ BwB}} \operatorname{Hom}\left(\varepsilon \quad , \qquad \right)$$

$$B \xrightarrow{j} G$$

$$\boxed{\mathrm{Hom}_G(j_* L, j_* M) = \mathrm{Hom}_B(j^* j_* L, M)}$$

$$j^* j_* L = \mathrm{Hom}_B(G, M)$$

$$= \prod_{B x B} \mathrm{Hom}_B(BxB, M)$$

$$\Big\Uparrow\!\!\wr$$

$$\mathrm{Hom}_{H_x}(B, M) = \mathrm{Hom}$$

$~~H_x = \{b \mid bx \in xB\} = B \cap x B x^{-1}~~$ $\quad H_x = \{b \mid xb \in Bx\}$

$$= B \cap x^{-1} B x$$

$\underline{b}x = x b'$ $\quad$ ~~$\varphi(bx) = \varphi(xb')$~~

x^{-1}

$$\varphi(x \underline{b}' b_1) = \varphi(b x b_1) = b\varphi(x b_1)$$

$$\begin{tikzcd}
 & B \\
B \cap x^{-1} B x \arrow[ur, hook, "\varepsilon_x = \mathrm{id}"] \arrow[dr, "\eta_x b = x \cdot b \cdot x^{-1}"] & \\
 & B
\end{tikzcd}$$

$$(\varepsilon_x)_* (\eta_x)^* M. \qquad ?$$

$$\therefore \quad \mathrm{Hom}_G(j_* L, j_* M) = \prod_{w}$$

We need an analogue of this formula for I_ξ. Assuming brutally that it holds ~~and~~ we find

$$\mathrm{Hom}_G(I_{\xi_1}, I_{\xi_2})' = \sum_{w \in W} \mathrm{Hom}_{B\cap w^{-1}Bw}(\xi_2, \xi_1)$$

as $B\cap w^{-1}Bw = MA(\underbrace{N\cap N^{w}}_{\text{acts trivially on } \xi})$

we conclude that

$$\mathrm{Hom}_{B\cap w^{-1}Bw}(\xi_2, \xi_1) = \mathrm{Hom}_{MA}(\xi_2, \xi_1^{w}).$$

and therefore obtain the formula

$$\boxed{\mathrm{Hom}_G(I_{\xi_1}, I_{\xi_2}) = \sum_{w} \mathrm{Hom}_{MA}(\xi_1, \xi_2^{w}).}$$

from which Bruhat's theorem follows.

Theorem of Bruhat: $\dim \mathrm{Hom}_G(I_{\xi_1}, I_{\xi_2}) \leq \sum_{w}$ - - - -

Generalize like mad: Idea is to ~~[illegible]~~ determine the category of $I_{\mathcal{J}}$.

$$\operatorname{Hom}_G(I_{\mathcal{J}_1}, I_{\mathcal{J}_2}) = \ ?$$

Assume $\mathfrak{m}$ connected!

$$\mathcal{M}(\mathfrak{g}, k) \xrightarrow{\text{res}} \mathcal{M}(B)$$

$$\begin{tikzcd} \mathcal{M}(\mathfrak{g}, k) \arrow[r, shift left, "\text{res}"] & \mathcal{M}(\cancel{b}) \arrow[l, shift left, "I"] \end{tikzcd} \qquad ?$$

$\mathcal{J}$

$$\operatorname{Hom}_{\mathfrak{g}}(\qquad)$$

$$V \longmapsto H$$

$$F : \mathcal{M}(\mathfrak{g}, k) \longrightarrow \mathcal{M}(\text{nea})$$

F right exact.

$$\begin{tikzcd} F : \mathcal{V} \arrow[r, shift left] & \mathcal{W} \arrow[l, shift left, "I"] \end{tikzcd}$$

$$\operatorname{Hom}(FV, \mathcal{J}) = \operatorname{Hom}(V, I_{\mathcal{J}}).$$

$I_{\mathcal{J}} =$ ~~k finite part of Hom~~

$\simeq$ k fin. $\operatorname{Hom}_M(\mathcal{U}(k), \mathcal{J})$

$\cong$

$\bigoplus \Lambda \otimes \operatorname{Hom}_M(\Lambda, \mathcal{J})$. ?

exact functor of $\mathcal{J}$ ~~not~~ compatible with direct limits $\therefore$ has an adjoint.

G left exact

$$\mathrm{Hom}(I(\xi), V) = \mathrm{Hom}(\xi, GV)$$

now you need a duality theorem

$$\boxed{H_q(n, V)' \simeq \underline{R^{n-q}G}(V^\sim)}$$

$$I(\xi) \simeq \bigoplus \Lambda \otimes \mathrm{Hom}_M(\Lambda, \xi) \quad \text{as} \quad k\text{-modules}$$

clearly an exact function of ξ

$$\simeq J \otimes_A \xi$$

So $\mathrm{Hom}_g(I(\xi), V) = \mathrm{Hom}_g(J \otimes_A \xi, V)$

$$\begin{tikzcd} \mathrm{Hom}_g(J \otimes_A \xi, V) \arrow[d, phantom, "\cong" rotate=90] \\ \mathrm{Hom}_A(\xi, \mathrm{Hom}_g(J, V)) \end{tikzcd}$$ ✓

$$\therefore\ GV = \mathrm{Hom}_g(J, V)$$

So $$R^qG(V) = \mathrm{Ext}^q_g(J, V).$$

$$\mathcal{I}(\mathcal{J}) = k\text{ fin } \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), \mathcal{J})$$

$$= k\text{ fin } \mathrm{Hom}_{\mathfrak{m}}(U(\mathfrak{k}), \mathcal{J})$$

$$=$$

$$\mathcal{I}(\mathcal{J}) = k\text{ fin } \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), \mathcal{J}) = J \otimes_{B} \mathcal{J}$$

What is J?

$$k\text{ fin } \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), \mathcal{J}) = \underline{J \otimes_{(\mathfrak{m}+\mathfrak{a})} \mathcal{J}}$$

$$\mathrm{Hom}_{\mathfrak{m}+\mathfrak{a}}(\underline{U(\mathfrak{a})U(\mathfrak{k})}, \mathcal{J}).$$

$$\underline{k\text{fin } \mathrm{Hom}_{\mathfrak{m}}(U(\mathfrak{k}), \mathcal{J})} \xleftarrow{\sim} R(K) \otimes_{M} \mathcal{J}$$ ╳ The compact trace.

~~$U(\mathfrak{k}) \otimes_{\mathfrak{m}}$~~

Calculate J for $\mathfrak{sl}(2, \mathbb{R})$.

$$I_\gamma = \bigoplus_{e^{2\pi\sigma}=\nu} \mathbb{C}\,\delta_\sigma$$

$$X\delta_\sigma = \frac{1}{\sqrt{2}}(\lambda+\sigma)\,\delta_{\sigma+1}$$

$$Y\delta_\sigma = \frac{1}{\sqrt{2}}(\lambda-\sigma)\,\delta_{\sigma-1}$$

$$H\delta_\sigma = \sigma\,\delta_\sigma.$$

Chev. form

$\bar{X}, \bar{Y}, \bar{H}$

$[\bar{H}, \bar{X}] = 2\bar{X}$

$[\bar{H}, \bar{Y}] = 2\bar{Y}$

$[\bar{X}, \bar{Y}] = \bar{H}$

$\therefore\ \bar{H} = 2H \qquad \bar{X}\delta_\sigma = (\lambda+\sigma)\delta_{\sigma+1}$

$\bar{X} = \sqrt{2}X \qquad \bar{Y}\delta_\sigma = (\lambda-\sigma)\delta_{\sigma-1}$

$\bar{Y} = \sqrt{2}Y \qquad \bar{H}\delta_\sigma = 2\sigma\,\delta_\sigma.$

Thus if

$I_\gamma = J \otimes_{MA} \xi$ we have

~~$J = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\,\delta_n$~~

$$J = \bigoplus_{n\in\mathbb{Z}} \underline{\delta_n}\,\mathbb{C}[A]$$

$J =$

~~$X\delta_\sigma$~~

$$J = \bigoplus_{\sigma\in\hat{K}} \delta_\sigma \quad U(\alpha)U(\mathfrak{n})$$

$$J = R(K)\,U(\alpha)U(\mathfrak{n})$$

$$J = \bigoplus_{\sigma \in \mathbb{C}} \delta_\sigma \cdot \mathbb{C}[A, N].$$

$$\boxed{\begin{aligned} H\delta_\sigma &= \sigma\delta_\sigma \\ X\delta_\sigma &= \tfrac{1}{\sqrt{2}}\delta_{\sigma+1}(A+\sigma) \\ Y\delta_\sigma &= \tfrac{1}{\sqrt{2}}\delta_{\sigma-1}(A-\sigma) \end{aligned}}$$

$$I_\xi$$

$$\boxed{J \otimes}$$

$$\mathrm{Hom}_{\mathfrak{g}}(I(\xi), V) = \mathrm{Hom}_{M,\mathfrak{a}}(\xi, \mathrm{Hom}_{\mathfrak{g}}(J, V)).$$

$$\|$$

$$\mathrm{Hom}_{\mathfrak{g}}(J \otimes_{M,\mathfrak{a}} \xi, V) = \mathrm{Hom}_{M,\mathfrak{a}}(\xi, \underline{\mathrm{Hom}_{\mathfrak{g}}(J, V)})$$

$$J = R(K) \otimes U(\mathfrak{a})$$

$$\varphi \in \mathrm{Hom}_{\mathfrak{g}}(J, V) =$$

~~$\varphi = (\varphi_\sigma)$ $\varphi_\sigma \in V_\sigma$~~

$$\varphi \longmapsto (\varphi\delta_\sigma).$$