Commutative Algebra for Singular Algebraic Varieties

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Worksheet 2: Extension of scalars, quotients and fibres

Hipotheses: A and B are commutative rings with unity, and $f : A \to B$ is a ring homomorphism.

Extension of scalars

Observe that any B-module N is, in particular, an A-module via f. Let M be an A-module. Note that $B \otimes_A M$ is naturally a B-module. We shall use the following properties:

- For any A-module M, $A \otimes_A M$ can be naturally identified with M. In particular, $A \otimes_A B \simeq B$.
- If $\{M_i\}_{i\in\Lambda}$ is a family of A-modules, then $\left(\bigoplus_{i\in\Lambda} M_i\right) \otimes_A B = \bigoplus_{i\in\Lambda} (M_i \otimes_A B)$.

1. Use the last property to show that if x is a variable, then $A[x] \otimes_A B \simeq B[x]$.

- Observe that B can be viewed as an A-module. The previous properties also hold when B is replaced by any A-module N.

Exact sequences

Consider an exact sequence of A-modules:

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Then the sequence of B-modules

$$M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0$$

is exact.

2. Use the previous fact to show that if M is a finitely generated A-module, then $B \otimes_A M$ is a finitely generated B-module.

3. Let $(A = k[x]_{\langle x\rangle}, m = \langle x\rangle, k)$ be a local ring. Observe that the sequence

$$0 \to m \to A \to k \to 0$$

is exact, but

$$m \otimes_A m/m^2 \to A \otimes_A m/m^2$$

is not injective.

4. Let $I \subset A$ be an ideal, and consider the exact sequences

$$0 \to I \to A \to A/I \to 0$$

$$I \otimes_A B \to A \otimes_A B \to A/I \otimes_A B \to 0$$

Show that the image of $I \otimes_A B$ in $A \otimes_A B = B$ can be identified with IB, i.e., the ideal expanded by I in B. Conclude that

$$A/I \otimes_A B \simeq B/IB.$$

5. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[x]/\langle x^2+1\rangle$. *Hint: Use the exact sequence* $0 \to \langle x^2+1\rangle \to \mathbb{R}[x] \to \mathbb{C} \to 0$.

6. Let $f(X) = X^3 + 6X + 12 \in \mathbb{Z}[X]$. Show that $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[X]/\langle f(X)\rangle = \mathbb{Z}_2[X]/\langle X^3\rangle$

7. Show that $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 = 0$, that $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_4 = \mathbb{Z}_2$, and that $\langle 2\rangle \otimes_{\mathbb{Z}} \mathbb{Z}_2 \neq 0$.

Flatness I

An A-module M is said to be *flat* if for any exact sequence of A-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

the tensored sequence

$$0 \to M_1 \otimes_A M \to M_2 \otimes_A M \to M_3 \otimes_A M \to 0$$

is again exact.

Facts

- Let $S \subset A$ be a multiplicative set. Then A_S is a flat A-module.
- Let (R, m, k) be a local ring, and let $(\hat{R}, \hat{m}, k)$ be its m-adic completion. Then $\hat{R}$ is a flat R-module.

8. Show that $\mathbb{Z}_3$ is not a flat $\mathbb{Z}$-module.

9. Show that a free A-module is flat. In particular, if x is a variable, $A[x]$ is a flat A-module.

10. If K is a field, then any K-module is flat.

Note: A ring homomorphism $A \to B$ is said to be flat if B is a flat A-module. For instance, the homomorphisms defined by localizations $A \to A_S$ are flat.

Fibers

• Let $I \subset A$ be an ideal

11. Note that there is a bijective correspondence between the ideals of A/I and the ideals of A that contain I. Observe that

$$V(I) := \{\mathfrak{p} \in \text{spec}(A) : \mathfrak{p} \supset I\}$$

is closed in spec(A). So the primes in $V(I)$ are in one to one correspondence with primes in spec(A/I).

12. Show that the ring homomorphism $A \to B$ induces the ring homomorphism:

$$A/I \to B \otimes_A A/I = B/IB.$$

13. What is spec (B/IB)? Conclude that the induced ring homomorphism:

$$A/I \to B \otimes_A A/I$$

is the restriction of f to the closed set defined by I in spec(A). In other words, $A \to B$ induces a continuous map spec$(A) \leftarrow$ spec(B), show that $V(IB)(\subset \text{spec}(B))$ is the inverse image of $V(I) \subset \text{spec}(A)$.

• Let $S \subset A$ be a multiplicative closed set

14. Given a homomorphism of rings $\alpha : A \to B$ and a multiplicative set S in A, there is a natural homomorphism, say $\alpha_S : A_S \to B_{\alpha(S)}$. Recall that there are inclusions spec$(A_S) \subset$ spec(A), and spec$B_{\alpha(S)} \subset$ spec(B). Show that spec$B_{\alpha(S)} \subset$ spec(B) is the set of primes in spec(B) mapping to primes of spec(A) which are in spec(A_S).

Fact

In what follows we will make use of the following fact: $B \to B_{\alpha(S)}$ can be identified with $B \to B \otimes_A A_S$.

15. Let $\mathfrak{p} \subset A$ be a prime ideal, and let $S = A \setminus \mathfrak{p}$. Observe that α induces a natural ring homomorphism:

$$A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{p}} = B_{f(S)}.$$

16. What is spec $(B \otimes_A A_{\mathfrak{p}})$?: If $\mathfrak{q} \in$ spec $(B \otimes_A A_{\mathfrak{p}})$, what is the contraction to A?

Taking quotients commutes with localization

17. Let $I \subset A$ be an ideal and let $S \subset A$ be a multiplicative set. Let $\overline{S}$ be the image of S in A/I. Show that $(A/I)_{\overline{S}} = A_S/I_S$. (See the remark in problem 14.)

• Conclusion

18. Let $\mathfrak{p} \subset A$ be a prime ideal, and let $k(\mathfrak{p})$ be the residue field of A at $\mathfrak{p}$, i.e., $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Show that a homomorphism $A \to B$ induces a homomorphism

$$k(p) \to B \otimes_A k(\mathfrak{p})$$

and that spec $(B \otimes_A k(\mathfrak{p}))$ is the fiber of the map

$$\mathrm{spec}(B) \to \mathrm{spec}(A)$$

at the point $\mathfrak{p} \in \mathrm{spec}(A)$.

Remark

Observe that by exercise 17, there are two different constructions that lead us to find the fiber over the prime $\mathfrak{p}$:

• First we can localize, and then take the quotient:

$$\begin{array}{ccccc} B & \longrightarrow & B \otimes_A A_{\mathfrak{p}} & \longrightarrow & B \otimes_A A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p}) = B \otimes_A k(\mathfrak{p}) **[l] \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & A_{\mathfrak{p}} & \longrightarrow & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = k(\mathfrak{p}); \end{array}$$

• Or we can take the quotient, and then localize

$$\begin{array}{ccccc} B & \longrightarrow & B \otimes_A A/\mathfrak{p} = B/\mathfrak{p}B & \longrightarrow & B/\mathfrak{p}B \otimes_{A/\mathfrak{p}} (A/\mathfrak{p})_{\mathfrak{p}} = B \otimes_A A/\mathfrak{p} \otimes_{A/\mathfrak{p}} (A/\mathfrak{p})_{\mathfrak{p}} = B \otimes_A k(\mathfrak{p}) \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & A/\mathfrak{p} & \longrightarrow & (A/\mathfrak{p})_{\mathfrak{p}} = k(\mathfrak{p}). \end{array}$$

19. Consider $A = \mathbb{R}[X] \subset B = \mathbb{R}[X,Y]/\langle X^2 + Y^2 - 1\rangle$. Indicate how may primes of B dominate A at the prime $\mathfrak{p}$ for:
i) $\mathfrak{p} = \langle X \rangle$.
ii) $\mathfrak{p} = \langle X - 1 \rangle$.
iii) $\mathfrak{p} = \langle X + 2 \rangle$.
Indicate if $B \otimes_A k(\mathfrak{p})$ is reduced in each of the previous cases.

20. Consider $A = k[X] \subset B = k[X,Y]/\langle XY - 1\rangle$. Show that the fiber over $\mathfrak{p} = \langle X \rangle$ is empty, namely, that $B \otimes_A (k[X]/\langle X \rangle) = 0$.

Generic fiber

21. Assume now that A is an integral domain. Then $(0) \subset A$ is a prime ideal. This is called the *generic point of* A. Note that this point is dense in $\operatorname{spec}(A)$. Observe that the quotient field K of A is nothing but $A_{(0)}$. Prove that $A \to B$ induces a homomorphism

$$K \to B \otimes_A K.$$

The set $\operatorname{spec}(B \otimes_A K)$ is the fiber over the generic point of A and it is called the *generic fiber* of $A \to B$.

19. Consider $A = \mathbb{R}[X] \subset B = \mathbb{R}[X,Y]/\langle X^2 + Y^2 - 1\rangle$. Indicate how may primes does the generic fiber have. What happens if we consider the fiber over an algebraic closure of $\mathbb{R}(X)$ by base change?

A useful result

Proposition [1, Proposition 1.10]**.** *Let A be a ring and let $I_1, \ldots, I_n \subset A$ be ideals. Consider the natural ring homomorphism:*

$$\phi : A \to A/I_1 \oplus \ldots \oplus A/I_n.$$

Then:
(i) *ϕ is injective if and only if $I_1 \cap \ldots \cap I_n = (0)$;*
(ii) *ϕ is surjective if and only if I_i is coprime with I_j for $i \neq j$ (i.e., for all $i \neq j$, $I_i + I_j = A$).*

Some examples

20. Let k be a field, and let $p(x) \in k[x]$ be a polynomial. Write

$$p(x) = q_1(x)^{\alpha_1} \cdots q_r(x)^{\alpha_r}$$

with $q_i(x) \in K[x]$ irreducible, $\alpha_i \in \mathbb{N}_{\geq 1}$, and $q_i(x) \neq q_j(x)$ if $i \neq j$.
(i) Show that:

$$k[x]/\langle p(x)\rangle \simeq k[x]/\langle q_1(x)^{\alpha_1}\rangle \oplus \ldots \oplus k[x]/\langle q_r(x)^{\alpha_r}\rangle.$$

(ii) Observe that each summand, $k[x]/\langle q_i(x)^{\alpha_i}\rangle$ is a local ring with maximal ideal $\langle \overline{q_i(x)}\rangle$.
(iii) Each summand is either a field, or a non-reduced local ring of dimension zero.

Let k be a field, and let $A = k[x,y]/\langle x \cdot y\rangle$.

The total quotient field of a reduced ring

21. Show that A is reduced, that it has two minimal primes, and that $\operatorname{spec}(A)$ has two irreducible components.

22. Show that $A_{\langle \overline{y}\rangle} \simeq k(x)$, the field of rational functions of the affine line $\{y = 0\} \subset \mathbb{A}^2_k$.

23. Let $S = A \setminus (\langle \overline{x}\rangle \cup \langle \overline{y}\rangle)$. Show that S is a multiplicative set and that $A \to A_S$ is injective.

24. Show that

$$A_S \simeq k(x) \times k(y),$$

where $k(x)$, the field of rational functions of the affine line $\{y = 0\} \subset \mathbb{A}^2_k$, and $k(y)$ is the the field of rational functions of the affine line $\{x = 0\} \subset \mathbb{A}^2_k$.

Observe that the A_S is the total quotient field $K(A)$ of A. Exercise 24 says that $K(A)$ is the product of the quotient fields of the coordinate rings of each of the irreducible components of $\operatorname{spec}(A)$.

One more interesting application of the Proposition

25. Suppose k is a field and that B is a zero dimensional k-algebra of finite type. According to a theorem of Noether, B is a finite extension of k. Since B is noetherian, it has only a finite number of minimal primes. Since B is zero dimensional, the minimal primes are also maximal. Thus B is a *semi-local ring* (i.e. *spec*(B) consists of finitely many closed points).

Let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the maximal ideals of B. We are going to show that that B is a direct sum of s local rings.

(i) The nilradical, $\mathfrak{N}$, of B is $\cap_i \mathfrak{q}_i$. Since B is noetherian, show that there is a power l of $\mathfrak{N}$ such that $\mathfrak{N}^l = 0$.

(ii) Observe that $\mathfrak{q}_1 \dots, \mathfrak{q}_s$ are pairwise coprime. This implies that $\cap_i \mathfrak{q}_i = \prod_i \mathfrak{q}_i$. Use this fact to show that $\cap_i \mathfrak{q}_i^l = 0$.

(iii) Using the Proposition, show that

$$B \simeq B/\mathfrak{q}_1^l \oplus \dots \oplus B/\mathfrak{q}_r^l.$$

Observe that $B_{\mathfrak{q}_i} = B/\mathfrak{q}_i^l$, and that this local ring is either a field or a non-reduced artinian local ring.

References

[1] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.