Patterns in the primes

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Mathematicians are **lazy**. This means we can simplify a problem about integers to one about primes.

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Theorem (Example: Fermat's last theorem)

There are no integer solutions to $x^n + y^n = z^n$ *with* $n > 2$ *apart from the obvious ones when* $xyz = 0$.

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5. Fermat: There are no solutions for $n = 4$. □

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$$\pi(x) = \#\{primes\ \le x\} \approx \int_2^x \frac{dt}{\ln t}.$$

$$\pi(10^{10}) = 455,052,511, \qquad \int_2^{10^{10}} \frac{dt}{\ln t} = 455,055,613.8...$$

Difference 3102.8... (< 0.0007%).

My favourite formula

Instead of counting primes with weight 1, it is easier to compensate for the density by counting with weight $\ln p$.

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Theorem (Riemann's explicit formula)

If x is not an integer, then

$$\sum_{\substack{n,p \\ p^n < x}} \ln p = x - \sum_{\substack{\rho \\ \zeta(\rho)=0}} \frac{x^\rho}{\rho} - \ln(2\pi).$$

Here $\zeta(s)$ is the 'Riemann zeta function', and the sum is over zeros of the zeta function.

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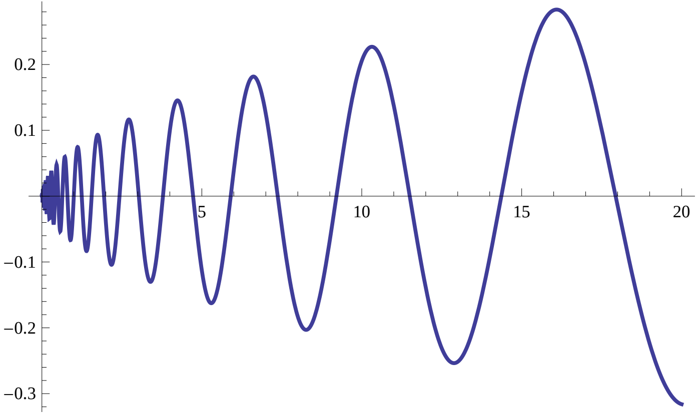
Riemann's formula

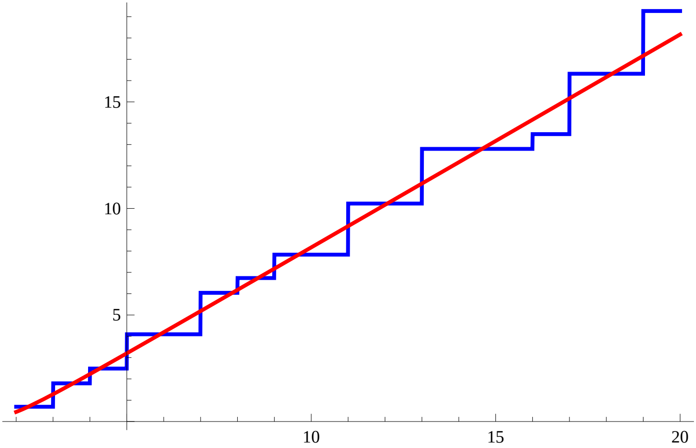
Therefore the zeros tell us exactly where the primes are!

Our step function is a sum of 'waves':

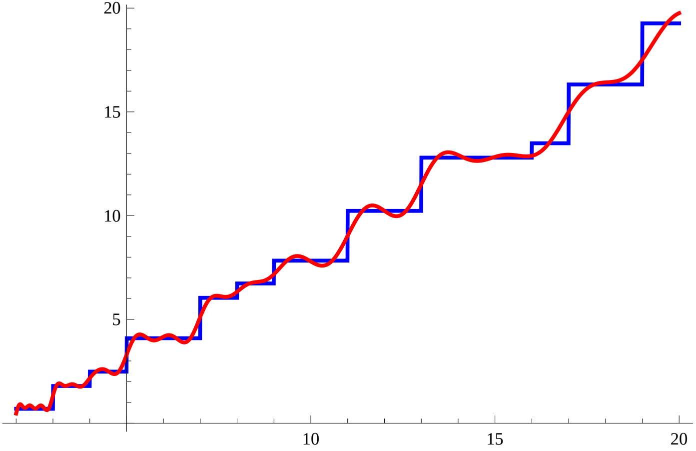


'The music of the primes'

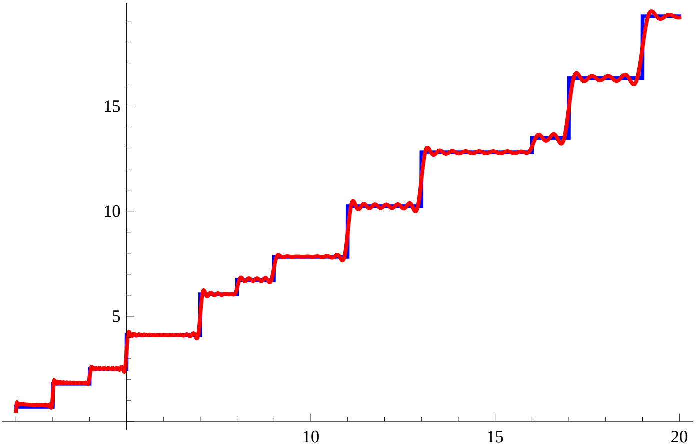
No Zeros



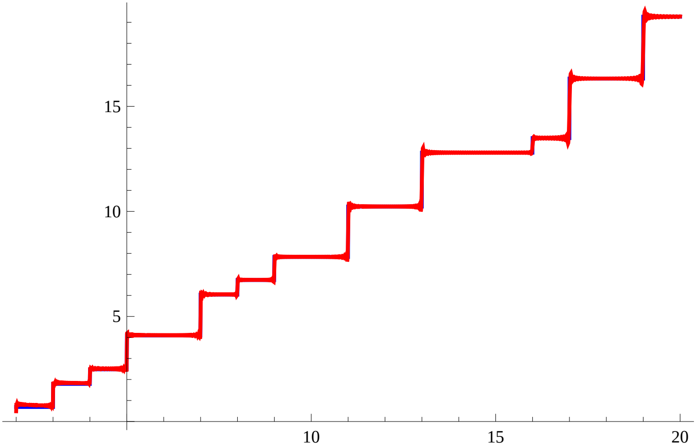
10 Zeros



100 Zeros



1000 Zeros



Riemann's Hypothesis

The size of x^{ρ} is $x^{\Re(\rho)}$.

Conjecture (Riemann's Hypothesis, $1, 000, 000)

All the non-trivial zeros of $\zeta(s)$ have real part 1/2.

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All the non-trivial zeros of $\zeta(s)$ have real part 1/2.

This means all the terms x^ρ have size $\sqrt{x}$, which is much smaller than x.

Corollary

Assume RH. Then for all $x > 2$

$$\left|\pi(x) - \int_2^x \frac{dt}{\ln t}\right| < 4\sqrt{x}\ln x$$

This would completely explain why $\int_2^x dt/\ln t$ is such a good approximation! This explains the large-scale structure!

It isn't just me who's excited

(Hilbert)

"If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?"

(Montgomery)

"So if you could be the Devil and offer a mathematician to sell his soul for the proof of one theorem - what theorem would most mathematicians ask for? I think it would be the Riemann Hypothesis."

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Corollary

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Proof.

$$\begin{aligned} \text{Average gap} &= \frac{\sum_{p_n<x}(p_{n+1} - p_n)}{\#\{p_n \leq x\}} \\ &= \frac{p_N - 2}{\pi(x)} \approx \frac{x}{x/\ln x} \\ &\approx \ln x. \end{aligned}$$

□

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- There are lots of pairs of primes which differ by 2:
  $(3, 5)$, $(5, 7)$, $(11, 13)$, $\dots$, $(1031, 1033)$, $\dots$,
  $(1000037, 1000039)$, $\dots$, $(1000000007, 1000000009)$, $\dots$

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Conjecture (Twin prime conjecture)

There are infinitely many pairs of primes (p, p') which differ by 2.

This is one of the oldest problems in mathematics, and is very much open!

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But this can't be right, as n and $n+1$ can't both be prime!

$$\#\{\text{twin primes} \ \le 10^8\} = 440312, \qquad \int_2^{10^8} \frac{dt}{(\ln t)^2} = 333530.2...$$

Difference 106781.8... (about **24.2**%)

Second Attempt

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Guess (Second attempt)

$$\#\{twin\ primes\ \leq x\} \approx 2\int_2^x \frac{dt}{(\ln t)^2}$$

Worse!

Third Attempt

If n and $n + 2$ are prime, n must be 2 more than a multiple of 3, and so 1 less than a multiple of 6.

- If we randomly picked a number n of the form $6k - 1$ of size x, then the probability n is prime is about $3/\ln x$.

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Guess (Third attempt)

$$\#\{\textit{twin primes} \leq x\} \approx \frac{3}{2}\int_2^x \frac{dt}{(\ln t)^2}$$

Error $\approx$ **13%**. Better!

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- If n and $n+2$ were 'independent', then the probability neither were a multiple of p is $(p-1)/p \times (p-1)/p$.
- So we were off by a factor $\frac{p(p-2)}{(p-1)^2}$.

Guess (Fourth attempt)

$$\#\{\textit{twin primes } \leq x\} \approx 2C_2 \int_2^x \frac{dt}{(\ln t)^2}$$

with $C_2 = \prod_{p>2} p(p-2)/(p-1)^2$.

$\#\{\text{twin primes } \leq 10^8\} = 440312, \qquad 2C_2 \int_2^{10^8} \frac{dt}{(\ln t)^2} = 440367.8...$

Difference 55.8... (this is $<$ **0.2**%). Success!

Other patterns

We can look at more than just gaps of size 2.

Conjecture (De Polignac)

For every positive integer h, there are infinitely many pairs of primes which differ by $2h$.

Again, we guess the number less than x is roughly $C_h x/(\ln x)^2$ for some constant C_h.

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In particular:

Theorem

There are infinitely many pairs (p_1, p_2) of primes such that $|p_1 - p_2| \leq 246$.

Other patterns II

We can also look for **triples** of primes $n, n+h_1, n+h_2$ for some fixed shifts h_1, h_2

1. If $h_1 = 2$, $h_2 = 4$ then $(3, 5, 7)$ is the only triple. (one of $n, n+2, n+4$ must be a multiple of 3)
2. If $h_1 = 2$, $h_2 = 6$ then there are many such triples.

Conjecture

There are infinitely many n such that n, $n+h_1, \dots, n+h_k$ are prime if there isn't an obvious reason why they can't be.

'Obvious reason' means one is always a multiple of some prime for all n.

Theorem

There exists $h_1, \dots, h_k$ such that n, $n+h_1, \dots, n+h_k$ are all primes for infinitely many n.

Optimistic extensions

If we assume a well-believed technical conjecture about primes in arithmetic progressions, then we can get close to the twin prime conjecture!

Theorem

Assume 'GEH'. Then there are infinitely many pairs (p_1, p_2) *of primes with* $|p_1 - p_2| \leq 6$.

Goldbach's conjecture

This conjecture also allows us to say something about another old conjecture

Conjecture (Goldbach's conjecture)

Every even number can be written as the sum of at most two primes.

Theorem

Assume 'GEH'. Then ***at least*** *one of the following is true:*

1. *There are infinitely many twin primes*
2. *For every large even integer N, one of N, $N+2$ or $N-2$ is the sum of two primes.*

Of course we expect both to be true!

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4. On Wikipedia it had been suggested that one could choose p, q such that $(p - 1)/2$ and $(q - 1)/2$ are prime.
5. If there are only 10 (say) 1024-digit primes p such that $(p - 1)/2$ is prime, then this is a VERY bad idea! Bob would die before Alice finds one!

A slight generalization of our model predicts there are many such primes.

Long path to go

It is an exciting time for prime number theory!

Any questions?