Universidad Autónoma de Madrid

Commutative Algebra for Singular Algebraic Varieties

Some tools of commutative algebra

O. E. Villamayor U.

Obergurgl, June 2012

Localization

Let A be a ring and

$S \subset A$ a multiplicative set ($1 \in S$ and if $s_1, s_2 \in S$ then $s_1 \cdot s_2 \in S$).

$A_S = \left\{ \frac{a}{s}, a \in A, s \in S \right\}$ where $\frac{a}{s} = \frac{a'}{s'}$ in $A_S \Leftrightarrow t(s'a - a's) = 0$ in A.

The homomorphism $\begin{array}{ccc} A & \xrightarrow{\phi} & A_S \\ a & \longmapsto & \frac{a}{1} \end{array}$ satisfies

a $\phi(S) \subset U(A_S)$,

b **Universal Property of the Localization**

If $A \xrightarrow{\beta} B$ maps S in $U(B)$ then there is a unique β':

$$\begin{array}{ccc} A & \xrightarrow{\quad\beta\quad} & B \\ & {\scriptstyle\phi}\searrow \quad \circlearrowleft \quad \nearrow{\scriptstyle\beta'} & \\ & A_S & \end{array}$$

Notation

- *If $S = A \setminus \mathfrak{p}$, $A_S = A_{\mathfrak{p}}$*
- *If $S = \{1, f, \ldots, f^n, \ldots\}$, $A_S = A_f$.*

A topological space attached to a ring, spec

As a set,

$$\mathrm{spec}(A) = \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } A\}$$

A topology is defined. Let $I \subset A$ an ideal,

$$V(I) := \{\mathfrak{p} \in \mathrm{spec}(A) : \mathfrak{p} \supset I\}$$

is a closed subset in $\mathrm{spec}(A)$.

Example

- $\mathrm{spec}(A/I)$"="$V(I) \subset \mathrm{spec}(A)$,
- $\mathrm{spec}(A_S) \subset \mathrm{spec}(A)$, the set of prime ideals $\mathfrak{p} \subset A$ s. t. $\mathfrak{p} \cap S = \emptyset$.
- $\mathrm{spec}(A_f)$ is an open set of $\mathrm{spec}(A)$.

Krull dimension

- We can construct chains of primes in $\mathrm{spec}(A)$:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \ldots \subsetneq \mathfrak{p}_m \text{ (of length } m\text{)},$$

- $\mathrm{spec}(A)$ has a finite number of minimal primes, since A a Noetherian ring.

Krull dimension of A

$\dim_{\mathrm{Krull}} A =$ is the length of the maximal chain of prime ideals of A.

Continuous functions

Let $A \xrightarrow{\beta} B$ be a homomorphism.

$$\mathfrak{q} \subset B \text{ a prime ideal} \Rightarrow \beta^{-1}(\mathfrak{q}) \subset A \text{ is a prime ideal.}$$

We can define

$$\begin{array}{ccc} \mathrm{spec}(A) & \xleftarrow{\pi} & \mathrm{spec}(B) \\ \beta^{-1}(\mathfrak{q}) & \longleftarrow\!\shortmid & \mathfrak{q} \end{array}$$

The preimage of a closed set is a closed set

$$\begin{array}{ccc} \mathrm{spec}(A) & \xleftarrow{\pi} & \mathrm{spec}(B) \\ \cup & & \cup \\ V(I) & & V(IB) \end{array}$$

Interesting case

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ \downarrow & & \downarrow \\ A_S & \longrightarrow & B_{\beta(S)} \end{array}$$

$$\begin{array}{ccc} \mathrm{spec}(A) & \xleftarrow{\pi} & \mathrm{spec}(B) \\ \cup & & \cup \\ \mathrm{spec}(A_S) & & \mathrm{spec}(B_{\beta(S)}) \end{array}$$

Fibers

We can take the localization and then take the quotient

$$
\begin{array}{ccccc}
B & \longrightarrow & B \otimes_A A_{\mathfrak{p}} & \longrightarrow & B \otimes_A A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p}) \; = B \otimes_A k(\mathfrak{p}) \\
\uparrow & & \uparrow & & \uparrow \\
A & \longrightarrow & A_{\mathfrak{p}} & \longrightarrow & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = k(\mathfrak{p})
\end{array}
$$

Or we can take the quotient, and then localize

$$
\begin{array}{ccccc}
 & & B/\mathfrak{p}B & & B/\mathfrak{p}B \otimes_{A/\mathfrak{p}} (A/\mathfrak{p})_{\mathfrak{p}} = B \otimes_A k(\mathfrak{p}) \\
 & & \| & & \| \\
B & \longrightarrow & B \otimes_A A/\mathfrak{p} & \longrightarrow & B \otimes_A A/\mathfrak{p} \otimes_{A/\mathfrak{p}} (A/\mathfrak{p})_{\mathfrak{p}} \\
\uparrow & & \uparrow & & \uparrow \\
A & \longrightarrow & A/\mathfrak{p} & \longrightarrow & (A/\mathfrak{p})_{\mathfrak{p}} = k(\mathfrak{p})
\end{array}
$$

Finite extensions

Definition

$A \longrightarrow B$ finite $\Leftrightarrow$ B is finite as an A-module.

$\Leftrightarrow$ $B = A[b_1, \ldots, b_r]$ where b_i is integral over A.

Interesting cases

1.

$$\begin{array}{ccc} A & \xrightarrow{\text{finite}} & B \\ \downarrow & & \downarrow \\ A/I & \xrightarrow{\text{finite}} & B \otimes_A (A/I) = B/IB \end{array}$$

2.

$$\begin{array}{ccc} A & \overset{\text{finite}}{\hookrightarrow} & B \\ \downarrow & & \downarrow \\ A_S & \overset{\text{finite}}{\hookrightarrow} & B \otimes_A A_S = B_S \end{array}$$

Example

$$\mathbb{R}[X] \hookrightarrow \mathbb{R}[X,Y]/\langle Y^2 - X \rangle$$

Let $A \subset B$ be a finite extension,

Facts:

1. $\operatorname{spec}(A) \xleftarrow{\pi} \operatorname{spec}(B)$ is surjective.
2. $\pi(\text{maximal ideal}) = \text{maximal ideal}$.
3. $\dim(A) = \dim(B)$.

Fibers of finite extensions

$$\begin{array}{ccc} A & \xhookrightarrow{\text{finite}} & B \\ \downarrow & & \downarrow \\ A_{\mathfrak{p}} & \xhookrightarrow{\text{finite}} & B \otimes k(\mathfrak{p}) \\ \downarrow & & \downarrow \\ k(\mathfrak{p}) & \xhookrightarrow{\text{finite}} & B \otimes_A k(\mathfrak{p}) \end{array}$$

$$\dim_{\text{Krull}} B \otimes_A k(\mathfrak{p}) = \dim_{\text{Krull}} k(\mathfrak{p}) = 0$$

1. $B \otimes_A k(\mathfrak{p})$ has a finite number of maximal ideals,
2. $B \otimes_A A_{\mathfrak{p}}$ is a semi-local ring.

A closer look at the fibers of a finite extension

Example

$$k[X] \xhookrightarrow{\text{finite}} k[X,Y]/\langle (Y^2 - X)(Y - X)\rangle$$

- Fibers over closed points
  - $\mathfrak{p} = \langle X - 1 \rangle$
  - $\mathfrak{p} = \langle X - 2 \rangle$
- Generic fiber
  - $\mathfrak{p} = (0)$

Facts

k a field. $k \hookrightarrow L$ a finite extension. Then

- maximal ideals of $L = \{\mathfrak{n}_1, \ldots, \mathfrak{n}_s\}$
- $L = L_{\mathfrak{n}_1} \bigoplus \cdots \bigoplus L_{\mathfrak{n}_s}$

Conclusion

Fibers of finite extensions break into direct sums of local rings.