Commutative Algebra for Singular Algebraic Varieties
Some tools of commutative algebra

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Localization

Let $A$ be a ring and $S \subset A$ a multiplicative set ($1 \in S$ and if $s_1, s_2 \in S$ then $s_1 \cdot s_2 \in S$).

$A_S = \left\{ \frac{a}{s}, a \in A, s \in S \right\}$ where $\frac{a}{s} = \frac{a'}{s'}$ in $A_S \iff t(s'a - a's) = 0$ in $A$.

The homomorphism $A \xrightarrow{\phi} A_S$ satisfies $a \xrightarrow{a} \frac{a}{1}$

a $\phi(S) \subset U(A_S)$,

b **Universal Property of the Localization**

If $A \xrightarrow{\beta} B$ maps $S$ in $U(B)$ then there is a unique $\beta'$:

Notation

- If $S = A \setminus p$, $A_S = A_p$
- If $S = \{1, f, \ldots, f^n, \ldots\}$, $A_S = A_f$. 
A topological space attached to a ring, \( \text{spec} \)

As a set,

\[
\text{spec}(A) = \{ p : p \text{ is a prime ideal of } A \}
\]

A topology is defined. Let \( I \subset A \) an ideal,

\[
V(I) := \{ p \in \text{spec}(A) : p \supseteq I \}
\]

is a closed subset in \( \text{spec}(A) \).

Example

- \( \text{spec}(A/I) = V(I) \subset \text{spec}(A) \),
- \( \text{spec}(A_S) \subset \text{spec}(A) \), the set of prime ideals \( p \subset A \) s. t. \( p \cap S = \emptyset \).
- \( \text{spec}(A_f) \) is an open set of \( \text{spec}(A) \).

Krull dimension

- We can construct chains of primes in $\text{spec}(A)$:

  $$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \ldots \subsetneq \mathfrak{p}_m \text{ (of length } m\text{),}$$

- $\text{spec}(A)$ has a finite number of minimal primes, since $A$ a Noetherian ring.

**Krull dimension of $A$**

$\text{dim}_{\text{Krull}} A = \text{is the length of the maximal chain of prime ideals of } A.$
Continuous functions

Let $A \xrightarrow{\beta} B$ be a homomorphism.

$q \subset B$ a prime ideal $\Rightarrow \beta^{-1}(q) \subset A$ is a prime ideal.

We can define

$$\text{spec}(A) \xleftarrow{\pi} \text{spec}(B)$$

$$\beta^{-1}(q) \xleftarrow{} q$$

The preimage of a closed set is a closed set

$$\text{spec}(A) \xleftarrow{\pi} \text{spec}(B)$$

$$\bigcup V(I) \bigcup V(IB)$$
Interesting case

\[
\begin{array}{c}
A \xrightarrow{\beta} B \\
\downarrow \quad \downarrow \\
A_S \quad \quad B_\beta(S) \\
\end{array}
\]

\[
\begin{array}{c}
spec(A) \xleftarrow{\pi} spec(B) \\
\bigcup \quad \bigcup \\
spec(A_S) \quad \quad spec(B_\beta(S)) \\
\end{array}
\]
Fibers

We can take the localization and then take the quotient

\[ B \to B \otimes_A A_p \to B \otimes_A A_p \otimes_{A_p} k(p) = B \otimes_A k(p) \]

\[ A \to A_p \to A_p/pA_p = k(p) \]

Or we can take the quotient, and then localize

\[ B/pB \to B \otimes_A A/p \to B \otimes_A A/p \otimes_{A_p} (A/p)_p = B \otimes_A k(p) \]

\[ A \to A/p \to (A/p)_p = k(p) \]
Finite extensions

Definition

\[ A \longrightarrow B \] finite \( \iff \) \( B \) is finite as an \( A \)-module.
\( \iff \) \( B = A[b_1, \ldots, b_r] \) where \( b_i \) is integral over \( A \).

Interesting cases

1.

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A/I & \longrightarrow & B \\
\end{array}
\]

\[
A/I \] finite \( \iff \) \( B \otimes_A (A/I) = B/IB \)

Example

\[
\mathbb{R}[X] \longrightarrow \mathbb{R}[X, Y]/\langle Y^2 - X \rangle
\]
Let $A \subset B$ be a finite extension,

**Facts:**

1. $\text{spec}(A) \xleftarrow{\pi} \text{spec}(B)$ is surjective.
2. $\pi(\text{maximal ideal}) = \text{maximal ideal}.$
3. $\dim(A) = \dim(B)$.

**Fibers of finite extensions**

\[
\begin{array}{ccc}
A & \xrightarrow{\text{finite}} & B \\
\downarrow & & \downarrow \\
A_p & \xrightarrow{\text{finite}} & B \otimes A k(p) \\
\downarrow & & \downarrow \\
k(p) & \xrightarrow{\text{finite}} & B \otimes A k(p)
\end{array}
\]

1. $B \otimes_A k(p)$ has a finite number of maximal ideals,
2. $B \otimes_A A_p$ is a semi-local ring.

$\dim_{\text{Krull}} B \otimes_A k(p) = \dim_{\text{Krull}} k(p) = 0$
A closer look at the fibers of a finite extension

Example

\[ k[X] \xrightarrow{\text{finite}} k[X, Y]/\langle (Y^2 - X)(Y - X) \rangle \]

▷ Fibers over closed points
  - \( p = \langle X - 1 \rangle \)
  - \( p = \langle X - 2 \rangle \)

▷ Generic fiber
  - \( p = (0) \)

Facts

\( k \) a field. \( k \xrightarrow{-} L \) a finite extension. Then

▷ maximal ideals of \( L = \{ n_1, \ldots, n_s \} \)

▷ \( L = L_{n_1} \oplus \cdots \oplus L_{n_s} \)

Conclusion

Fibers of finite extensions break into direct sums of local rings.