{h}

~~[illegible]~~

$$\begin{tikzcd} \mathrm{mod}(R/I) \arrow[r, "i_*"] & \mathrm{mod}(R) \arrow[l, shift right=3, "i^*"'] \arrow[l, shift left=3, "i^!"] \arrow[r, "j^*"] & \mathcal{M}(I) \arrow[l, shift right=3, "j_!"'] \arrow[l, shift left=3, "j_*"] \end{tikzcd}$$

$$R/I = \varinjlim_{\substack{\mathfrak{a}\ \text{fg} \\ \text{left ideal} \\ \subset I}} \mathrm{Hom}_R(R/I, R/\mathfrak{a})$$

$$\varinjlim_{\mathfrak{a}} {}_I(R/\mathfrak{a}) \xrightarrow{\sim} R/I$$

$$\exists \mathfrak{a} \ni \quad \begin{tikzcd} R/\mathfrak{a} \arrow[r, two heads] \arrow[d] & R/I \arrow[l, bend right] \\ R/\mathfrak{a}_1 & \end{tikzcd}$$

Let's organize the final preparation.
So let's see what has to be done.

$$\mathcal{M}_n = \mathrm{mod}(R)/\mathrm{null}(R, I)$$

$$\mathcal{M}_t = \mathrm{mod}(R)/\mathrm{tors}(R, I)$$

characterization of torsion modules.

1) $\forall M' < M : {}_I(M/M') \neq 0$
2) $\mathrm{Hom}(M, N) = 0$ for all N st. ${}_I N = 0$.
3) For all inj Q st ${}_I Q = 0$, $\mathrm{Hom}(M, Q) = 0$
4) For all $\alpha = (a_n)$ in I, $m \in M$ $\exists n$ $a_n \cdots a_1 m = 0$
5) For all F ~~flat~~ R^{op} mod st $F = FI$, $F \otimes_R M = 0$.

{5} What do we know?

F firm flat

$$0 \to F \to \underbrace{K \otimes_R F \to \bigoplus_p E(R/p) \otimes_R F}_{\text{solid injectives}} \to 0$$

Conversely suppose we take a surjection

$$0 \to M \to K^{(I)} \twoheadrightarrow \bigoplus_{p_\alpha} E(R/p_\alpha) \to 0$$

$E(R/p_\alpha)$ not flat, but it should have flat dimension ≤ 1 since we have

$$0 \to R_{p_\alpha} \to K \to E(R/p_\alpha) \to 0$$

Thus M is flat and firm.

~~If M is flat &~~

7/20 Anyway we know this much:

Any Q solid injective is canonically an extension

$$0 \to \bigoplus E(R/p_\alpha) \to Q \to Q \otimes_R K \to 0$$

of a height 0 injective by a height 1 inj.

Any firm flat F is canonically the kernel

$$0 \to F \to F \otimes_R K \to \bigoplus_p F \otimes_R E(R/p) \to 0$$

of a ~~[illegible]~~ surjection from a height 0 injective to a height one injective

$\alpha\ \beta\ \gamma\ \delta\ \varepsilon\ \zeta\ \theta\ \iota\ \kappa\ \lambda\ \mu\ \nu\ \xi\ o$

$\pi\ \rho\ \sigma\ \tau\ \upsilon\ \phi\ \chi\ \psi\ \omega$

{t} Let's see if we can find the image of

Let's try to prove firm modules form an ~~abelia~~ abelian category.

$$K \longrightarrow \bigoplus E(R/p) \longrightarrow E(k)$$

should be a short exact sequence of firm modules. I need to check for any firm N that

$$0 \longrightarrow \operatorname{Hom}(N, K) \longrightarrow \operatorname{Hom}(N, \bigoplus E(R/p)) \longrightarrow \operatorname{Hom}(N, E(k))$$

is exact. Write N as quotient $F_1 \to F_0 \to N \to 0$

Then

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\operatorname{Hom}(N, \bigoplus E(R/p)) \arrow[r] \arrow[d] & \operatorname{Hom}(N, E(k)) \arrow[d] \\
\operatorname{Hom}(F_0, \bigoplus E(R/p)) \arrow[r] \arrow[d] & \operatorname{Hom}(F_0, E(k)) \arrow[d] \\
\operatorname{Hom}(F_1, \bigoplus E(R/p)) \arrow[r] & \operatorname{Hom}(F_1, E(k))
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & * \arrow[d] & * \arrow[d] & * \arrow[d] \\
0 \arrow[r] & * \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[d] \\
0 \arrow[r] & * \arrow[r] & * \arrow[r] & *
\end{tikzcd}$$

Can suppose F is ~~firm~~ flat

$$\begin{tikzcd}
G \arrow[r, two heads] \arrow[d] & F \arrow[d] & \\
K \arrow[r] & \bigoplus E(R/p) \arrow[r] & E(k)
\end{tikzcd}$$

$\{u\}$ I my situation we have h.d ≤ 2.
So the kernel of a map of ferin flats is always flat.

F ferin flat complex
can be replaced by ~~an~~ a solid injective complex.

$$0 \to F \to F \otimes_R K \to \bigoplus_p F \otimes_R E(R/p) \to 0$$

So in this example any flat is ~~torsion~~ null free.

Go other way as follows. Given ~~an~~ ~~injective~~ a solid injective complex Q one has

$$0 \to \Gamma(Q) \to Q \to Q \otimes_R K \to 0$$

and this splits so that

$$Q = \text{h-fibre}\left(Q \otimes_R K \to \Gamma(Q) \right)$$

Take V a contractible K-vector space complex, mapping onto $\Gamma(Q)$

Then I can split

$$0 \to \Gamma(Q) \to Q \oplus V \to Q \otimes_R K \oplus V \to 0$$

in such a way that the boundary map is surjective. Then I get

$$F = \mathrm{Ker}\left((Q \otimes_R K) \oplus V \twoheadrightarrow \Sigma \Gamma(Q) \right)$$

I know

$$0 \to F \to Q \otimes_R K \oplus V \to \Sigma \Gamma(Q) \to 0$$

$\{V\}$ So F gives Fibre $(Q \otimes_R K \oplus V \twoheadrightarrow \Sigma \Gamma(Q))$

$$\cong$$

$$Q \oplus V.$$

So let's try to formulate this.
Given Q you have

$$0 \to \Gamma(Q) \to Q \to Q \otimes_R K \to 0$$

You choose a contractible K vector space V with a surjection $\Sigma^{-1} V \twoheadrightarrow \Gamma(Q)$. Then form

$$\cancel{\text{Cone}}(V \to Q)$$

$$IR \to P \xrightarrow{f} P/IP$$

YES $\boxed{\text{OKAY.}}$

$$pf = p - [d, h]$$
$$= p - p \text{ in degrees } \leq n$$

$$0 \to \Gamma(V) \to Q \to Q \otimes_R K \to 0$$

$$Q = \text{Fib}\,(Q \otimes_R K \to \Sigma \Gamma(V))$$

$$Q = Q \otimes_R K \oplus \Gamma(V)$$

$$Q \oplus V = (Q \otimes_R K \oplus V) \oplus \Gamma(V)$$

where $d: V \to \Sigma \Gamma(V)$ ~~is surj~~ is 0

$$\simeq (Q \otimes_R K \oplus V) \oplus \Gamma(V)$$

where $d: V \to \Gamma(V)$ is surjective

$$\leftarrow \quad \text{Ker}\{(Q \otimes_R K) \oplus V \to \Sigma \Gamma(V)\}$$

gives

$$\overset{-1}{F}$$