# Commutative Algebra for Singular Algebraic Varieties

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**Worksheet 1: Basic tools: localizations, blow-ups**

## On the localization of a ring at a multiplicative set

A subset $S$ in $A$ is a multiplicative set if $1 \in S$ and $S$ closed by the product. Let $\beta : A \to B$ be a homomorphism. Observe that $\beta(S)$ is also a multiplicative set in $B$.

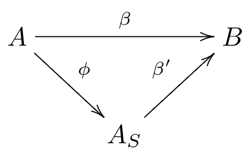
A first example is given by $S = U(A)$, the units of $A$, and in this case any homomorphism $\beta : A \to B$ maps $U(A)$ to $U(B)$ (the units of $B$).

Recall that $A_S = \{\frac{a}{s}, a \in A, s \in S\}$ is a ring, that $\frac{a}{s} = \frac{a'}{s'}$ in $A_S$ if and only if $t(s'a - a's) = 0$ in $A$ for some $t \in S$; and that $\phi : A \to A_S$, $\phi(a) = \frac{a}{1}$ is an homomorphism. The homomorphism $\phi : A \to A_S$ has the following properties:

**a)** The multiplicative set $S$ maps into $U(A_S)$ (the units of $A_S$) via $\phi$.

**b)** The homomorphism $\phi : A \to A_S$ is universal with the property stated in a):

**Universal Property of the localization.** *If an homomorphism $\beta : A \to B$ maps the multiplicative set $S$ in $U(B)$ then there is a commutative diagram of homomorphisms*




*and $\beta'$ is unique with this property.*

**1.** Show that if $\phi' : A \to (A_S)'$ also has the universal property, then there is a unique isomorphism of $A$-algebras between $A_S$ and $(A_S)'$. This uniqueness enables us to identify $\phi : A \to A_S$ with $\phi' : A \to (A_S)'$.

**2.** If $S = U(A)$, then $\phi : A \to A_S$ is $id : A \to A$.

**3.** If $S \subset T$ in $A$, then $(A_S)_T = A_T$.

## Some (useful) properties

**4.** $A_S = 0$ if and only if $0 \in S$.

**5.** If $S$ in $A$ is a multiplicative set and $\beta : A \to B$ be a homomorphism, then there is a natural homomorphism say $\beta_S : A_S \to B_{\beta(S)}$. (FACT: if $\beta : A \to B$ is injective, so is $\beta_S : A_S \to B_{\beta(S)}$.

**6.** An element $a \in A$ is in $\ker \phi$ if and only if there is $s \in S$ such that $s \cdot a = 0$. Conclude that $\phi$ is injective if and only if $S$ does not contain zero divisors.

**7.** Find $\ker \phi$ for $S = \{\overline{1}, \overline{3}\}$ in $A = \mathbb{Z}/6\mathbb{Z}$.

**8.** If $A$ is a domain and $S \subset A \setminus \{0\}$, then $A_S$ is a domain and $A \subset A_S$ (a localization of a domain is a domain). In particular, if $K$ is the quotient field of $A$, and $S \subset A \setminus \{0\}$, then $A_S \subset K$ (i.e., a localization of a domain is a subring of the quotient field).

**Facts:**

• Let $\mathfrak{p} \subset A$ be a prime ideal. Then $T = A \setminus \mathfrak{p}$ is a multiplicative set, and we will use the notation $A_T = A_{\mathfrak{p}}$.

• Let *spec*$(A)$ be the set of prime ideals of a ring $A$ with the Zariski topology. Then

$$spec(A_S) = \{\mathfrak{p} \in spec(A),\ \mathfrak{p} \cap S = \emptyset\}.$$

**9.** If $\mathfrak{p} \in spec(A_S)$, show that $(A_S)_{\mathfrak{p}} = A_{\mathfrak{p}}$.

**9.** Describe *spec*$(A_{\mathfrak{p}})$. Show that $A_{\mathfrak{p}}$ is a local ring.

**Some (useful) examples**

**11.** Let $k$ be a field, and let $k[x, y]$ be the polynomial ring in two variables with coefficients in $k$. Describe: $spec(k[x, y]_{\langle x,y\rangle})$, $spec(k[x, y]_{\langle x\rangle})$, $spec(k[x, y]_{(0)})$.

**12.** Let $a \in A$. Then $S = \{1, a, a^2, \ldots, a^n, \ldots\}$ is a multiplicative set. Then we will use the notation $A_S = A_a$, or $A_S = A[\frac{1}{a}]$. Show that $A \to A_a$ is the same as $A \to A[X]/\langle aX - 1\rangle$ (always using the precious identification).

**13.** Given elements $a, b$ in $A$, then

$$A_{ab} = (A_a)_b = (A_b)_a.$$

**14.** Let $a \in A$. Show that $spec(A_a) \subset spec(A)$ is a Zariski open subset. We consider $\mathrm{Spec}(A_a)$ as the restriction of $\mathrm{Spec}(A)$ to the Zariski open set $spec(A_a)$.

**15.** If $A$ is a domain, then $(0)$ is prime and $A_{(0)}$ is the quotient field $K$ of $A$. Recall that $A \subset K$. Moreover, for any prime $\mathfrak{p}$, $A \subset A_{\mathfrak{p}} \subset K$[1].

**Some facts about reduced rings**

**16.** Let $A$ be a noetherian ring and let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ be the minimal prime ideals. Show that $A$ is reduced if and only if the natural morphism

$$A \longrightarrow A/\mathfrak{p}_1 \oplus \cdots \oplus A/\mathfrak{p}_r$$

is injective. Let $a \in A$ and consider

$$a \longmapsto (a_1, \ldots, a_r).$$

Show that $a$ is a zero divisor if and only if $a_i = 0$ for some $i \in \{1, \ldots, r\}$.

**17.** If $A$ is reduced, and $S$ is the complement of the minimal primes of $A$, then $A \to A_S$ is injective (see exercise 16). In this case $A_S$ is said to be the *total quotient ring of* $A$. Show that the total quotient ring of a reduced ring is semilocal, of dimension zero. Describe the prime ideals of $spec(A_S)$ as a subset of $spec(A)$.

**Blow-ups**

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[1]Suppose $A$ is the coordinate ring of an affine variety $X$. By definition this is the ring of regular functions on $X$, and $K$ is the field of rational functions on $X$. If $\mathfrak{p}$ is the maximal ideal corresponding to some point $\xi \in X$, then $A_{\mathfrak{p}}$ is the ring of regular functions on $\mathfrak{p}$, which sits between $A$ and $K$

**18.** Let $B$ be an $A$-algebra, let $f \in A$ suppose that $B \subset A_f$. Show that $f$ is a non-zero divisor in $B$.

Let $I = \langle f_1, \dots, f_r \rangle$ be an ideal in $A$. For each $i = 1, \dots, \leq r$, let

1 (0.0.1)
$$A_i = A\left[\frac{f_1}{f_i}, \dots \frac{f_r}{f_i}\right] (\subset A_{f_i})$$

denote the smallest $A$-subalgebra containing the collection of elements $\left\{\frac{f_1}{f_i}, \dots \frac{f_r}{f_i}\right\}$ (by abuse of notation, in (0.0.1) we denote by $A$ the image of $A$ in $A_{f_i}$).

**19.** Show that the extended ideal $IA_i$ is principal. Moreover, prove that $IA_i$ is a free $A_i$-module of rank one.

**Discussion**

With the same notation as before, given two indices, $i, j \in \{1, \dots, r\}$, there is a natural identification of the localizations,
$$A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right]\left[\frac{f_i}{f_j}\right] = A\left[\frac{f_1}{f_j}, \dots, \frac{f_r}{f_j}\right]\left[\frac{f_j}{f_i}\right] (\subset A_{f_i f_j})$$
where $A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right]\left[\frac{f_i}{f_j}\right]$ denotes the localization of $A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right]$ at the element $\frac{f_j}{f_i}$, or say
$$A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right]\left[\frac{f_i}{f_j}\right] := A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right]_{\frac{f_j}{f_i}}$$

The *blow-up of $A$ at $I$* is the morphism of schemes
$$\mathrm{Spec}(A) \stackrel{\pi}{\leftarrow} X$$
where $X$ is the scheme obtained by patching the affine charts $A_i = A[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}]$ described above, and $\pi$ is defined by patching the morphisms of affine schemes
$$\mathrm{Spec}(A) \longleftarrow \mathrm{Spec}\left(A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right]\right)$$
(i.e., the homomorphisms $A \to A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right] (\subset A_{f_i})$). Note that the total transform of $I$ in $X$ is a locally free ideal in $X$.

**21.** Show that if $I = \langle f \rangle$ is a principal ideal, then the blow-up is an affine morphism and $X = \mathrm{Spec}(A_1)$ where $A_1$ is the image of $A$ in $A_f$.

**22.** Show that the blow-up of $A$ at a nilpotent ideal is the empty set. Describe the blow-up of $\mathbb{Z}_4$ at the ideal expanded by $\overline{2}$.

**23.** Show that the blow-up of $\mathbb{Z}/6\mathbb{Z}$ at the ideal spanned by $\overline{2}$ is $\mathbb{Z}/3\mathbb{Z}$.

**24.** Let $k$ be a field and let $k[x, y]$ be the polynomial ring in two variables with coefficients in $k$. Construct the blow-up at $\langle x, y \rangle$:
$$\mathbb{A}^2_k \stackrel{\pi}{\longleftarrow} X.$$
Let $C := \{x^2 - y^3 = 0\}$.

(i) What is $\pi^{-1}(C)$?

(ii) What is $\overline{\pi^{-1}(C \setminus (0, 0))} \subset X$?