Commutative Algebra for Singular Algebraic Varieties

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Worksheet 4: Separability; Étale extensions

Separability

Let K be a field. A polynomial $p(x) \in K[x]$ is said to be *separable over* K if $\langle p(x), p'(x)\rangle = K[x]$ (here $p'(x)$ denotes the formal derivative of $p(x)$ respect to the variable x). Observe that if $p(x)$ is separable over K, then for any field extension $K \subset L$, also $\langle p(x), p'(x)\rangle = L[x]$. In particular, it will be separable over an algebraic closure of K.

Question: What does it mean for a polynomial to be separable?

Separability: The case of *irreducible* polynomials

Let $p(x) \in K[x]$ be an irreducible polynomial over K.

1. Show that the following conditions are equivalent:

(a) $p(x)$ is **not** separable over K;
(b) $p'(x) = 0$;
(c) $p(x) \in K[x^p]$, where $p = \mathrm{char}(K) > 0$.

Hint: Recall that $K[x]$ is a principal ideal domain; since $p(x)$ is irreducible, $\langle p(x), p'(x)\rangle \neq K[x]$ if and only $p(x)$ divides $p'(x)$. Note also that $deg(p(x)) > deg(p'(x)$.

2. Let $L = K[x]/\langle p(x)\rangle$. Note that L is a field, since $p(x)$ is irreducible. Show that

$$L_{\{p'(x)\}} = \begin{cases} L & \text{if and only if } p(x) \text{ is separable over } K \\ 0 & \text{if and only if } p(x) \text{ is } \textbf{not} \text{ separable over } K \end{cases}$$

3. Conclude from exercise 1 that any irreducible polynomial over a field of characteristic zero is separable.

4. Show that any irreducible polynomial over $\mathbb{F}_p$ is separable. More generally, any irreducible polynomial over a finite field K is separable. *Hint: If $p = char(K)$, then the Frobenius homomorphism $a \to a^p$ is an isomorphism on K.*

5. Let $K = \mathbb{F}_3(x)$. Show that the polynomial $T^3 - x \in K[T]$ is not separable.

A field K is said to be perfect if any irreducible polynomial $p(x) \in K[x]$ is separable. In particular any field of characteristic zero is separable, and any finite field is separable.

6. Let K be a field of characteristic $p > 0$. Show that K is perfect if and only if the Frobenius $F : K \to K$ is surjective.

Separability: The general case

Let $p(x) \in K[x]$, and write $p(x) = q_1(x)^{\alpha_1} \cdots q_s(x)^{\alpha_s}$ with $q_i(x) \in K[x]$ irreducible, $q_i \neq q_j$ if $i \neq j$, and $\alpha_i \in \mathbb{N}_{\geq 1}$ for $i = 1, \ldots, s$.

7. Show that $p(x)$ is **not** separable over K if and only if $q_i(x)$ divides $p'(x)$ for some $i \in \{1, \ldots, s\}$. (*Hint: If* $\langle p(x), p'(x)\rangle \neq K[x]$*, then* $\langle p(x), p'(x)\rangle \subset \langle q_i\rangle$ *for some* $i \in \{1, \ldots, s\}$*.*)

8. Conclude that $p(x)$ is separable over K if and only if $\alpha_1 = \ldots = \alpha_s = 1$ and $q_i(x)$ is separable for $i = 1, \ldots, s$.

9. Let $L = K[x]/\langle p(x)\rangle$. Note that

$$L = (k[x]/\langle q_1(x)^{\alpha_1}\rangle) \oplus \ldots \oplus (k[x]/\langle q_s(x)^{\alpha_s}\rangle).$$

Now, localize at $p'(x)$,

$$L_{p'(x)} = (k[x]/\langle q_1(x)^{\alpha_1}\rangle)_{p'(x)} \oplus \ldots \oplus (k[x]/\langle q_s(x)^{\alpha_s})_{p'(x)}.$$

Show that:

(a) $p(x)$ is separable over K if and only $L_{p'(x)} = L$. Note that in this case L is a direct sum of fields.

(b) When $p(x)$ is not separable over K,

$$L_{p'(x)} = (k[x]/\langle q_{i_1}(x)\rangle)_{p'(x)} \oplus \ldots \oplus (k[x]/\langle q_{i_r}(x))_{p'(x)}$$

where $q_{i_1}(x), \ldots, q_{i_r}(x)$ are the only separable factors of $p(x)$ which in addition have $\alpha_{i_j} = 1$ for $j = 1, \ldots, r$. Hence, $L_{p'(x)}$ is either 0, or a direct sum of fields.

Separable extensions

Let $K \to L$ be an algebraic extension of fields. An element $a \in L$ is said to be *separable over* K if its minimal polynomial over K is separable. The extension $K \to L$ is said to be *separable* if all elements of L are separable over K.

10. Using the previous exercises, is not difficult to see that $K \subset L$ is separable if and only if for any field extension $K \to K'$, the ring $L \otimes_K K'$ is reduced. Consider, for instance, the field extension $\mathbb{F}_3(x) \subset \mathbb{F}_3(x)[y]/\langle y - x^3\rangle$. Note that $\mathbb{F}_3(x)[y]/\langle y - x^3\rangle \otimes_{\mathbb{F}_3(x)} \mathbb{F}_3(x)[y]/\langle y - x^3\rangle \simeq \mathbb{F}_3(x)[y]/\langle (y^{1/3} - x)^3\rangle$.

Some conclusions

- From the alternative description in exercise 10, it follows that separability is stable by base change.
- The Theorem of the Primitive Element states that any finite separable extension of a field K is simple (i.e., it is of the form $K[x]/\langle p(x)\rangle$ for some irreducible polynomial $p(x) \in K[x]$).
- A separable extension $K \subset L$ is a special case an *étale* extension in the particular case in which the base ring is a field.

Flatness II

Let A be a ring. Recall that

- Any free A-module is A-flat.
- If B is an A-algebra, then B is A-flat, if it is flat as an A-module. In particular a ring homomorphism $A \to B$ is flat if B is a flat A-algebra.
- The localitzation of A at a multiplicative set $S \subset A$, A_S, is A-flat.

11. Show that if B is an A-flat algebra, and C is a B-flat algebra, then, C is a flat A-algebra, i.e., the composition of flat homomorphisms is flat.

12. Show that flatness is stable by base change: if $A \to B$ is flat, and C is an A-algebra, then $C \to B \otimes_A C$ is flat.

13. Let $p(t) \in A(t)$ be a monic polynomial. Show that $A \to A[t]/\langle p(t)\rangle$ is flat. Observe that $A \to (A[t]/\langle p(t)\rangle)_{p'(t)}$ is also flat.

Facts about flat extensions

Theorem. [1, Theorem 8, p.48] Let B a finitely generated flat A-algebra. Then the induced morphism

$$spec(B) \to spec(A)$$

is open (i.e., the image of an open set is open). Sea B una A-algebra finitamente generada y plana sobre A. Entonces el morfismo es abierto (imagen de un abierto es abierto).

Theorem. [1, Theorem 53, p.159] Let B a finitely generated A-algebra. Then the set of points of $spec(B)$ where $A \to B$ is flat is open.

Étale extensions

Let k be a field, and let A be a k-algebra of finite type. The ring A could be, for instance, the coordinate ring of an affine variety over k. In some situations, it could be convenient to enlarge the field k to a bigger one, say L, in a way that the algebraic properties of A remain after changing the base field. Observe the following diagram:

$$\begin{array}{ccc} A & \longrightarrow & A \otimes_k L = A' \\ \uparrow & & \uparrow \\ k & \longrightarrow & L. \end{array}$$

Let $\mathfrak{p} \subset A$ be a prime, and denote by $k(\mathfrak{p})$ the residue field of the local ring $A_{\mathfrak{p}}$. Then:

$$\begin{array}{ccc} A' = A \otimes_k L & \longrightarrow & L \otimes_k k(\mathfrak{p}) \\ \uparrow & & \uparrow \\ A & \longrightarrow & k(\mathfrak{p}). \end{array}$$

- If $k \to L$ is finite, then $A \to A'$ is a finite extension, and there is a finite number of primes in A' dominating $\mathfrak{p} \subset A$ (observe that $L \otimes_k k(\mathfrak{p})$ is a finite extension of $k(\mathfrak{p})$, and hence semi-local) . This would say that A' is a finite cover of A.
- If, in addition, $k \to L$ is separable, then $L \otimes_k k(\mathfrak{p})$ is reduced, and therefore a finite direct sum of fields.

Let us examine closely the last assertion. Localize at $\mathfrak{p}$:

$$\begin{array}{ccc} A_{\mathfrak{p}} \otimes_k L = A' \otimes_A A_{\mathfrak{p}} & \longrightarrow & L \otimes_k k(\mathfrak{p}) \\ \uparrow & & \uparrow \\ A_{\mathfrak{p}} & \longrightarrow & k(\mathfrak{p}). \end{array}$$

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s \subset A'$ be the prime ideals dominating $\mathfrak{p}$. Denote by $k(\mathfrak{q}_i)$ the residue field of each $\mathfrak{q}_i$. Then, on the one hand, since $L \otimes_k k(\mathfrak{p})$ is a zero dimensional (reduced) ring,

$$L \otimes_k k(\mathfrak{p}) = k(\mathfrak{q}_1) \oplus \ldots \oplus k(\mathfrak{q}_s);$$

and

$$k(\mathfrak{q}_i) = (L \otimes_k k(\mathfrak{p}))_{\mathfrak{q}_i}.$$

On the other hand

$$L \otimes_k k(\mathfrak{p}) = A' \otimes_A A_{\mathfrak{p}} / (\mathfrak{p} A' \otimes_A A_{\mathfrak{p}})$$

and

$$k(\mathfrak{q}_i) = \left(A' \otimes_A A_{\mathfrak{p}} / (\mathfrak{p} A' \otimes_A A_{\mathfrak{p}})\right)_{\mathfrak{q}_i} = A'_{\mathfrak{q}_i} / \mathfrak{p} A'_{\mathfrak{q}_i}.$$

Thus $\mathfrak{p} A'_{\mathfrak{q}_i} = q_i$.

A local homomorphism of local rings, $(A, m, k) \to (B, n, K)$, is said to be étale if the following conditions hold:

(a) The homomorphism is flat;
(b) $mB = n$;
(c) The extension $k \to K$ is separable.

A ring homomorphism $A \to B$ is étale if it is étale at the localization at each maximal ideal $m \subset A$. Observe that $A \to B$ is étale if it is flat with étale fibres.

14. Show that $k[x] \to k[x,y]/\langle y^2 - x\rangle$ is étale locally at $\langle x-1\rangle \subset k[x]$ but it is not étale at $\langle x\rangle$.

15. Show that if $(A, m, k) \to (B, n, K)$ is étale, then the Hilbert-Samuel function of both local rings coincide. *Hint: Compare the graduate rings at the maximal ideals.*

16. Let $p(t) \in A[t]$ be a monic polinomial. Observe that $A \to A[t]/\langle p(t)\rangle$ may not be étale (see exercise 13). However, after localizing at $p'(t)$, we get, either a zero ring or else,

$$A \to (A[t]/\langle p(t)\rangle))_{p'(t)}$$

is étale. *Hint: You only have to show that the fibres are étale.*

17. Show that étale extensions are stable by base change.

Facts: Let $A \to B$ be a ring homomorphism. Then:

- There is an open (maybe empty) set of *spec*(A) where it is flat;
- The set of points of *spec*(A) where it is étale is open (maybe empty);
- If $A \to B$ is étale, then a suitable localization of B is of the form $(A[t]/\langle p(t)\rangle)_{p'(t)}$ for some monic polynomial $p(t) \in A[t]$. These are called *standard étale.*

Étale topology

18. Show that two different étale extensions of a ring A can be dominated by a third.

References

[1] H. Matsumura, *Commutative algebra.* Second edition. Mathematics Lecture Note Series, 56. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980. xv+313 pp.