A. Topics in K Theory and Cyclic Co-Homology, Michaelmas Term 1989


Editor’s remark The lecture notes were taken during lectures at the Mathematical Institute on St Giles in Oxford. There have been subsequent corrections, by whitening out writing errors. The pages are numbered, but there is no general numbering system for theorems and definitions. For the most part, the results are in consecutive order, although in one course the lecturer interrupted the flow to present a self-contained lecture on a topic to be developed further in the subsequent lecture course. The note taker did not record dates of lectures, so it is likely that some lectures were missed in the sequence. The courses typically start with common material, then branch out into particular topics. Quillen seldom provided any references during lectures, and the lecture presentation seems simpler than some of the material in the papers.


**Commonly used notation**

- \( k \) a field, usually of characteristic zero, often the complex numbers
- \( A \) an associative unital algebra over \( k \), possibly noncommutative
- \( \mathbb{A} = A/k \) the algebra reduced by the subspace of multiples of the identity
- \( \Omega^n A = A \otimes (\mathbb{A} \otimes \ldots \otimes \mathbb{A}) \)
- \( \omega = a_0 da_1 \ldots da_n \) an element of \( \Omega^n A \)
\( \Omega A = \oplus_{n=0}^{\infty} \Omega^n A \) the universal algebra of abstract differential forms

e an idempotent in \( A \)

d the formal differential (on bar complex or tensor algebra)

\( b \) Hochschild differential

\( b', B \) differentials in the sense of Connes’s noncommutative differential geometry

\( \lambda \) a cyclic permutation operator

\( K \) the Karoubi operator

\( \circ \) the Fedosov product

\( G \) the Greens function of abstract Hodge theory

\( N \) averaging operator

\( P \) the projection in abstract Hodge theory

\( D \) an abstract Dirac operator

\( \nabla \) a connection

\( I \) an ideal in \( A \)

\( V \) vector space

\( M \) manifold

\( E \) vector bundle over manifold

\( \tau \) a trace

\( T(A) = \oplus_{n=0}^{\infty} A^{\otimes n} \) the universal tensor algebra over \( A \)
Topics in K Theory and Cyclic Cohomology

\[ HC^*(\mathbb{A}) = \{ \text{homomorphisms} \} \]
\[ \text{furry} = \text{furry} \]

Local theory of elliptic operators. Lie algebra cohomology of \( gl_n(\mathbb{A}) \)

\[ 0 \rightarrow I \rightarrow R \rightarrow A \rightarrow 0 \]

\[ \text{Theorem} \quad \text{If } R \text{ is a free algebra then we have exact sequences} \]

\[ \left( \mathcal{R} / (I, I^m) \right)^* \rightarrow \left( I^m / [I, I^m] \right)^* \rightarrow HC^{2m-1}(\mathbb{A}) \rightarrow 0 \]

Connes

\[ \big( \mathcal{R} / (I) \big)^* \rightarrow HC^0(\mathcal{R} / I^m) \rightarrow H^{2m}(\mathbb{A}) \rightarrow 0 \]

\[ \left[ I^m \mathcal{R} + [I, \mathcal{R}] \right] \]

J. Cuntz - gave proof of these exact sequences

Deal with the case of non-immersed extension.

\[ RA = \bigoplus_{n=0}^\infty A_n \]

\[ \left\{ \begin{array}{ll}
\text{unital theory} & J \in A \\
\text{non-unital theory} & \end{array} \right. \]
usual in homological algebra to take $I$-er. But the tools used are based on geometrical techniques in working with $\mathbf{A} = k \oplus \mathbf{A}$.

(category of nonunital algebras)

(category of unital algebras)

\[ \mathbf{A} \to \mathbf{A} \]

Nonunital category: $R\mathbf{A} = \mathbf{A} \otimes_k \mathbf{A}$

$0 \to \mathbf{A} \to R\mathbf{A} \to R\mathbf{A} \to 0$ universal extension of $\mathbf{A}$

$\mathbf{A} \to R\mathbf{A}$

$\mathbf{A}$!/ F! algebra homomorphisms which makes this diagram commute.

Cuntz proved this by cyclic formalism for the case $K = \mathbf{R} \otimes_k \mathbf{A}$.

$\mathbf{A}$ review of cyclic formalism: over field $k$

$A$ nonunital algebra, cochain of degree $n$ on $\mathbf{A}$ is a multilinear map

$\operatorname{Ext} \mathbf{A}(a_1, \ldots, a_n)$ on $\mathbf{A}$ with values in a vector space.
Key calculations:
\[
\begin{align*}
    b^2 &= (b')^2 = 0 \\
    (A - 1)N &= N(A - 1) = 0 \\
    b'(A - 1) &= (A - 1)b \\
    bN &= Nb
\end{align*}
\]

Double complex anticommutes. Rows are exact since the characteristic of \( k \) is zero.

Cochain complex with differential.

In summary, the cochain complex with differential.

E.g. \( \delta : A \to R \) is a linear map of 1-cochains

\[
\begin{align*}
    \delta(w) &= \delta(p^q) = \delta p - p \delta q \\
    w(a_1, a_2) &= p(a_1, a_2) + q(a_1, a_2) \\
    &= p(a_1, a_2) - p(a_1) q(a_2) \\
    \implies w &= 0 \implies \delta \text{ is an algebra homomorphism}
\end{align*}
\]

Bianchi identity:
\[
\delta w = \delta(\delta p + q^2) = \delta p - q \delta p
\]

\[
\delta w = b'w(a_1, a_2, a_3) = \delta p(a_1, a_2, a_3)
\]

\[
\text{i.e. } w(a_1, a_2, a_3) + \delta p(a_1, a_2, a_3)
\]

\[
= w(a_1, a_2) p(a_3) - p(a_1) w(a_2, a_3)
\]

Let \( \text{adj} p \) be the adjoint of \( p \) in \( \text{Hom}(A, R) \)

\[
\text{adj} p(b) = [b, a] = b a - a b \implies (\delta - \text{adj} p)(w) = 0
\]

where \( (, ) \) denotes the degree

\[
\delta w = \delta p - q \delta p = 0
\]

since \( \delta + \text{adj} p \) is a derivation.

\[
\delta(w^n) + p(w^n) - w^{n+1} = 0
\]

Bar construction \( B(A) \) differential graded co-algebra

\[
\text{Hom}(B(A), R)
\]
\[ p_2 w^n (a_1, ..., a_{2n+1}) - w^n (a_1, ..., a_{2n}) p(a_{2n+1}) \]
\[ = p(a_1) w(a_2, ..., a_{2n+1}) - w(a_1, a_2, ..., a_{2n}) p(a_{2n+1}) \]
\[ - w(a_1, a_2) w(a_3, ..., a_{2n}) p(a_{2n+1}) \]
\[ \text{Take } R = RA \text{ and } p \text{ the canonical inclusion } R \hookrightarrow RA. \text{ Consider the cochain} \]
\[ w = b' + b^2 \]
\[ w^n (a_1, ..., a_n) = w(a_1, a_2, ..., a_n) \]
\[ w^n (a_1, ..., a_{2n}) = p(a_0) w^n (a_1, ..., a_{2n}) \]

**Theorem:** A linear functional \( f \) on \( RA \) is a cocycle if and only if the associated cochain \( f \) is a cocycle
\[ \begin{cases} \quad b'_{2n} = \frac{1}{n} N f_{2n-1} \\ b'_{2n+1} = (1 - \Lambda) f_{2n+1} \end{cases} \]

**Proposition:** One has
\[ b'(w^n) = (1 - \Lambda)(pw^n)(a_1, ..., a_{2n}) \]
\[ = - [w^n (a_1, ..., a_{2n}) p(a_{2n+1})] \]
\[ b'(pw^n) - (1 + \Lambda) w^n (a_1, ..., a_{2n+1}) \]
\[ = [p(a_0) w^n (a_1, ..., a_{2n+1}) p(a_{2n+1})] \]

Yesterday we proved that
\[ (\delta + a d p) w^n = \delta w^n + p w^n - w^n p = 0 \]
i.e.
\[ b'(w^n)(a_1, ..., a_{2n+1}) = p(a_0) w^n (a_1, ..., a_{2n+1}) - w^n (a_1, ..., a_{2n}) p(a_{2n+1}) \]
\[ = [p(a_0) w^n (a_1, ..., a_{2n+1}) - p(a_{2n+1}) w^n (a_1, ..., a_{2n})] \]
\[ + \Lambda w^n (a_1, a_{2n+1}) \]

which proves the first identity.

**For the second formula**
\[ b'(w^n)(a_0, a_1, ..., a_{2n+1}) = p(a_0) w^n (a_1, ..., a_{2n+1}) \]
\[ - p(a_{2n+1}) w^n (a_1, ..., a_{2n+1}) \]
\[ + p(a_0) w^n (a_1, ..., a_{2n+1}) - p(a_{2n+1}) w^n (a_1, ..., a_{2n+1}) \]
\[(p(a, a_1) - p(a_0, a_1)) \omega^n(a_2, \ldots, a_{2n})
\]
\[\omega^n(a_0, a_1) \delta(a_2, \ldots, a_{2n})
\]
\[- p(a, a_1) \omega^n(a_0, a_1)
\]
\[\omega^n(a, a_1) \delta(a_2, \ldots, a_{2n})
\]

which gives us the second identity.

Remark: These identities hold for any linear map \( p : A \rightarrow R \) because each \( \omega \) induces

\[ p : A \rightarrow R
\]

\[R^A
\]

**Theorem.** Let \( R = RA, I = IA = \text{Ker}(RA \rightarrow A) \)

1) Let \( \tau \) be a linear functional on \( R \), let

\[ f_0 = \tau(\omega^n), f_{2n} = \tau(\omega^n)
\]

Then \( \tau \) is a base if and only if

a) \( b f_{2n} = (\lambda) f_{2n-1} \) \( n \geq 1 \)

b) \( f_{2n-1} = \frac{1}{n} f_{2n} \)

c) \( \lambda^2 f_{2n} = f_{2n} \)

2) Let \( \tau \) be a linear functional on \( I^n \)

Then \( \tau \) vanishes on \( [I, I^{m+1}] \)

(rem \( [I, I^{m-1}] \)) if and only if \( \tau \)

satisfies a), b) for \( n \geq m \) and

\[ \lambda^2 f_{2n} = f_{2n} \] \( n \geq m \)

(rem \( n \geq m \))

Hence

\[ f_{2n} = \tau(\omega^n)
\]

\[ f_{2n-1} = \tau(\omega^n)
\]

are defined for \( n \geq m \).

**Proof.**

a) Assume \( \tau \) a base. Apply \( \tau \) to the above identities. It will kill off the brackets so that
\[ b_1 \left( (w^m) \right) = \left( -1 \right)^m \left( w^m \right) \]

\[ b_2 \left( (w^m) \right) = \left( 0 \right)^m \left( w^m \right) \]

\[ \sum_{i=1}^{2n} \left( a_{i1} \rightarrow a_{i2} \right) = \sum (w(a_{i1}, a_{i2}), w(a_{i3}, a_{i4})) \]

\[ \sum_{i=1}^{2n} \left( a_{i1} \rightarrow a_{i2} \right) = \sum (w(a_{i1}, a_{i2}), w(a_{i3}, a_{i4})) \]

This is a true \[ \mathcal{A}_2 \]

\[ N_{f_{2n}} = \sum_{i=1}^{2n} \mathcal{A}_2 = f_{2n} \]

\[ N_{f_{2n}} = \sum_{i=1}^{2n} \mathcal{A}_2 = n (1+n) f_{2n} \]

(2) Reverse the argument. We obtain

\[ \sum (w(a_{i1}, a_{i2}), w(a_{i3}, a_{i4})) = 0 \]

\[ \sum (w(a_{i1}, a_{i2}), w(a_{i3}, a_{i4})) = 0 \]

Conclude \[ \text{I} (I^{\prime} F(a)) = \emptyset \]

i.e. \[ \text{I} (I^{\prime} F(a)) = \emptyset \]

But \[ \text{I} (I^{\prime} F(a)) \] generates \( R \). Hence \( \sum \) is a true.

// Similar to the proof of (1), especially in the unnumbed case.

It is hard to show that if \( \text{I} (I^{\prime} F(a)) = \emptyset \)

then \[ \sum (w(a_{i1}, a_{i2}), w(a_{i3}, a_{i4})) = 0 \]

Proof by identity

\[ b_1 \left( (w^m) \right) = \left( -1 \right)^m \left( w^m \right) \]

\[ b_2 \left( (w^m) \right) = \left( 0 \right)^m \left( w^m \right) \]

\[ b_1 \left( (w^m) \right) = \left( -1 \right)^m \left( w^m \right) \]

\[ b_2 \left( (w^m) \right) = \left( 0 \right)^m \left( w^m \right) \]

\[ b_1 \left( (w^m) \right) = \left( -1 \right)^m \left( w^m \right) \]

\[ b_2 \left( (w^m) \right) = \left( 0 \right)^m \left( w^m \right) \]

\[ b_1 \left( (w^m) \right) = \left( -1 \right)^m \left( w^m \right) \]

\[ b_2 \left( (w^m) \right) = \left( 0 \right)^m \left( w^m \right) \]

\[ b_1 \left( (w^m) \right) = \left( -1 \right)^m \left( w^m \right) \]

\[ b_2 \left( (w^m) \right) = \left( 0 \right)^m \left( w^m \right) \]

\[ b_1 \left( (w^m) \right) = \left( -1 \right)^m \left( w^m \right) \]

\[ b_2 \left( (w^m) \right) = \left( 0 \right)^m \left( w^m \right) \]
$H'(A, M) = \{ \text{Derivations } \mathcal{D} : A \to M \}/\text{Inner derivations} \in \text{End}_R A$

$H^1(A, M) = \{ \text{Isomorphism classes of } \mathcal{D} \text{ on } C^0(A, M) \}

\begin{align*}
\forall \mathcal{M} = k^* \text{ with multiplication by elements of } A, \\
C^\bullet(A, k^*) &= \text{complex of } k^\text{-valued cochains with the differential } (-b), \text{ exact at 0 of degree zero} \\
\end{align*}

$\forall \mathcal{M} = A^*$ dual $\{ f(\mathbf{a}_i) = f(\mathbf{a}_i) \}$

$C^n = (k \otimes_{k^*} \mathcal{M})^*$

\begin{align*}
&\mathbf{f}(\mathbf{a}_1, \ldots, \mathbf{a}_n) \\
&= \mathbf{f}(\mathbf{a}_1, \ldots, \mathbf{a}_n) + \sum_{i} (-1)^{i-1} \mathbf{f}(\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_i, \ldots, \mathbf{a}_n)
\end{align*}

\begin{align*}
\varphi(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \varphi(\mathbf{a}_1, \ldots, \mathbf{a}_n) \prod_{1 \leq i < j \leq n} \mathbf{a}_j - \prod_{1 \leq i < j \leq n} \mathbf{a}_j
\end{align*}

\begin{align*}
\varphi(\mathbf{a}_0, \ldots, \mathbf{a}_n) &= \{ \varphi(\mathbf{a}_0, \ldots, \mathbf{a}_n) \text{ if } \mathbf{a}_0 \in \mathcal{M} \\
&= \text{complex of cochains of degree } \geq 1 \text{ with differential } b
\end{align*}

$\tilde{\mathcal{M}} = k \otimes_{k^*} \mathcal{M}$

$\mathcal{O} \to \mathcal{A} \to \tilde{\mathcal{M}} \to k \to 0$

Dualizing gives $0 \to \mathcal{A}^* \to \tilde{\mathcal{M}}^* \to k \to 0$

Claim that $C^\bullet(A, \mathcal{M})$ is the complex.

Define $C^\bullet(A, \mathcal{M}) \to C^\bullet(A, \tilde{\mathcal{M}}^*)$

$\forall (\mathbf{a}_0, \ldots, \mathbf{a}_n) \\
\varphi(\mathbf{a}_0, \ldots, \mathbf{a}_n) = \{ \varphi(\mathbf{a}_0, \ldots, \mathbf{a}_n) \text{ if } \mathbf{a}_0 \in \mathcal{M} \\
\mathbf{a}_0 = 1$
\[(b\tilde{Y}|(\tilde{\alpha}_0, \ldots, \tilde{\alpha}_n)) = \tilde{Y}(\tilde{\alpha}_0, \ldots, \tilde{\alpha}_n) + \sum_{i=1}^{n} (-1)^{i} \tilde{Y}(\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{i-1}, \tilde{\alpha}_{i+1}, \ldots, \tilde{\alpha}_n)\]

If \(\tilde{\alpha}_0\) is restricted to \(\alpha\), we denote by \(\tilde{Y}\).
If \(\tilde{\alpha}_0 = 1\), then we have \((1-\lambda)\tilde{Y}\).

By diagram chasing one gets the following from (8), the Leray exact sequence:

\[Hc^0(A) = Hc^0(A, A) = \{ (\alpha, 0, 0, A) \}\]

\[\begin{array}{c}
  0 \\
  \rightarrow \end{array} Hc^1(A) \rightarrow H^1(A, A) \rightarrow Hc^0(A) \]

\[\begin{array}{c}
  S \\
  \rightarrow \end{array} Hc^2(A) \rightarrow Hc^2(A, A) \rightarrow Hc^1(A) \]

\[\begin{array}{c}
  S \\
  \rightarrow \end{array} Hc^3(A) \rightarrow Hc^3(A, A) \rightarrow Hc^2(A) \]

Theorem: Take \(R = RA\), \(L = IA\) the kernel of the canonical map of \(R\) onto \(A\).

\[(R/[R,R])^* \rightarrow (I^m/[I,I^m])^* \]

Then the sequence here is exact.

Proof: Let us recall that \(\tilde{C} \in (I^m/[I,I^m])^*\) is equivalent to a couple \(\tilde{f}_{2m} = (\tilde{f})\) for \(m \geq m\), \(\tilde{C} \in (I^m/[I,I^m])^*\) same as above but also \(\lambda \tilde{f}_{2m} = \tilde{f}_{2m}\).

Formulas:
\[\tilde{f}_{2m} = \tilde{C}(\omega)^n/n!\]
\[\tilde{f}_{2n} = \tilde{C}(\omega)\]

Conclusion: \(\tilde{f}_{2m} = (\lambda)\tilde{f}_{2m}\) above; \(\tilde{f}_{2m} = N\tilde{f}_{2m} \tilde{\alpha}\) above.

\[\begin{array}{c}
  b' \tilde{f}_{2m} \rightarrow T \tilde{f}_{2m} \rightarrow 1 \rightarrow \tilde{f}_{2m} \rightarrow \tilde{f}_{2m} \rightarrow 0 \rightarrow Hc(A) \]

Connes had constructed the map \(\tilde{C} \in (I^m/[I,I^m])^*\) to the \(N_{1/2m}\) class in \(Hc^2(A)\) Let form be the couple given by \(\tilde{C}\).
\[ f_{2m} \xrightarrow{N} Nf_{2m} \xrightarrow{\iota} 0 \]

\[ Nf_{2m} \text{ is a cyclic cocycle.} \]

Assume \( \iota \in (I^m/I, I^{m-1})^* \) comes from a cocycle on \( R \). Then \( f_{2m} \) extends to a cocycle \( f_{2m} \)

By diagram chasing, we can modify the cocycle \( f_{2m} \) and suppose \( \iota = f_{2m} = 0 \).

Then \( f_{2m} \) is a cyclic cocycle (degree \( 2m-2 \)) with \( b_{2m-1} = Nf_{2m} \).

The cyclic cohomology class of \( Nf_{2m} \) is \( 0 \).

Similarly, if \( \iota \in (I^m/I, I^{m-1})^* \) has its cohomology class \( 0 \), then \( \iota \) extends.

\[ \text{Subsequence of the Cone Map: Let } \iota \text{ be a cyclic cocycle } \]

\[ f_{2m} \xrightarrow{\iota} N \xrightarrow{f_{2m}} Q_{2m} \xrightarrow{1} 0 \]

... construct a cocycle \( \gamma(f) \) by diagram chase.

To check that it comes from a true map, one needs to check the symmetry conditions. Need \( N^* f_{2m} \).

One can do so by taking \( f_{2m} = \frac{1}{2m} Q_{2m} \).

\[ \text{Theorem (Even case)} \]

Let \( R = RA, I = IA = \text{Ker}(RA \rightarrow A) \)

\[ (SL^R/[E, SL^R] + I^m SL^R)^* \rightarrow (R/E, R^* + I^m)^* \]

\[ H^{2m}(R) \xrightarrow{\delta} 0 \]

\( SL^R \) is a bimodule of noncommutative differential over \( R \).

There is a canonical derivation \( d: R \rightarrow SL^R \)

which is a universal derivation: given \( D: R \rightarrow M \), \( D \) derivation, there is a unique
bimodule map \[ SL^iR \rightarrow M \] such that \( uv = 0 \).

Fact 1. One has an exact sequence of \( R \) bimodules
\[ \overset{\alpha}{\longrightarrow} R \otimes_R R \otimes_R R \otimes_R R \rightarrow SL^iR \rightarrow 0 \]

2) There is an exact sequence - standard normalised resolution
\[ \overset{\beta}{\longrightarrow} R \otimes_R R \otimes_R R \otimes_R R \rightarrow \overset{\gamma}{\longrightarrow} R \rightarrow 0 \]

One has exact sequence
\[ 0 \rightarrow SL^iR \overset{\delta}{\longrightarrow} R \otimes_R R \rightarrow 0 \]

\[ i(\delta v) = b'(1 \otimes v \otimes 1) = v \delta a, \delta = 0 \]

3) \[ \overset{\alpha}{\longrightarrow} R \otimes_R R \otimes_R R \rightarrow \overset{\beta}{\longrightarrow} R \rightarrow 0 \]

4) Equivalence between:
   i) times \( T \) on \( SL^iR \) on \( R \) bimodule
   ii) \( \eta \) -cocycle in complex \( C(R, R^*) \)
   iii) pair \( v_2 \in (R^*)^i, \eta \in R^* \) satisfying
   \[ b' \eta_2 = (i - 1) v_2 \]
   \[ b \eta_2 = 0 \]

5) If \( R = RA \), then \( SL^iR \cong R \otimes_R R \otimes_R R \)

6) \( SL^iR/[R, SL^iR] \cong R \otimes_R RA \)

(Obvious by universal property of \( SL^iR \).)
Recall \( R \leftarrow \oplus_{p \geq 1} A^p \) 

\[
\begin{align*}
\text{From (5) onwards } \quad R &= RA \\
\text{components } \quad w^n, \quad p = 2n, \quad \mu^n, \quad p = 2n+1.
\end{align*}
\]

\[
S^1/\langle R, R \rangle \leftarrow \oplus_{p \geq 1} A^p
\]

with components \[
\begin{align*}
w^{n-1}d \phi \quad p = 2n-1 \\
-\mu^n d \phi \quad p = 2n.
\end{align*}
\]

\[
(w^{n-1}d \phi)(a_1, a_2, \ldots, a_{2n-1}) = \\
\quad = w^{n-1}(a_1, a_2, \ldots, a_{2n-1})d \phi(a_{2n-1}) - \\
\quad - \mu^n(a_1, a_2, \ldots, a_{2n-1})d \phi(a_{2n-1}).
\]

\[
= \mu^n(w^{n-1}(a_1, a_2, \ldots, a_{2n-1})d \phi(a_{2n-1}).
\]

\( \therefore \) Equivalence between bases \( \iota' \) on \( S^1/\langle R \rangle \) and inhomogeneous cochains \( g = \{g_0 : \pi^1\} \) given by \[
\begin{align*}
g_{2n-1} &= \iota'(w^{n-1}d \phi) \\
g_{2n} &= \iota'(-\mu^n d \phi)
\end{align*}
\]

**General definition (any \( R \)):** If \( \iota' \) is a base on \( S^1/\langle R \rangle \), then \( \iota'd \) is a base on \( R \)

Such a base will be called null cobordant. A base \( \iota \) on \( R \) is null cobordant if and only if \( \iota \in \text{Im } B \) in the exact sequence 

\[
\begin{align*}
\iota \in H^0(R)
\end{align*}
\]
Theorem (cont) \( f = \text{coboundary of } h \):

\[
\begin{align*}
\alpha_n & = T'd(\omega^m) \\
\alpha_{2n} & = b'h_{2n_1} + (A1)h_{2n} \\
\alpha_{2n+1} & = -b'h_{2n+1} + \frac{1}{h_{2n+1}} Nh_{2n+1}
\end{align*}
\]

\[N \rightarrow f_{2n+1} \xrightarrow{\lambda^{-1}} f_{2n} \xrightarrow{\lambda^{-1}} f_{2n-1} \xrightarrow{\lambda^{-1}} N\]

Lemma: One has

\[h_{2n} = \sum_{i=0}^{n} \lambda^{2n-2i} g_{2n-1} \]

where

\[g_{2n} = \sum_{i=0}^{n} \lambda^{2n-2i} (g_i + f_{2n})\]

is a cochain depending only on \( g_{2n} \).

Recall: Definition: A brace \( T \) on \( R \) is null homotopic/cobordant if it is of the form

\[T = T' \]

for some brace \( T' \) on \( R \) as a \( R \)-bimodule.

Theorem? If \( T = T'd \) then the couple of \( T \) is the coboundary of the \( g \)-cochain associated with \( T' \).
\[ \begin{align*}
\text{(i)} \quad f_{n+1}(d'(\omega)) &= b'h_{2n-1} + (\Delta - 1)h_{2n} \\
\text{(ii)} \quad f_{n+1}(d'p'\omega) &= (b'h_{2n} + Nh_{2n+1})/n+1
\end{align*} \]

\[ dw = \delta dp + ddp + ddp \]
\[ w = \delta dp \in H \Rightarrow w \]
\[ dw = \sum_i w_i(\omega) w_{n-i} \]
\[ = \sum_i w_i(\omega) (\delta dp)(dp) w_{n-i} \]
\[ = (\delta dp) \sum_i w_i(\omega) w_{n-i} \]
\[ = (\delta dp)(\mu_{2n-1}) \]

\[ \delta m_{2n-1} = -p\mu_{2n-1} - m_{2n-1} + dw^n \]

\[ b'\mu_{2n-1} (\alpha_{12}, \alpha_{2n}) = \mu_{2n-1}(\alpha_{12}, \alpha_{2n}) \]
\[ + \sum_i w_i(\alpha_{12}, \alpha_{2n}) dp_{\alpha_{12}, \alpha_{2n}} \]
\[ + dw^n(\alpha_{12}, \alpha_{2n}) \]

Apply (i) to this expression. Now (i) commutes with (ii).
Given a time $t$ in $R$, a mapping $f$ in $\text{cycl}$. A cycle $f$ is such that $f_p = 0$ for $p \geq 2m + 1$:

$$f_{2m+1} = t(\rho^m)$$

Prove:

$$Z^{m-1}(A) \rightarrow H^{m-1}(A) \rightarrow H^{m}(A) \rightarrow 0$$

Cyclic cochain, cyclic complex.
\[
\mathcal{C}(2^{m-1}(A)) \xrightarrow{\delta} \mathcal{C}(2^m(A)) \to \mathcal{H}(2^m(A)) \to 0
\]

\[
(R/R')^* \to (R/R')^* \to \text{coh}(K)
\]

Define \( \nu \): given a cycle (2m) couple
\[
\nu_{2m} = 0 \quad \nu_{2m+1} = 0
\]

Thus there is a unique true \( \nu \) on \( R \) such that
\[
0 \quad \nu_{2m} \quad \text{and all other}
\]

\[
\nu_{2m+1} \quad \text{completely zero}
\]

\[
\nu \circ (w^{m-1}) = 0 \quad \nu_{2m} = 0 \quad \nu_{2m+1} \neq 0
\]

\[
\nu \circ (w^{m-1}, p^m) = \begin{cases} 0 & h \neq m \\ \nu_{2m+1} & h = m \end{cases}
\]

Let \( V(\nu_{2m+1}) = \nu \) (Note that \( \nu \circ (I^{m-1}) = 0 \))

To define \( \nu \), given a cycle (2m-1) cocycle \( \nu_{2m-1} \).

Let \( I \) be the cone on \( R/R' \) with
\[
\nu_{2m-1} = \nu(\omega^{m-1}d\nu)
\]

\[
\nu_m = \nu(-\omega^{m-1}d\nu) = \begin{cases} 0 & h \neq m \\ \nu_{2m} & h = m \end{cases}
\]

Note that \( \nu(I^{m-1}) = 0 \) since \( I^{m-1} \)

is the image of \( \omega^{m-1}d\nu \) and \( -\omega^{m-1}d\nu \)

Define \( \mu(\nu_{2m}) = \nu \)

Need to check that the square commutes.

This comes down to checking that
\[
+\nu \quad \nu_{2m} = \nu \quad \nu_{2m}
\]

For \( \nu \) : I'd

The (wedge is the boundary of the h-chain of \( I \). Recall the formulae

\[
\begin{align*}
\tilde{h}_{2m} &= \tilde{Z}(\nu(\omega^{m-1}d\nu)) \\
\tilde{h}_m &= \tilde{Z}(-\tilde{P}^{m-1}w^md\nu_{2m})
\end{align*}
\]

\[
\begin{align*}
\tilde{h}_{2m} &= \tilde{Z}(\nu(\omega^{m-1}d\nu)) \\
\tilde{h}_m &= \tilde{Z}(-\tilde{P}^{m-1}w^md\nu_{2m})
\end{align*}
\]

Lemmas: \( h_{2m-1} = \sum_i \lambda_{2m-1} \tilde{h}_{2m-1} \)

\[
\begin{align*}
\tilde{h}_{2m} &= \tilde{Z}(\nu(\omega^{m-1}d\nu_{2m-1} + \text{non-deg. pole}) \\
\tilde{h}_m &= \tilde{Z}(-\tilde{P}^{m-1}w^{m-1}d\nu_{2m-1})
\end{align*}
\]
Now \( g_{2m} = -\frac{1}{2m} \gamma_{2m} \) and all other \( g_p \) are zero.

Then only \( h_{2m} = \frac{1}{2m} \gamma_{2m} \).

We know that, by diagram chasing that there is a cochain \( h_i \) such that

\[ f \circ h_i = f_i \] for any \( i \)

The problem is to show that \( h_i \) can be chosen to be the \( i \)th cochain of \( \gamma_i \). The induction hypothesis states the smallest \( p \) such that \( f_p \neq 0 \).

Suppose \( p = 2m - 1 \)

\[ h_{2m} \rightarrow f_{2m-1} \rightarrow f_{2m-2} \rightarrow \cdots \rightarrow f_1 \rightarrow f_0 \]

To prove that \( w \) is surjective, we start with \( \gamma \) on the cycle of \( f \) and return on \( K \). Let \( f \) be the cycle of \( f \).

Diagram chasing for \( h_{2m} \) yields \( h_{2m} \) to be a cyclic cochain.
Suppose \( f = 2n \)

Find \( g \)

\[ h_n \rightarrow f_n \quad g_n \rightarrow 0 \]

\[ g_n \neq 0 \text{ for } n \text{ even} \]

\[ h_n \rightarrow f_n \quad g_n \rightarrow 0 \]

\[ h_{2n} = 0 \quad g_{2n} 

It suffices to show that \( g_n \) can be chosen so that when we apply \((A) \sum_{i=1}^{2n} a_i f_i = f_n \)

\[ \sum_{i=1}^{2n} a_i = 0 \]

But this follows from \( A f_n = f_n \)

Thus \( f_n = 0 \)

Universal means that any commutative \( A \rightarrow B \)

degree 2 subgroup \( S \)

\[ d_1 \exists \text{ such that } d_1 \text{ uniquely to a hom } S^A \rightarrow S \]

\[ d_2 \exists \text{ dependency on } \alpha \text{ in } \alpha \]

\[ \text{Prop. } \alpha \in \mathbb{A} \rightarrow \mathbb{S} \]

\[ (a_0, a_1, \ldots, a_n) 

\[ \alpha \rightarrow d_0 \cdot \ldots \cdot d_n 

Gives a vector space isomorphism for all \( \alpha \)

\[ \text{Prop. Surjectivity: let } L \subset \mathbb{S} \text{ be the subspace spanned by } a_0, a_1, \ldots, a_n \text{ where the range of } \alpha 

\[ \mathbb{A} = \mathbb{S} \text{ for all of } \mathbb{S} \text{ under } \alpha \]

\[ \text{Since } \mathbb{A} \text{ is generated by the elements } a_i \text{ for } a_i \in \mathbb{A} \text{ we note } \]

\[ a \in (a_0, a_1, \ldots, a_n) \in L \]

\[ d_0 a_0 + a_1 d_1 = 0 \]

To prove injectivity, put \( S^A = \mathbb{A} 

\[ d(a_0, a_1, \ldots, a_n) = (1, a_0, a_1, \ldots, a_n) \]
This defines a complex $(\Omega, d)$. Use the fact that the space $\text{Hom}(L, M)$ of $M$-algebra into the differential $d(a) = a_{n+1} - (-1)^{n} a_{n}$.

We have a homomorphism $A \rightarrow \text{Hom}(L, M)$

$\alpha \rightarrow \lambda$ (left multiplication by $\lambda$)

$d\alpha \rightarrow [d, \lambda] = d\alpha - \lambda d\alpha$

This induces a homomorphism of $D$-algebras

$\Omega \rightarrow \text{Hom}(L, M)$

This makes $\Omega$ into a left $D$-module over $\Omega A$

$\alpha_{0}d\alpha_{1} \ldots d\alpha_{n} \rightarrow \alpha_{0}[d, \alpha_{1}] \ldots [d, \alpha_{n}]$

We have an involution $\tau : L \rightarrow L$

$(\alpha_{0}[d, \alpha_{1}] \ldots [d, \alpha_{n}]) = \tau$

$[d, \alpha_{1}] \ldots [d, \alpha_{n}] = \lambda_{n}, \lambda_{n+1}, \ldots$

$a(\alpha, 1) = d\alpha_{n} = (\lambda_{n+1})$

$\alpha_{n}d\alpha_{1} \ldots d\alpha_{n-1} = \alpha_{n}d\alpha_{1} \ldots d\alpha_{n-1}$

This produces a map $\lambda_{n} \rightarrow \lambda_{n} = \text{inv}$.

Thus there are two canonical homomorphisms

$A \rightarrow \Omega A$

which have a universal property.

Consequences

1) $A \rtimes QA \phi : \phi \rightarrow \phi^{*}$ injective

2) There is an automorphism of order $2$ on $QA$

such that $\phi^{2} = 1, \phi \phi = 1$

$\Omega A$ is a superalgebra

$\Omega A = (QA, \Omega QA)$

$\phi^{2} = 1, \phi \phi = 1$

odd (even) decomposition

Put $a = \text{even (odd) component of } a$

$a^{2} = (\text{even (odd) components of } a^{2})$

$a = \text{even (odd) components of } a$

\[ a = \frac{a + a_{2}}{2} \]

\[ a = \frac{ia - ia_{2}}{2} \]

End: $A = k + kE, \quad e^{2} = e$

Work out $\Omega A$

The Cuntz Algebra $QA = A \rtimes A$ is the free product of $A$ with itself.
Quillen

\[
\begin{align*}
(q_1 q_2)^+ &= q_1^+ q_2^+ + q_1 q_2^- \\
(q_1 q_2)^- &= q_1^- q_2^- + q_1^- q_2^+ \\
\end{align*}
\]

Verify:

1) The linear maps \( a \mapsto a^+ \), \( a \mapsto a^- \) from \( A \) to \( Q \) are unital pairs, satisfying the above relations.

2) \( QA \) is the subalgebra generated by \( A \) in the ring such that

\[ \text{Hom}_A(\mathbb{R}_+ t) \cong \text{Hom}_Q(\mathbb{R}_+ t) \]

\[ 1 \in A \]

\[ 1 = \frac{1}{t} (1 - t) = 0 \]

\[ 1^t = 1 \]

Define map \( \theta : A \to QA \)

\[ (a_0, a_n) \mapsto a_0^+ a_1^- \cdots a_n^- \]

Limit:

\[ \lim_{t \to 0} \theta(a) = a \]

Prop. 1: The sum of three maps is an isomorphism:

\[ \text{Hom}_A(\mathbb{R}_+ t) \cong QA \]

Prop. 2: The map is a vector space isomorphism:

\[ QA \cong \mathbb{R}A \]

1. \( a \in A \) vector space isomorphism

\[ QA \cong \mathbb{R}A \]

2. With respect to this isomorphism, the algebra structure on \( QA \) corresponds to the following product on \( \mathbb{R}A \):

\[ a \cdot q_1^+ q_2^- = a_0^+ a_1^- \cdots a_n^- \]

Proof of \( \text{Surjectivity} \):

Let \( L \) be the span of \( a_0^+ a_1^- \cdots a_n^- \). Show that \( L \) is a left ideal. Use the fact that \( L \) is generated by \( a_0^+ a_1^- \cdots a_n^- \) where \( a_i \in \mathbb{R}A \) for \( i = 0, \ldots, n \).

\[ a_0^+ a_1^- \cdots a_n^- + (a_0^+ a_1^- \cdots a_n^-) = (a_0^+ a_1^- \cdots a_n^-) + (a_0^+ a_1^- \cdots a_n^-) = 0 \]

\[ a^+ \in \mathbb{R}A \]

Similarly, we can show for \( a^- \), give similar argument. Note that \( L \)

\[ L \subset QA \]

\[ \text{Invertibility: We define an action of QA on \( SL_n \) (i.e., a left module structure), and we use this action on the element \( L \in \mathbb{R}A \) to obtain a map \( QA \to SL_n \), which is unique to the map \( L \to QA \).}

Let us define the map, using the universal property of \( QA \), so that homomorphism,

\[ QA \to \text{Hom}(\mathbb{R}A, SL_n) \]

\[ a \mapsto (a_0^+ a_1^- \cdots a_n^-) \mapsto (1-d) a (1-d) \]

\[ (a_0^+ a_1^- \cdots a_n^-) = a + da - da d \]

\[ a : b \mapsto ab da \]

\[ (1-d) a (1-d) = a - da - da d \]

\[ a = a - da d \]

What happens to \( a^+ q_1^- \cdots a_n^- \)?

\[ \eta \in (a_0^+ a_1^- \cdots a_n^-) \mapsto \eta = (a_0^- a_1^+ \cdots a_n^+) \]
J-adic filtration $\mathfrak{a}^n = \mathfrak{a} \cap \mathfrak{a}^n \cap \mathfrak{a}^{n-1}$ is commutatively isomorphic to $\mathfrak{a}^n$ as graded $A$-algebras.

Let $J = \mathfrak{a}$. Then $A$ is a Noetherian graded $A$-algebra.

Complements: $A = \mathfrak{a} / \mathfrak{a}^2 \times \mathfrak{a} / \mathfrak{a}^2 \times \cdots$.

This shows (2).

If we denote a grading by the usual notation $A = A_0 \oplus A_1 \oplus \cdots$, then $\text{Hom}_A(A_0, A) = R(A)$, where $R(A) = R_{\mathfrak{a}}(A)$.

For $\mathfrak{a} \neq \mathfrak{a}^2$, let $f(y)$ be the monic polynomial in $A$ such that $f(y) = 0$.

Let $T(\gamma) = T \circ f(y)$.

Now consider the map $A \rightarrow H$, where $H = A \rightarrow \gamma$ and $\gamma$ is a graded $A$-module.

This extends to a homomorphism $A^1 \rightarrow A^1$.

Exercises:

1. Show that $\gamma(A) = 0$.

2. Show that $\gamma(A)$ is a graded $A$-module.

3. Show that $\gamma(A)$ is a graded $A$-module.

4. Show that $\gamma(A)$ is a graded $A$-module.

5. Show that $\gamma(A)$ is a graded $A$-module.

6. Show that $\gamma(A)$ is a graded $A$-module.

7. Show that $\gamma(A)$ is a graded $A$-module.

8. Show that $\gamma(A)$ is a graded $A$-module.

9. Show that $\gamma(A)$ is a graded $A$-module.

10. Show that $\gamma(A)$ is a graded $A$-module.
well defined because $\mathcal{QA} \simeq SLA$. 
How can we do this? 
$\mathcal{QA} \simeq SLA$ 

$\mathcal{QA}$ is graded, $\mathcal{LH}$ is graded algebra, $\mathcal{QA}$ has the 
underlying vector space $V$. We take

$\mathcal{QA} \rightarrow \text{Hom}_k(\mathcal{LH}, \mathcal{LH})$

$(a \mapsto (a \cdot \xi \cdot a^{-1}) = a \cdot \xi \cdot a^{-1}$

However, the filtration $\mathcal{J}$ of $\mathcal{QA}$ has a

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3. The important identity

Even means that \( (F \bar{F}) = \bar{Z}(Z) \)

Odd means that \( (F \bar{F}) = -\bar{Z}(Z) \).

Obviously an even, true. To verify the symplectic identity, we suppose \( y \) are of the same parity.

\[
\tau (\bar{X} \bar{Y}) = \tau (\bar{F} \bar{Y}) = \tau (Y \bar{F})
\]

\[
= \tau (\bar{F} \bar{F} \bar{Y})
\]

\[
= (-1)^{\bar{Y} \bar{Y}} \tau (\bar{Y})
\]

\[
= (-1)^{\bar{Y} \bar{Y}} \tau (Y)
\]

\[
= (-1)^{\bar{Y} \bar{Y}} \tau (Y)
\]

Proposition. Let \( \tau \) be an even symplectic on \( QA \).

Then put \( \tau (\bar{X} \bar{Y}) = \tau (\bar{X} \bar{Z}) \).

\[
\tau (\bar{X} \bar{Y}) = \tau (\bar{Y} \bar{Z})
\]

\[
\tau (\bar{X} \bar{Y}) = \tau (\bar{Y} \bar{Z})
\]

1) \( b \bar{s} = (-1)^{\bar{y}} \bar{s} \bar{s}
\]

2) \( b \bar{k} = \bar{k} \bar{k} = -k \bar{k} = \bar{s} \bar{k} = k s = s \bar{b} \)

Proof: by calculation. Let \( b \) be an even.

Let \( b \) be even.

Let \( b \) be even.

Kawamata's Operator \( K \) on \( \mathcal{M} = A \otimes \mathcal{M} \rightarrow \mathcal{M}^\circ \).

\[ K(a_0, a_1, \ldots, a_n) = (-1)^n (a_0, a_1, \ldots, a_n) \]
\[ Q = Q \cdot A = A \cdot A^{-\cdot c} \quad \text{for } n \in n' \]

makes \( Q \cdot A \) a superalgebra.

\[ \begin{align*}
\lambda a &= a^+ + a^- \\
\lambda a &= a^+ + a^-
\end{align*} \]

\[ Q \cdot A = \bigoplus_{n \in n'} (A \otimes A_{\odot n}) \]

\[ a_+ a^- - a^- a_+ \equiv (a_a, a_{a_1}, a_{a_2}, \ldots) \]

\[ f_n \in (A \otimes A_{\odot n})^* \]

normalized, threshold conditions.

\[ J = \{ a^\cdot a \} \]

\[ J^m \approx \{ a^\cdot a \}_{n \in n_m} \]

\[ f_n(a_a, a_{a_1}, a_{a_2}, \ldots) = \lambda (a_a, a_{a_1}, a_{a_2}, \ldots) \]

\[ \text{Proposition: } f \text{ vanishes on } [Q, J^m] \text{ if and only if} \]

\[ \begin{align*}
\lambda f_n &= \sum_{m=1}^{n_m} \left( a^\cdot a \right)_{m=1}^{n_m} \\
\lambda f_n &= \sum_{m=1}^{n_m} \left( a^\cdot a \right)_{m=1}^{n_m}
\end{align*} \]

\[ \begin{align*}
\lambda f_n &= \sum_{m=1}^{n_m} \left( a^\cdot a \right)_{m=1}^{n_m} \\
\lambda f_n &= \sum_{m=1}^{n_m} \left( a^\cdot a \right)_{m=1}^{n_m}
\end{align*} \]
if and only if
\[ b_{n} = \frac{2}{n} f_{n} \quad n \neq m \]
\[ k_{n} = f_{n} \quad n = m \]

The non-unital version of these equations underlies the splitting of \( f_{n} \)
\[ \tau_{n} \to \tau_{n} \to \tau_{n} \quad n \neq m \]
\[ \tau_{n} \to \tau_{n} \quad n = m \]

and the above equations become
\[ \begin{align*}
  b'_{n} &= (1 - \lambda) f_{n} \quad n \neq m \\
  b'_{m} &= \frac{2}{m} N \epsilon_{m} \\
  \lambda b_{m} &= \epsilon_{m}
\end{align*} \]

(identities)
\[ (b_{n} - (1 + \lambda) s f_{n+1})(a_{0}, \ldots, a_{m}) = \tau \left( a_{0}, \ldots, a_{n}, a_{m+1} \right) \]
\[ (1 - \lambda) f_{n} \in \mathfrak{a}_{0}, \ldots, a_{m} \]}

Proof of the Proposition: Assume that \( E \) is a superbase.
\[ KE = 0 \]
\[ \Rightarrow KE = f_{n} \] (orthogonal)
\[ s_{n}: s_{0} f_{n} = k s_{n} f_{n} = s_{n} A f_{n} = \lambda s_{n} f_{n} \]
\[ (1 + \lambda) s_{n} f_{n} = 2 s_{n} f_{n} = \frac{2}{n} N s_{n} f_{n} \]

Conversely, it is basically the same.

Verification of the identities:

\[ \text{Hom}(B(A), B) \]
\[ \text{B}(A) \text{ has constant } = \text{PGL} \text{ independent with different } b'. \]

\[ \text{Hom}(B(A), B) \]
\[ \text{all homogeneous elements } B \text{ with degree } \geq 1 \]

\[ \text{Introduce a Z grading on Hom}(B(A), B): \]
\[ \text{Hom}(B(A), B) \]
\[ \text{Algebra structure defined via } \]
\[ fg = m_{n}(f_{n} g) \]
\[ = \sum_{n} m_{n}(f_{n}) \sum (a_{0}, a_{n}) \]
\[ = \sum_{n} m_{n}(f_{n}) \left( a_{0}, a_{n} \right) \]
\[ = \left( \begin{array}{c}
  a_{0} \\
  a_{n}
\end{array} \right) \]
\[ = \left( \begin{array}{c}
  a_{0} \\
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\[ = \left( \begin{array}{c}
  a_{0} \\
  a_{n}
\end{array} \right) \]

\[ \delta f = -(-1)^{n+1} f \]
This gives a graded algebra.

\[ q: A \to Q^+ \quad p(a) = a^+ \]

\[ q: A \to Q^- \quad q(a) = a^- \]

Even parity and \( q \) is even.

Claim \((bq + qq - q\delta p)(\alpha, \alpha) = 0\)

\[ (bq + qq - q\delta p)(\alpha, \alpha) = q^2(\alpha, \alpha) + q(\alpha, q(\alpha)), \]

\[ = q^2 - q^{-2}(\alpha, \alpha) = q\alpha - a_i \]

\[ (bq + qq - q\delta p)(\alpha, \alpha) = -q(\alpha, q(\alpha)) + q(q(\alpha), \alpha) \]

\[ = -q\alpha_i + q^2(\alpha, \alpha) - q^2 - q^{-2}(\alpha, \alpha) = 0 \]

Recall \( X \) is defined on \( A \otimes A^* \subset Q \)

\[ = q^{-2} - q^{2} a_i - q^{-2} a_i = 0 \]

\[ = q^{-2}(\delta \text{ad}_p)q^{-2} = 0 \]

\[ = (bq + qq - q\delta p)(q^n) = 0 \]

To prove that \( b(q^n) (a_0, \ldots, a_n) = (a_0 a_1 a_2 \ldots a_n) \)

\[ = (a_0 a_1 a_2 \ldots a_n) + (-1)^n (a_0 a_1 a_2 \ldots a_n) \]

\[ = (a_0 a_1 a_2 \ldots a_n) + (-1)^n a_0 a_1 a_2 \ldots a_n \]

\[ = (a_0 a_1 a_2 \ldots a_n) + (-1)^n a_0 a_1 a_2 \ldots a_n \]

\[ = (a_0 a_1 a_2 \ldots a_n) + (-1)^n a_0 a_1 a_2 \ldots a_n \]

More generally, \( b(q^n) (a_0, \ldots, a_n) = \)
Two cases to consider: Ungraded case: let \( \hat{\tau} \in \mathcal{E}(X) \) ungraded, operate here on \( Z(X) \).

1. Two cases to consider: Ungraded case: let \( \hat{\tau} \in \mathcal{E}(X) \) ungraded, operate here on \( Z(X) \).

Then \( \hat{\tau} \) is an even supertrace (strong)

\[
\tau(x) = \tau(x^t) \quad (x \in J^m)
\]

2. Two cases to consider: Ungraded case: let \( \hat{\tau} \in \mathcal{E}(X) \) ungraded, operate here on \( Z(X) \).

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\[
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\]
cycle: \[ f_n(a_0, ..., a_n) = \text{tr}(F a_0 a_1 ... a_n) \]
\[ = (\text{tr}(F)) \text{tr}(a_0 a_1 ... a_n) \]

In the graded case, we obtain an odd cycle
\[ f_n(a_0, ..., a_n) = \text{tr}(F a_0 a_1 ... a_n) \]
\[ = (\text{tr}(F)) \text{tr}(a_0 a_1 ... a_n) \]

These cycles are \( K \)-invariant
(normalized threshold cycles).

Couple condition
\[ b_f = \frac{1}{2} \text{tr}(F a_f) \quad \text{if } f \neq e \]

Goal: To obtain a homology formula. Say that if
\( F \) is defined, the corresponding cycles are cohomology. (Coupled change by a constant).
Consider a re-parameterized family
\( F = \{ F_\tau : \tau \in \mathbb{R} \} \)

Lemma: There is a family of unitary operators
\( U \in \mathcal{U}(\mathbb{C}; e^0) \) for which the co-parameterized family such that
\( U_0 = I \) and
\( U F u^{-1} \) is the constant unitary family with value \( F_0 \).
\[ U F_0 u^{-1} = F_0 \]
\[ U F_0 u^{-1} F_0 + F_0 u^{-1} = 0 \]

Proof. Solve the differential equation
\[ u = u (t F F) \]
\[ u_0 = I \]
\[ 0 = (F) F = F^* F \]
\[ \Rightarrow F^* F \text{ is skew adjoint} \]
\[ \Rightarrow U_t \text{ is a unitary operator} \]
\[ (U F u^{-1})^* = u F^* u^{-1} + u F u^{-1} \]
\[ = u (t F F) u^{-1} \text{ and } u F u^{-1} + u F u^{-1} \]
\[ = u (F - C F C^{-1} F) \quad = 0 \]
\[ \Rightarrow u F u^{-1} \text{ is constant} \]

Since \( F \) anticommutes with \( u^{-1} \), then
\[ F_0 = u F_0 \text{ anticommutes with } u^{-1} u^{-1} \]

Replace \( F \rightarrow \mathbb{L}(H) ) F = F_0 \) by
\[ \mathbb{L} = \mathbb{L}(H) \]
\[ u F u^{-1} = F_0 \text{ fixed} \]
\[ a \rightarrow u a u^{-1} \]

Let \( a \) be constant unitary and varying \( c \)
\[ a \rightarrow \mathbb{L}(H) (a \rightarrow u a^{-1}) \]
\[ E \rightarrow F_0 u^{-1} \]

But \( L \rightarrow \mathbb{L} u^{-1} \) (anticommutes with \( F_0 \))
\[ (a)^* = (u a^{-1})^* = E L, (a) \]
\[ (L a)^* = F_0^* L (a) \]
\[ = -E L, (a) \]
\[ (a^+)^* = \left[ L_a, a \right] \]
\[ (a^-)^* = \left[ L_a, a^+ \right] \]

This motivates the following situation: super

Let \( R L Q \) denote the free \( R \)-bimodule

with \( R L Q = Q \otimes R \)

\textbf{Lemma.} There is a unique (degree zero) homomorphism \( \varphi : Q \otimes R \to Q \) such that

\[ \varphi(1 \otimes a) = \frac{g}{a} \otimes a^+ \]

\[ \varphi(a \otimes 1) = \frac{g}{a} \otimes -a^- \]

\textbf{Proof:} Verify the relations. Check consistency: for \( a \)

\[ \varphi(a^+ a) = \varphi(a a^+) \]

Recall from \( (\text{BIL}, L) \), a differential super-bimodule.

We can consider \( (\text{BIL})_L, \text{QIL} \) as a differential

super-bimodule.

\[ a \mapsto a^+ \quad \text{odd} \]

\[ a \mapsto a^- \quad \text{even} \]

\[ [a, b] = ab - ba \quad \text{supercommutator} \]

\[ D_a = L_a + q \]

\[ D_{a^+} = L_a + q^+ \quad \text{supercommutator} \]

\[ \text{Put} \quad \mu_n = \sum_{i=0}^n q^i L^{n-i} q^{-i} \quad \text{odd} \]

\[ \mu_n(a_1, \ldots, a_n) = \sum_{i=0}^n (-1)^i q^{i-1} a_{n-i} L^{n-i} q^{-i} \quad \text{even} \]

\[ d[w_n] = \sum_{i=0}^{n-1} \omega(w_{n-i}) w^i \]

\[ = \sum_{i=0}^{n-1} \omega^i (\text{Braid}_p) w^i \]

\[ D[g_{n+1}] = \sum_{i=0}^n q^i (p_q) q_{n-i}^+ \]

\[ = \sum_{i=0}^n q^i (\text{Braid}_p) q_{n-i}^+ = (\text{Braid}_p) \mu_n \]

Recall \( \left\{ \sum (\text{Braid}_p) q = 0 \right\} \subset \left\{ g = 0 \right\} \quad \text{odd} \]

\[ \left\{ (L_g)^2 = g^2 \right\} \subset \left\{ g = 0 \right\} \quad \text{even} \]

\textbf{Proof.} Let \( \tau' \) be a supertrace on \( R L Q \)

\textbf{More generally a supertrace defined on} \( \sum \left( \text{Braid}_p \right) q \)

\[ \tau'(D(p q)) = (-b \tau((p q) \mu_n)) + \frac{2}{n+1} (b \tau((p q) \mu_n)) \]

\[ (\tau' D(p)) \text{supercommutes to the supertrace } \tau' \text{ on } Q, \text{ i.e. } \text{in the (ordinary) exterior algebra} \]

\[ \tau'(p \mu_n) \]
\[ \bar{e}^n = \sum_{i=0}^{n} e_i^i \bar{q}^i \quad \text{Apply } \bar{T} \]

\[ \bar{T} \left( \sum_{i=0}^{n} (-1)^i \bar{q}_i \right) \cdot (a_0, \ldots, a_n) = \]

\[ (-1)^{n+1} \bar{T} \left( \sum_{i=0}^{n} (-1)^{i} a_0^i q_0^i \right) \]

\[ (-1)^i \bar{T} \left( \sum_{i=0}^{n} a_i \bar{q}_i - a_i \right) \]

\[ (-1)^i \bar{T} \left( \sum_{i=0}^{n} a_0^i q_0^i \right) \]

\[ (-1)^i \bar{T} \left( \sum_{i=0}^{n} a_i \bar{q}_i - a_i \right) \]

\[ (-1)^i \bar{T} \left( \sum_{i=0}^{n} a_0^i q_0^i \right) \]

In the graded case we get \( f_n = 0 \) for all \( n \) because operators commute with \( \bar{T} \). Instead we consider the cochain

\[ f_n(a_0, \ldots, a_n) = \bar{T} \left( \sum_{i=0}^{n} (-1)^{i} a_0^i q_0^i \right) \]

Now \( f_n = 0 \) as the operator we are taking the trace of commutes with \( \bar{T} \); hence we get an odd cochain. \( (f_{2n})_{2n-1} \)

Prop: The above cochains are cocycles which are \( K \)-invariant.

Homology: Consider a homology of Fuchsian modules where the homomorphism \( A \to \mathbb{Z}(H) \) is fixed but \( F = (F_i) \) varies smoothly.

To show that \( \bar{T} = \bar{T}F \) is a coboundary.

Put \( L = \bar{T}F \) anti-commutes with \( F \).

Demoting \( D \bar{v} = \bar{v} + \bar{v} - \bar{v} \bar{v} \)

if \( \bar{v} = (c_i) \) is a family of operators.

\[ D(F) = \bar{T}F + \frac{1}{2} \bar{F}F + \frac{1}{2} \bar{F}F \]

\[ = \bar{T} + \frac{1}{2} \bar{F}F - \frac{1}{2} F = 0 \]

\[ Da = a + la - al \]

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$D(FanF) = F(Da)F = F(aF - F(a)F)$

$\therefore 2D(p(a)) = Lq - a\Lambda + (LEaF) + F(p(a))$

$D(p(a)) = Lq - a\Lambda + (LTaF) + F(p(a))$

$D(p(a)) = Lq - a\Lambda$

$Dq = Lq - a\Lambda$

$Lq = Lq - a\Lambda$

$2F(x, a, y, z) = 2F(Dx, a, y, z) = 6(F(Dx, a, y, z))$

$6(F(Dx, a, y, z)) = 6D(F(x, a, y, z))$

$6D(F(x, a, y, z)) = 6D(F(x, a, y, z))$

$\therefore 6D(F(x, a, y, z)) = 6D(F(x, a, y, z))$
$2$ is a map of free $R$-modules induced by

$$uv^* ightarrow (u^{*}v, u^*)$$

jective

Recall that the homology functor for a tower of $R$-

modules is given by

$$\mathcal{H}_n(T) = \sum_{i=0}^{n-1} \mathcal{H}_i(T)$$

Write $w = y^2$

$$\sum_{i=0}^{n} y^{2i} (y^c + y^d) y^{2n-i}$$

What is the meaning of $y = 2$

$$\sum_{i=0}^{n} y^{2i} (y^c + y^d) y^{2n-i}$$

Question: What about bases $T$ on $R$ of the

such that $K_T = -T$ exist

(same as admissible bases vanishing on $\mathbb{P}^1$)

To show such a $T$ is uninteresting from the

cyclic cohomology viewpoint.

Let $T$ be any trace on $\mathbb{R}$-Vector

$$\mathcal{H}_n(T) = \mathcal{H}_n(x, y) \mapsto 1+2 \mathcal{H}_n(x, y)$$

such that $T$ has a copy

of a copy of $\mathbb{R}$ vector space $\mathbb{R}$, namely that it comes

from a Hilbert space

Claim that $T$ has a copy, namely that it comes

from a Hilbert space.

But because $\mathcal{H}_n = 0$

(To see this: $\mathcal{H}_n = (\mathcal{H}_n(x), \mathcal{H}_n(y))$

in the commutative picture

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathcal{H}_n = (-\mathcal{H}_n, 0)$$

cohomology of $-\mathcal{H}_n$ consists of

$$(\mathcal{H}_n(x), \mathcal{H}_n(y)) \mapsto (\mathcal{H}_n(x), \mathcal{H}_n(y))$$

$R = \mathbb{Q}$

$X = \mathbb{C}$

$\mathcal{H}_n = \mathcal{H}_n(x, y)$

$D^+ = L - L - a_1$

$\mathcal{H}_n \rightarrow \mathcal{H}_n$

$D^- = \mathcal{H}_n(x, y)$

$\mathcal{H}_n = \mathcal{H}_n(x, y)$
\[
\text{consider } R\text{-bimodule map with } \delta_i = i' \delta_i.
\]

\[
R \otimes \mathcal{L} \xrightarrow{\delta_i} R \otimes \mathcal{L} \xrightarrow{\delta_i} R
\]

\[
R \otimes \mathcal{L} \xrightarrow{\delta_i} R \otimes \mathcal{L} \xrightarrow{\delta_i} R
\]

\[
\mu_{i,i} = \int\mathcal{L} \cdot \mathcal{L} \cdot \mathcal{L} \cdot \mathcal{L}
\]

\[
\mu_{i,i} = \int\mathcal{L} \cdot \mathcal{L} \cdot \mathcal{L} \cdot \mathcal{L}
\]

Conclusion that \( \mathcal{L} \) can be identified with the operation \( \delta \) on \( R \otimes \mathcal{L} \).

\[
\text{compose is that } \mathcal{L} \text{ is } \mathcal{L}-\text{module over } \mathcal{L}
\]

\[
\text{even question: Start with } \mathcal{L} \text{ by forming on } \otimes \mathcal{L} \text{. Then we get a module } \mathcal{L} \text{ on } \otimes \text{ whose couple } \mathcal{L} \text{ is the coboundary of }
\]

\[
\mathcal{L} \text{ by the homotopy formula for spaces in } \mathcal{L}.
\]

\[
\text{The couple of this time } \mathcal{L} \text{ is the couple of the tuple } \mathcal{L} = \mathcal{L} \text{ on } \otimes \text{ it which by the homotopy formula for spaces in } \mathcal{L}
\]

\[
\text{is the couple of the formula of } \mathcal{L} \text{ by the homotopy formula for spaces in } \mathcal{L}
\]

\[
\text{What is the relation between } \mathcal{L} \text{ and } \mathcal{L} \text{?}
\]

Answer: \( \mathcal{L} = \mathcal{L} \text{ because we have shown } \mathcal{L} \text{ to be the coboundary of } \mathcal{L} \text{ by the homotopy formula for spaces in } \mathcal{L}. \)

\[
\mathcal{L} = \mathcal{L} \text{ and } \mathcal{L} \text{ is the coboundary of } \mathcal{L} \text{ by the homotopy formula for spaces in } \mathcal{L}
\]

\[
\text{By } R(\mathcal{L}) = R(\mathcal{L}) \text{ universal property}
\]
\[ S^h A \cong A \otimes \!(\! t) \] on \( F = (A, b, s = d, x, y) \)

On \( S^h A \) we have left multiplication by \( S^h A \)
also the action of \( A^2 \) given by
\[
\begin{align*}
(a \mapsto t(1, a), b(1, d) &= a \cdot d \\
\ell(a) \mapsto (1, a) d &= a \cdot d \\
\alpha \mapsto \tau(a) d &= a \cdot d \\
\gamma \mapsto \sigma(a) d &= a \cdot d
\end{align*}
\]

\( h[d] = h \otimes h \) with \( d = 0 \)
\( h[F] = h \otimes h \) with \( F = 1 \)

For all \( A \) is contract \( S = S^h B \) \( A = B \)
\( A \times (h[a]) = S L + Ld = S \otimes h[a] \)

where \( a \cdot w = a \cdot w + (-1)^{|a|} w \cdot d \in \) the core product

\( A \times (h[F]) = Q \otimes Q F = Q \otimes h[F] \)

where \( \mathcal{K} \) is the core product

In \( Hom_\mathbb{A}(S, S) \) consider the subalgebra generated by \( A \) (left multiplication) by \( d \) and \( F \),
\[
\begin{align*}
F(a) &= (-1)^{|a|} w \\
\mathcal{L} &= \text{left multiplication by } d
\end{align*}
\]

\[ R = S \otimes A \leq S \text{ as left multiplication}. \]

\[ R \otimes d \otimes SF \otimes SD \] is a subalgebra of \( \text{Hom}(S, S) \)

\[
\begin{align*}
\alpha \mapsto \tau(a) d &= a \cdot d \\
\gamma \mapsto \sigma(a) d &= a \cdot d
\end{align*}
\]

Hence the map is injective,

Injective: \( (\omega_0 + \omega_2, \omega_1, \epsilon + \omega_3, dF) \in S \).

For all \( \gamma \), then

\( \gamma = 1 \)
\( \omega_0 + \omega_2 = 0 \)
\( \gamma = d \)
\( (\omega_0 - \omega_2) d = 0 \)

If \( A \) is the ring of endomorphism by \( d \) is

injective on \( S \).

Have \( \omega_0 = \omega_2 = 0 \).

\( \gamma = a \)
\( \omega_1 d = \omega_2 d = 0 \)
\( \gamma = a d \)
\( \omega_2 d = \omega_1 d = 0 \)

\( \omega_0 = \omega_2 = 0 \).

\[
\begin{align*}
S &= S^h A \leq S \text{ as left multiplication}. \]

\[ R \otimes d \otimes SF \otimes SD \] is a subalgebra of \( \text{Hom}(S, S) \)

Claim: this map is injective and an isomorphism.

Proof: \( R \otimes d \) is a subalgebra of \( \text{Hom}(S, S) \)
\( d \in S \) \( \) \( SF \) \( SD \) \( \in S \).

Hence the map is injective,

Injective: \( (\omega_0 + \omega_2, \omega_1, \epsilon + \omega_3, dF) \in S \).

For all \( \gamma \), then

\( \gamma = 1 \)
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\( \omega_2 d = \omega_1 d = 0 \)

\( \omega_0 = \omega_2 = 0 \).
the universal $\Lambda[A]$ is a $\mathbb{Z}$-graded algebra with $A$ in degree 0 and an element $d$ of degree 1 with $d^2 = 0$

1. (graded algebra with degree 0 $d$
    
2. $\mathbb{Q}[A]$ algebra with degree 1 generated by $d$

These categories are the same.

$\mathbb{Q}[A] = \text{universal } \mathbb{Q} \text{ algebra with } A \text{ in degree 0}.$

Adjunct an element $d$

$\Lambda[A] \otimes k[d] = \Lambda[A] \otimes k[d]$

in the universal ungraded algebra with $A$ in degree 0.

$\Lambda[A] \otimes k[d] \leq \Lambda[A] = A$

Formulas:

$\text{Res}_{\Lambda[A]} \otimes R \otimes k[d] = \text{Hom}(\mathbb{R}, R) = (\Lambda[A] \otimes k[d]) \otimes k[d]$

To identify $Q \otimes \text{Hom}(\mathbb{R}, R)$, in this subalgebra recall that

$\begin{align*}
    c a & \mapsto (\text{scale } a) = (2A a) \\
    c a & \mapsto (\text{scale } a) \\
    a + b & \mapsto a + b \\
    a - b & \mapsto a - b \\
\end{align*}$

$Q[A] \leq \Lambda[A] \otimes k[d]$

$Q[A] \otimes k[d] \leq (\Lambda[A] \otimes k[d]) \otimes k[d]$

$Q[A] = \gamma(2x)$ matrices over $\mathbb{R}$

$\Lambda[A]$ subalgebra of the algebra of
Let \( R \in \text{End}_R \mathcal{O} \) be a right \( R \)-module.

\[
\begin{align*}
(a \cdot \mathrm{deg}) (w \cdot dy) &= 2w \cdot \mathrm{deg} w + ady \cdot \mathrm{deg} dy \\
&= uw - da w + dny + dny
\end{align*}
\]

\[
(a \cdot \mathrm{deg}) (\omega) = \begin{pmatrix} a - ad & -da \\ a & 0 \end{pmatrix}
\]

\[
Q \in M_2(R)
\]

Conversely, we have an embedding \( \mathcal{O} \leq M_2(Q) \).

Formula (Lovre, Lu, Penfitt, ...) \[
\begin{align*}
&(a \rightarrow (ca \ 0) \\
&(0 \ 2a)
\end{align*}
\]

\[
dw \rightarrow \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}
\]

Obtain as follows: \( \mathcal{O} \rightarrow L(H) \rightarrow F \\
\begin{align*}
a^t &\rightarrow (\omega) \\
\end{align*}
\]

Define \( \mathcal{O} \rightarrow R \otimes R \mathcal{E} \)

Next we have the numerical property of \( R \mathcal{E} R \) get a map.

Conclude that \( Q \mathcal{E} R \) with \( A \mathcal{E} R \) identifies.