RESOLUTION OF SINGULARITIES PART (4B) INDUCTIVE PROCEDURE FOR GLOBAL DESINGULARIZATION

This is a continuation to the first portion of PART IV. Please make some references also to the PART I and PART II made available by the courtesy of the Clay Math.

1. Reduction from $/^q$ to $/^p$

We focus our attention to the case in which the given E is a $/q$-exponent in the A-site $\mathfrak{Z} = (\mathcal{I}, \mathcal{Z}, \varpi)$ furnished with an NC-data Γ.

The embedding dimnesions $embd(E)$ and $embd(E, \Gamma)$ of the A-site $\mathfrak{Z}$ are the first important invariant used in our strategy towards resolution of singularities of E. The first result was that there exists a finite sequence of permissible blowups such that

(1) $embd(E)$ and $embd(E, \Gamma)$ are monotone nonincreasing,
(2) if $embd(E, \Gamma) > embd(E)$ then we can make $embd(E, \Gamma)$ strictly smaller
(3) and finally we reach the state in which the two embed numbers coincide.

Thus we may and will begin with the assumption that $\mathfrak{Z}$ is (E, Γ)-saturated as well as E-saturated and $embd(E, \Gamma) = embd(E)$, say $= d(0)$.

Definition 1.1. The *cotangent bundle* $CT_{\mathfrak{Z}}$ of an A-site $\mathfrak{Z}$ is the collection of the cotangent bundles $\{CT(i) = CT_{\varpi(i)}\}$ of the $\varpi(i), i \in \mathcal{I}$, which are associated with the pullback map $CT(m) : CT(j) \to CT(i)$ associated with the morphisms $\varpi(m) : \varpi(i) \to \varpi(j)$ for $(m : i \to j) \in mor(\mathcal{I})$.

The given $/^q$-exponent E will be renamed $\mathcal{G}(0)$ to indicate our starting point. Locally at a "closed" site-point $P = (i, \xi)$ we pick an abc-expression of $\mathcal{G}(0)$ as $\mathcal{G}(0, i, \xi) = (\mathbf{g}(0, i, \xi) \| /^q)$ where

$$\mathbf{g}(0, i, \xi) = z(0, i, \xi)^{q\mathbf{a}(0,i,\xi)} v(0, i, \xi)^{\mathbf{b}(0,i,\xi)} g(0, i, \xi)$$

with a cofactor $v(0, i, \xi)^{\mathbf{b}(0,i,\xi)}$ and with a residual factor $g(0, i, \xi)$ whence $ord_\xi(g(0, i, \xi)) = resord_\xi(\mathcal{G}(0, i, \xi))$. Recall that we are choosing a regular system of parameters $x(0, i, \xi) = (w(0, i, \xi), v(0, i, \xi), \omega(0, i, \xi))$ of $R_\xi = \mathcal{O}_{\varpi(i),\xi}$ in the manner of standard abc-expression.

We are now focusing our attention to local problems at $P = (i, \xi)$ and we will often drop $(0, i, \xi)$ from the parameters $w(0, i, \xi), v(0, i, \xi), \omega(0, i, \xi)$ for the sake of notational simplification. As a matter of fact we will need to modify these parameters many times in search of their better selection in order to elucidate much deeper properties of $\mathcal{G}(0)$. Our process will be named "$\sharp/^q$-filtering of $\mathcal{G}(0)$.

The cofactor parameters $v = v(0, i, \xi)$ will be subject to repeated modification which replace each component by unit multiplication. A change of v by this kind of modification will be said to be *allowable*. They are local equations of the globally defined NC-data Γ throughout the A-site $\mathfrak{Z}$, consisting of the normal crossing $\Gamma(i)$ in each smooth ambient scheme $\varpi(i)$ with $i \in \mathcal{I}$. We let $V(0, i, \xi) = v^{\mathbf{b}(0,i,\xi)} g(0, i, \xi)$ which we call the *core ideal* of the chosen *abc*-expression of $\mathcal{G}(0)$ at P.

We want to set up a global strategy of reducing singularities and examine its local effect in terms of an *abc*-expression of a given exponent $E = \mathcal{G}(0)$ and NC-data $\Gamma = \Gamma(0)$ in the A-site $\mathfrak{Z} = \mathfrak{Z}(0)$. We let $R = \mathcal{O}_{\mathfrak{Z}}$ of which the stalk $\mathcal{O}_{\varpi(i),\xi}$ at $P = (i, \xi)$ is denoted by R_ξ.

Let us note that there exists a globally defined $/^q$-exponent $\mathcal{V}(0)$ in the A-site $\mathfrak{Z}$ as follows.

$$\begin{aligned} \mathcal{V}(0) \ &= \ \{\mathcal{V}(0, i), i \in \mathcal{I}\} \ \ \textit{such that} \\ (V(0, i, \xi) \| /^q) \ &= \ \mathcal{V}(0, i)_\xi \ \ \textit{which is the stalk at } \xi \end{aligned} \tag{1.1}$$

The reason for the existence is that the multiplication by elements of $\rho^e(\mathcal{O})_{\mathfrak{Z}}$ commutes with equivalent ambient reductions as well as etale morphisms which belong to the category $\mathfrak{Z}$.

We now proceed to build up what we will call the $\sharp/^q$*-filtering* of $\mathcal{G}(0)$, firstly locally about a chosen closed point $\xi \in \varpi(i)$. Its globalization will be examined afterward through the A-site $\mathfrak{Z}$.

2. $/^p$-DECOMPOSITION, $\sharp$ VS $\flat$

We continue with the case in which the exponent E in the A-site $\mathfrak{Z}$ is a $/^q$-exponent. Furthermore this time we restrict our interest to the case of $q = p$, which may appear too special but actually very important key step in our strategy toward resolution of singularities in the characteristic $p > 0$. Moreover we will be assuming $d = \tilde{d}$ in the sense of Eq.(**??**) with reference to Def.(**??**).

$$(2.1) \qquad \textit{Begin with } \mathfrak{Z}, \ E\textit{-saturated as well as } (E, \Gamma)\textit{-saturated.}$$

Our task of reduction of singularitiy is firstly to be concentrated within a neiborhood of a connected component C of $Sing(E[d])$. Thus the centers of permissible blowups must be all chosen inside the inverse images of C so that they do not affect any of the other connected components at all. We perform the same on other components, one after another, untill we reach the state in which the final transform of E has its saturated *embedding dimension* strictly smaller than the original d. Our induction will be based upon the embedding dimensions.

Definition 2.1. We define $E[C\triangleright] \supset E[C]$ to be the $/^p$-exponent in $\mathfrak{Z}[d\triangleright] \supset \mathfrak{Z}[d]$ which are obtained from $E[d\triangleright] \supset E[d]$ by imposing unit to every one of those ideals within a neighborhood of $Sing(E[d\triangleright]) \setminus C[d\triangleright]$, so that we have $C[d\triangleright] = Sing(E[C\triangleright])$.

With the dimension level $d = \tilde{d}$ of Eq.(**??**) we will assume with no loss of generality that

$$(2.2) \qquad \mathfrak{Z} = \mathfrak{Z}[d\triangleright], \ E = E[d\triangleright] \ \textit{and} \ \Gamma = \Gamma[d\triangleright]$$

The connected component C of $Sing(E[d])$ will be selected in such a way that

$$(2.3) \qquad embd(E[C\triangleright]) = embd(E[d\triangleright]) = embd(E) = d$$

Note that such C always exists. Let $C\triangleright \supset C$ be the unique extension of C into $\mathfrak{Z}[d\triangleright]$ and $C\triangleright$ is then a connected component of $Sing(E[d\triangleright])$. incidentally we set aside the other connected components with $<$ in Eq.(2.3). Note that $=$ or $<$ thereof remains unchanged by any saturation.

Definition 2.2. We next define the "checked associate" of $E[C\triangleright] \supset E[C]$, defined firstly for $E[C]$ in $\mathfrak{Z}[d]$ as follows and by unique extension into $\mathfrak{Z}$.

$$\begin{aligned} \mathcal{G}(C,0) &= \mathfrak{z}^{-p\beta(C,0)}E[C] \\ \textit{with}\ \ \beta(C,0) &= \alpha(C,0)-\gamma(C,0) \\ \textit{out of the } \Gamma\textit{-maximal divisor}\ &\mathfrak{z}^{p\alpha(C,0)}\ \ \textit{of}\ \ E[C] \\ \textit{where}\ \beta(C,0)\in\mathbb{Z}_0^t\ \ &\textit{and}\ \ 0\le\gamma(C,0)_k<1,\ \forall k \end{aligned} \tag{2.4}$$

This is in accord with the notation of Th.(**??**) by terms of the ideals $\mathfrak{z}$ of Γ in the sense of Def.(**??**).

Definition 2.3. $\mathfrak{z}^{p\gamma(C,0)}$ is called the Γ-cofactor of dimension level d of the exponnent $E[C]$ along C. We rewrite it as $v(C,0)^{\mathbf{c}(C,0)}$ with $0<\mathbf{c}(C,0)_k<p$. Namely we discard all the zero components out of $p\gamma(C,0)$ and write the result by $\mathbf{c}(C,0)$, while $v(C,0)$ denotes the necessary subsystem of $\mathfrak{z}$. We call $v(C,0)$ the *cofactor parameters* of $E[C]$.

The $\mathfrak{z}^{-p\beta(C,0)}E[C]$ means the $/^p$-exponent $(\mathfrak{z}^{-p\beta(C,0)}\mathcal{E}[C]\,\|\,/^p)$ with respect to a representation $E[C]=(\mathcal{E}[C]\,\|\,/^p)$ in $\mathfrak{Z}[d]$, so chosen as to have $\mathcal{E}[C]$ divisibe by $\mathfrak{z}^{p\alpha(C,0)}$ in $\mathcal{O}_{\mathfrak{Z}[d]}$. Clearly we have

$$\begin{aligned} &\mathfrak{z}^{p\gamma(C,0)}\ \ \textit{is the maximal}\ \Gamma\textit{-divisor of}\ \ \mathcal{G}(C,0) \\ &\textit{and it will also be denoted by}\ \ \mathfrak{z}(C,0)\ \ \textit{for short.} \end{aligned} \tag{2.5}$$

We write

$$\mathcal{G}(C,0)=(\mathbf{g}(C,0)\|\,/^p)\quad\textit{with}\ \ \mathbf{g}(C,0)=\mathfrak{z}(C,0)\mathbf{h}(C,0) \tag{2.6}$$

where $\mathbf{h}(C,0)$ is called "residual factor" of $\mathcal{G}(C,0)$, as well as the same of $E[C]$, along C. Here it should be kept in mind that there remains $/^p$-ambiguity in $\mathbf{h}(C,0)$ and hence in $\mathbf{g}(C,0)$, that is, up to "$/^p$-equivalence".

The $\mathfrak{z}(C,0)$ is a global ideal in $\mathfrak{Z}[d]$, which is the Γ-cofactor of $\mathcal{G}(C,0)$, as well as the same of $E[C]$ along C. For each closed point $P=(i,\xi)\in C$ we choose a *-full idempotent differential operator $\mathfrak{d}^*(z)$ with respect to a regular system of parameters z of R_ξ including generators of those $\mathfrak{z}_k$ vanishing at ξ. Recall that the operator $\mathfrak{d}^*(z)$ is locally defined as follows. Given $z=(z_1,\cdots,z_n)$ we have idempotent differential operators which are $\rho^e(R_\xi)$-endomorphisms of $\rho(R_\xi)$, denoted by $\mathfrak{d}^\alpha,\alpha\in\mathbb{Z}_0^n$ with $0\le\alpha_j<p^e$ for all j, such that $\mathfrak{d}^\alpha(z^\beta)=z^\beta$ if $\beta=\alpha$ and $=0$ if otherwise. Then $\mathfrak{d}^*(z)$ is the sum of those $\mathfrak{d}^\alpha$ except the identity. Incidentally $*$ differential operators usually means to vanish onon $\rho^e(R_\xi)$. $\mathfrak{d}^*(z)$ depends upon the choice of z but only up to differences in $\rho^e(R_\xi)$.

A decomposition of the $/^p$-exponent is firstly "locally defined" at each $P = (i, \xi) \in \mathfrak{Z}_{cl}$ as follows.

(2.7) *Decompose $\mathcal{G}(C,0)_P$ into the following two parts*

$$\mathcal{G}^\sharp(C,0)_P \;=\; \Big(\mathfrak{z}(C,0)\mathfrak{d}^*(z)\big(\mathfrak{z}(C,0)^{-1}\mathbf{g}(C,0)\big) \parallel /^p\Big)$$

and

$$\mathcal{G}^\flat(C,0)_P \;=\; \Big(\mathfrak{z}(C,0)(id - \mathfrak{d}^*(z))\big(\mathfrak{z}(C,0)^{-1}\mathbf{g}(C,0)\big) \parallel /^p\Big)$$

where each of the two parts is well defined as a global $/^p$-exponent in the A-site $\mathfrak{Z}$ although the above representation ideals are only up to addition of elements of $\rho(R_\xi)$.

Theorem 2.1. *The decomposition of Eq.(2.7) has the following properties.*

1. *(1) There exist globally defined $/^p$-exponents $\mathcal{G}^\sharp(C,0)$ and $\mathcal{G}^\flat(C,0)$ in the A-site $\mathfrak{Z}[d]$ such that Eq.(2.7) is their local presentations at every $P \in \mathfrak{Z}[d]_{cl}$.*
2. *(2) In the local expression Eq.(2.7) of $\mathcal{G}^\flat(C,0)$, its ideal is its cofactor multiplied by an ideal in $\rho(\mathcal{O}_{\mathfrak{Z}})$.*
3. *(3) $\mathcal{G}^\flat(C,0)$ is trivial, i.e., $= ((0)\|/^p)$, if $\gamma(C,0) = (0)$ and hence the cofactor $\mathfrak{z}(C,0)$ is unit.*
4. *(4) In general $\mathcal{G}^\flat(C,0)$ is "stable" in the sense that after every equence of permissible blowups its transform has ideal which is NC-divisor times ideals in $\rho(\mathcal{O}_{\mathfrak{Z}})$. Here the NC-divisor is either non-trivial Γ-divisor or the ideal of a smooth irreducible hypersurface which has normal crossings with Γ-system.*

We then proceed to define $\mathcal{G}(C,1)$ after Eq.(2.7).

(2.8)

$$\mathcal{G}(C,1) \;=\; \mathcal{G}^\sharp(C,0) \;=\; (\mathbf{g}(C,1) \parallel /^p)$$
$$\textit{with}\;\; \mathbf{g}(C,1) \;=\; v(C,0)^{\mathbf{c}(C,1)}\mathbf{h}(C,1)$$
$$\textit{where}\;\; \mathbf{c}(C,1) \in \mathbb{Z}_0^s,\; s = |v(C,0)|,\;\; \textit{is chosen}$$
$$\textit{to be the maximum having residual}\;\; \mathbf{h}(C,1) \in \mathcal{O}_{\mathfrak{Z}}$$
$$\textit{We write}\; \mathfrak{z}(C,1) \;=\; v(C,0)^{\mathbf{c}(C,1)}\; \textit{for short}$$

As for $\mathbf{c}(C,1)$ we may not have the component-wise strict inequalities of Eq.(2.4) which was imposed only upon the initial $\mathbf{c}(C,0)$. $\mathcal{G}(C,1)$ is then decomposed into $\mathcal{G}^\sharp(C,1)$ and $\mathcal{G}^\flat(C,1)$ as follows.

The next task is there a further decomposition of $\mathcal{G}^\sharp(C,0)$. We will repeat a "similar" process to produce $\mathcal{G}(C,k), k = 1, 2, \cdots$, successively, starting from $\mathcal{G}(C,0)$ of Eqs.(2.4)-(2.6) and making use of its decomposition of Eq.(2.7). However we will work under a certain new constrant, that is to maintain the original cofactor parameters $v(C,0)$ of Def.(2.3) chosen for the initial exponent $E(C)$ of Eq.(2.2) after Def.(2.1). Namely we define and express

(2.9) *Decompose* $\mathcal{G}(C,1)_P$ *into the two parts*

$$\mathcal{G}^\sharp(C,1)_P \;=\; \Big(\mathfrak{z}(C,1)\mathfrak{D}^*(z)\big(\mathfrak{z}(C,1)^{-1}\mathbf{g}(C,1)\big) \parallel /^p\Big)$$

and

$$\mathcal{G}^\flat(C,1)_P \;=\; \Big(\mathfrak{z}(C,1)(id-\mathfrak{D}^*(z))\big(\mathfrak{z}(C,1)^{-1}\mathbf{g}(C,1)\big) \parallel /^p\Big)$$

Theorem 2.2. *There exist globally defined* $/^p$*-exponents* $\mathcal{G}^\sharp(C,1)$ *and* $\mathcal{G}^\flat(C,1)$ *in* $\mathfrak{Z}[d]$ *having their local presentations Eq.(2.9) at every* $P \in \mathfrak{Z}[d]_{cl}$ *as follows.*

1. *(1) In the local expession Eq.(2.9) of* $\mathcal{G}^\flat(C,1)$*, its ideal is* $\mathfrak{z}(C,1)$ *multiplied by an ideal in* $\rho(\mathcal{O}_{\mathfrak{Z}})$*.*
2. *(2) In general* $\mathcal{G}^\flat(C,1)$ *is "stable" in the same sense as in Th.(2.2).*
3. *(3)* $\mathcal{G}^\flat(C,1)_P$ *is trivial and* $\mathcal{G}^\sharp(C,1)_P = \mathcal{G}(C,1)_P$ *if and only if* $ord_P(\mathcal{G}^\sharp(C,1)) = ord_P(\mathcal{G}(C,1))$ *and* $\mathbf{c}(C,1) = \mathbf{c}(C,0)$

Definition 2.4. Define $\mathcal{G}(C,2) = \mathcal{G}^\sharp(C,1)$ if either $ord_P(\mathcal{G}^\sharp(C,1)) > ord_P(\mathcal{G}(C,1))$ or $\mathbf{c}(C,1) > \mathbf{c}(C,0)$. This is the case when $\mathcal{G}^\sharp(C,1) \neq \mathcal{G}(C,1))$

If $\mathcal{G}^\sharp(C,1) = \mathcal{G}(C,1))$ then the decomposition of $\mathcal{G}(C,1))$ similar to Eq.(2.9) is trivial and useless. In this case we need to emply a new opeation in order to obtain a better way of defining $\mathcal{G}(C,k), k > 1$, We will do this in the next section,

3. Cotangential flags local-global relation

Assume that we have already defined $\mathcal{G}(C,k), k \geq 1$, as follows (eg. k=1 from the previous section) and we will restart from there.

$$\begin{gathered} \mathcal{G}(C,k) = (\mathbf{g}(C,k) \,\|\, /^{p}) \\ \textit{with some } k \geq 1 \\ \textit{such that } Sing(\mathcal{G}(C,k) \subset C \textit{ is a closed subscheme} \\ \textit{and } \mathcal{G}(C,k) \;=\; \mathcal{G}^{\sharp}(C,k)). \end{gathered} \tag{3.1}$$

We also have $\mathfrak{z}(C,k) = v(C,0)^{\mathbf{c}(C,k)}$ defined in the same way as Eq.(2.8) with the original cofactor variables $v(C,0)$.

Definition 3.1. Pick any $P = (i,\xi) \in C_{cl}$ and define the following numbers.

$$\begin{aligned} \mu(C,k,P) \;&=\; ord_P(\mathcal{G}(C,k)) - ord_P(\mathfrak{z}(C,k)) \quad \textit{and} \\ \mu(C,k) \;&=\; max_{P\in C}\{\,\mu(C,k,P)\,\},\ \textit{say } \mu \textit{ for short} \\ \theta(C,k) \;&=\; [\mu/p],\ \textit{say } \theta \textit{ for short} \end{aligned} \tag{3.2}$$

Definition 3.2.

$$\begin{gathered} \textit{Define } \; S(\mu,C,k) \;=\; S(max,C,k) \;\; \textit{to be the subscheme} \\ \textit{such that } \; S(\mu,C,k)_{cl} \;=\; \{\,P \in C_{cl} \,\big|\, ord_P(\mathcal{G}(C,k)) = \mu\,\} \end{gathered}$$

And denoting $\rho^{j}(\mathcal{O}_{\mathfrak{Z}})$ by $\mathcal{O}(j)$ for short for $\forall j$, we then define

Definition 3.3.

$$\textit{for every } \; \nu > 0 \;\; \textit{in particular } \; \nu = \mu$$

$$\begin{gathered} \mathcal{D}(\nu,C,k,\iota) \;= \\ \mathfrak{z}(C,k)\, Diff^{(\nu-p^{\iota}-1)}_{\mathcal{O}(0)/\mathcal{O}(\theta)}\; Der_{\mathcal{O}(0)}\Big(\mathfrak{z}(C,k)^{-1}\mathbf{g}(C,k)\Big) \\ \textit{where } \; 0 \leq \iota \leq \theta \end{gathered} \tag{3.3}$$

$$\textit{Define the coherent ideal}$$

$$\begin{gathered} \mathcal{D}(\nu,C,k) \;=\; \textstyle\sum_{|\mathbf{m}|=\nu} \prod_{\iota=0}^{\theta} \mathcal{D}(C,k,\iota)^{\mathbf{m}_{\iota}} \\ \textit{where } \; |\mathbf{m}| \;=\; \textstyle\sum_{\iota} \mathbf{m}_{\iota} \end{gathered} \tag{3.4}$$

Note that $\mathcal{D}(\nu,C,k), \forall \nu \geq \mu$, have their supports inside $S(max,C,k)$.

Remark 3.1. Let us review the "local" version of the cotangential flag structure of $I = \mathbf{g}(C,k) \subset R_{\xi}$ at a closed point $P = (i,\xi) \in C$. It is a sequence of κ_{ξ}-submodules of

$$L = M_{\xi}/M_{\xi}^{2} \subset gr_{\xi}(R_{\xi}) = \kappa_{\xi}[L]$$

with $M_\xi = max(R_\xi)$ and $\kappa_\xi = R_\xi/M_\xi$, and it is as follows.

$$(0) \;=\; L(0) \;\subset\; L(1) \;\subset\; \cdots \;\subset\; L(\theta) \;\subset\; L \tag{3.5}$$

which are characterised by the following properties:

(1) There exist $f(\iota)_i \in I$ and $\partial(\iota)_i \in Diff^{d-p^\iota}_{R_\xi/\mathbb{K}}$ such that

$$\begin{gathered} ord_\xi(f(\iota)_i) \;=\; p^\iota \;\; \textit{for all } i \textit{ and} \\ \exists w(\iota)_i \in L(\iota) \textit{ such that } in_\xi(f(\iota)_i) - w(\iota)_i^{p^\iota} \in \kappa_\xi[L(\iota-1)] \\ \textit{and } \; \{w(\iota)_i, \forall i\} \textit{ freely generate } L(\iota)/L(\iota-1) \end{gathered} \tag{3.6}$$

(2) The initial $in_\xi(I)$, that is ($I \mod M_\xi^{\mu+1}$), is a homogeneous ideal of degree μ and

$$in_\xi(I) \;\subset\; \kappa[\,\rho^\iota(\,L(\iota)\,), 0 \leq \iota \leq \theta\,] \tag{3.7}$$

Here is an important point in the local-global relation with regards to the cotagential flag structure.

Theorem 3.1. *The $\mathcal{D}(C,k)$ is a global coherent ideal in the A-site $\mathfrak{Z}$. The local initial $in_P(\mathbf{g}(C,k))$ of Eq.(3.7) is contained in the initial of the global coherent ideal $\mathcal{D}(C,k)$ of Eq.(3.3).*

4. IDEALISTIC SUM-DIVISION

$$\mathcal{G}(C,k+1) \;=\; \Big(\mathcal{G}(C,k)/\big(\rho^{\theta}(R_\xi)[\mathcal{D}(C,k,P)]\big)\Big)$$
$$\textit{meaning}$$
$$ord_\xi\Big(\;\mathbf{g}(C,0)(i)\,/\,\big(\rho(R_\xi)+\mathcal{D}(C,0,i)^{\mu}\big)\;\Big)$$
$$\textit{where } R_\xi = \mathcal{O}_{\varpi(i),\xi} \textit{ for the given } P=(i,\xi)$$

(4.1)
$$\textit{We decompose } \mathcal{G}^{\sharp}(C,1) \quad \textit{into}$$
$$\mathcal{G}^{\dagger}(C,1)_P \;=\; \big(\mathfrak{z}^{\dagger}(C,1)\,\|\,/^{p})_P\big)$$
$$\textit{where } \mathfrak{z}^{\dagger}(C,1) \;=\; \mathbf{g}^{\sharp}(C,1) \sqcap \rho^{\theta}(R_\xi)[\mathcal{D}(C,1)\mathbf{g}^{\sharp}(C,1)]$$
$$\textit{and}$$
$$\mathcal{G}^{\dagger}(C,1)_P \;=\; \big(\mathfrak{z}^{\dagger}(C,1)\,\|\,/^{p})_P\big)$$
$$\textit{where } \mathfrak{z}^{\ddagger}(C,1) \;=\; \mathbf{g}^{\sharp}(C,1) \boxminus \rho^{\theta}(R_\xi)[\mathcal{D}(C,1)\mathbf{g}^{\sharp}(C,1)]$$

We write $\mu(0)$ for the μ above in order to distinguish it from the numbers $\mu(k), k=1,2,\cdots$ which we define below. Similarly we write $\mathcal{G}(0)$, $\mathbf{g}(0)$ and $\mathcal{D}(0)$ for $\mathcal{G}(C,0)$, $\mathbf{g}(C,0)$ and $\mathcal{D}(C,0)$ respectively. We will write $\mathbf{S}$ for $\mathbf{S}(C,0)$.

now for each k we proceed to define inductively the following symbols.

$$\begin{aligned} \mu(k) &= min_{P\in \mathbf{S}}\{ord_P(\mathcal{G}(k-1)/\mathcal{D}(k-1)^{\mu(k-i)}\} \\ \mathcal{D}(k,i) &= \mathcal{D}^{\sharp}(k,i) + \mathcal{D}^{\flat}(k,i) \\ &\mathcal{D}^{\sharp}(k,i)\ Diff_{\varpi(i)}^{(\mu(k)-2)}\mathfrak{d}^* Der_{\varpi(i)}(\nabla(k,i)) \\ &\mathcal{D}^{\flat}(k,i)\ Diff_{\varpi(i)}^{(\mu(k)-2)}(id-\mathfrak{d}^*) Der_{\varpi(i)}(\nabla(k,i)) \\ &\mathcal{D}(C,1,P)\ for\ \big(\mathcal{D}(C,0)\big)(P) \\ &wih\ \nabla(k,i) = M_P^{\mu(k)}\bigcap\Big(\mathbf{g}(k-1)+\mathcal{D}(k-1,i)^{\mu(k-1)}\Big) \end{aligned} \tag{4.2}$$

where M_P denotes

For $\ell \geq p$ we have a "subscheme" $Sing(\mathcal{G}(C,0),\ell) \subset \mathfrak{Z}[d]$ such that

$$Sing(\mathcal{G}(C,0),\ell)_{cl} = \{P\in\mathfrak{Z}[d]_{cl}\,\big|\,ord_P(\mathcal{G}(C,0))\geq \ell\} \tag{4.3}$$

where $ord_P(\mathcal{G}(C,0)) = min_{\mathbf{g}(C,0)}\{ord_{\xi}(\mathbf{g}(C,0))\}$ for $P=(i,\xi)$. Note that the equality Eq.(4.3) is only for "closed points" and it is not true in general if we include nonclosed points. As a matter of fact we may have "generic down" phenomena, that is, for a generic point Q of the subscheme $Sing(\mathcal{G}(C,0),\ell)$ we may have $ord_Q(\mathcal{G}(0))$ strictly smaller than ℓ. However for the special case of $\ell=p$ we have proven that $Sing(\mathcal{G}(C,0),p)$ of Eq.(4.3) is true without $\{cl\}$. Thus $Sing(\mathcal{G}(C,0),p) = Sing(\mathcal{G}(C,0))[d] = Sing(\mathcal{G}(C,0)[d])$ as subschemes of $\mathfrak{Z}[d]$.

We define the following maximum order and its locus.

$$\begin{aligned} &maxd(\mathcal{G}(C,0)) = max\{\ell\big|\,Sing(\mathcal{G}(C,0),\ell)\neq\emptyset\}\quad and \\ &Sing(\mathcal{G}(C,0),max) = Sing(\mathcal{G}(C,0),\mu)\quad with\quad \mu = maxd(\mathcal{G}(C,0)) \\ &\mathcal{D}(C,0,P) = \big(\mathcal{D}(C,0)\big)(P) \end{aligned} \tag{4.4}$$

(4.5)

If $maxd(\mathcal{G}(C,0)) = \infty$ then $Sing(\mathcal{G}(C,0) = \mathfrak{Z}[d]$ and $\mathcal{G}(C,0)$ must be zero exponent. In what follows we will be excluding the trivial case.

Write $\mathbf{S}(C,0)$ for $Sing(\mathcal{G}(C,0), max)$ and $\mu(C,0)$ for $maxd(\mathcal{G}(C,0))$.

We will define the cotangential derivative, derivative for short,

$$\mathcal{D}(C,0) \;=\; \mathcal{D}^{\sharp}(C,0) \;+\; \mathcal{D}^{\flat}(C,0) \tag{4.6}$$

of $\mathcal{G}(C,0)$ along $\mathbf{S}(C,0)$ and its image $\tau(C,0)$ into the cotangent bundle $CT_{\mathfrak{Z}}$ of $\mathfrak{Z}$ as follows. Let $\mu = \mu(C,0)$.

$$\begin{gathered} \textit{define}\;\; \mathcal{D}^{\sharp}(C,0,i) \;=\; Diff^{(\mu-2)}_{\varpi(i)}\mathfrak{d}^{*}Der_{\varpi(i)}(\mathbf{g}(C,0)) \\ \textit{with a *-full idempotent } \mathfrak{d}^{*} \;\; \textit{and} \\ \textit{define}\;\; \mathcal{D}^{\flat}(C,0,i) \;=\; Diff^{(\mu-2)}_{\varpi(i)}(id-\mathfrak{d}^{*})Der_{\varpi(i)}(\mathbf{g}(C,0)) \\ \textit{let}\;\; (\tau(C,0))(i) \subset CT_{\varpi(i)} \;\; \textit{be the image of } \mathcal{D}(C,0,i). \\ \textit{In other words} \\ (\tau(C,0))(i) \;=\; \Big(\mathcal{D}(C,0,i) \quad \mathrm{mod} \;\; (max(\mathcal{O}_{\varpi(i)}))^{2}\Big) \end{gathered} \tag{4.7}$$

$$\textit{We also write } \tau(C,0,i) \textit{ meaning } (\tau(C,0))(i) \tag{4.8}$$

Note that $\tau(C,0)$ is a coherent subspace (not necessarily a subbundle) of $CT_{\mathfrak{Z}}$ with support $\mathbf{S}(C,0)$. Note that the action of $Der_{\varpi(i)}$ is to eliminate the $/^{p}$-ambiguity of the representative $\mathbf{g}(0)$ of $\mathcal{G}(0)$. Hence $\mathcal{D} = \{\mathcal{D}(0,i), i \in \mathcal{I}\}$ is globally well-defined as "ideal" in $\mathfrak{Z}$. Moreover we have $ord_{\xi}(\mathcal{D}(0,i)) \le 1$ everywhere and it is equal to 1 if and only if P belongs to $\mathbf{S}(C,0) = Sing(\mathcal{G}(C,0),\mu)_{cl}$.

We next define the *cotangential rank*, or *cotrk* for short, of $\mathcal{G}(C,0)$ at each closed point $P=(i,\xi)$ as follows.
(4.9)
$$cotrk_P(\mathcal{G}(C,,0)) = rank(\tau(C,0,i,\xi)) \quad with \quad \tau(0,i,\xi) = \big(\tau(C,0,i)\big)(\xi)$$

where $\big(\tau(C,0,i)\big)(\xi)$ denotes the fiber at ξ of $\tau(C,0,i) \subset CT_{\varpi(i)}$ and rank refers to that of module over the residue field at $\xi \in \varpi(i)$. Note that $cotrk_P(\mathcal{G}(C,0))$ is upper semicontinuous in Zariski topology of $\mathfrak{Z}_{cl}$. We define the number $\mathbf{r}(C,0) = mincrk(\mathcal{G}(C,0))$ and subscheme $Sing(\mathcal{G}(C,0),\mu(C,0),\mathbf{r}(C,0)))$ of the A-site $\mathfrak{Z}$ as follows.

(4.10)
$$\begin{aligned} \mathbf{r}(C,0) &= mincotrk(\mathcal{G}(C,0)) = min\{cotrk_P(\mathcal{G}(C,0)) \big| P \in \mathbf{S}(C,0)\} \\ \mathbf{S}_{cotrk} &= Sing(\mathcal{G}(C,0,\mu(C,0),\mathbf{r}(C,0)))_{cl} \\ &= \{P \in \mathbf{S}(C,0) \big| cotrk_P(\mathcal{G}(C,0)) = \mathbf{r}(C,0)\} \end{aligned}$$
where $Sing(\mathcal{G}(C,0,\mu(C,0),\mathbf{r}(C,0)))$ *is a subscheme of* $Sing(\mathcal{G}(C,0,\mu(C,0))) \subset \mathfrak{Z}$

We have $\mathbf{r}(C,0) > 0$ because we have excluded the case of $Sing(\mathcal{G}(C,0)) = \emptyset$.

We write $codiv(E)(C,0)$ meaning the Γ-cofactor $\mathfrak{z}^{p\gamma(C,0)}$ of $E(C,0)[d]$ which is a globally defined ideal in $\mathfrak{Z}[d]$. Let us then define the following "ideal" in $\mathfrak{Z}[d]$

$$\mathcal{D}(C,0)^{(\mu)(C,0)} = \big(\mathcal{D}(C,0)^{\mu(C,0)}\big) \cap div(E(C,0)) = div(E(C,0))J(C,0) \ with$$

Writing $G(C,0) = (div(E(C,0))h(C,0)\| /^p)$ with various residual factor ideal $h(C,0)$, we define

(4.11)
$$\begin{aligned} \mu(C,1)_P &= max_{h(C,0)}\{ord_\xi\big(h(C,0)(i) \, mod \, J(0)\big)\} \\ &nd \ \ \mu(C,1) = max_P(\mu(C,1)_P) \\ &for \ all \ P = (i,\xi) \in \mathbf{S}(C,1)_{cotrk} \end{aligned}$$

Note that for any point $P=(i,\xi) \in \mathbf{S}(C,0)_{cl}$ we have $ord_P(\mathcal{G}(C,0)) = p(maxd(\mathcal{G}(C,0))) > p$ and $dim_\xi(\varpi(i)) = d$.

Locally at such a point P we have a presentation of $\mathcal{G}(C,0) = (\mathbf{g}(C,0)\|/^p)$ as follows.

$$\mathbf{g}(C,0,i,\xi) = v(i,\xi)^{\gamma(C,0,i)}h(i,\xi)$$

where $v(i,\xi)$ is the system of generators of the ideals of those $\Gamma_k(i) \ni \xi$.

Locally at $P=(i,\xi) \in \mathbf{S}(C,0)_{cl}$, we will need to consider *allowable changes* of cofactor parameters $v = v(0,i,\xi)$. With a chosen v we rewrite $v^{\gamma(0,i,\xi)}$ for $v^{\gamma(0,i,\xi)}$ simply for later notational convenience. It is a cofactor of $\mathcal{G}(0)$ as well as that of

(4.12)
$$\mathcal{G}(C,0,i)_\xi = (v^{\gamma(0,i,\xi)}h(C,0,i,\xi)\| /^q)$$

Remark 4.1. From now on we drop the letter C from all the symbols because it is fixed throughout from nw on.

With respect to the range of allowable changes of v we define the following maximum numbers.

$$\text{(4.13)}\quad \mathbf{d}(1,i,\xi) \;=\; \max_v\{\ \nu \,\big|\, V(0,i) \subset (max(R_\xi))^\nu + \rho(R_\xi)[v]\ \}$$
$$\textit{and}\ \ \mathbf{d}(1,i) \;=\; \max_{\xi\in\varpi(i)}\{\ \mathbf{d}(1,i,\xi)\ \}$$

The next lemma gives another way of defining the same numbers by means of idempotent differential operators.

Lemma 4.1. *Locally at a closed site-point $P = (i,\xi)$ let us pick any system σ such that (σ, v) is a regular system of parameters of R_ξ. We consider the following $*$-full idempotent operator.*

$$\text{(4.14)}\qquad \mathfrak{d}_{\sigma/v} \;\in\; Diff_{R_\xi/\rho(R_\xi)[v]}\ \ \textit{with respect to}\ \ \sigma$$
$$\textit{and define}\ \ G(1,i,\xi,\sigma/v) = \mathfrak{d}_{\sigma/v} G(0,i,\xi,\sigma/v)$$

We then claim

$$\text{(4.15)}\qquad \mathbf{d}(1,i,\xi) \;=\; ord_\xi(G(1,i,\xi,\sigma/v))$$

Proof. By the definition of $\mathbf{d}(1,i,\xi)$ we have $G(0,i) \subset (max(R_\xi))^d + \rho(R_\xi)[v]$ with $d = \mathbf{d}(1,i,\xi)$. Applying $\mathfrak{d} = \mathfrak{d}_{\sigma/v}$ we obtain $\mathfrak{d}G(0,i) \subset \mathfrak{d}(max(R_\xi))^d$ which implies $\mathfrak{d}G(0,i) \subset (max(R_\xi))^d$. Hence we have $ord_\xi(\mathfrak{d}_{\sigma/v}G(0,i)) \geq \mathbf{d}(1,i,\xi)$. Moreover $(id - \mathfrak{d}_{\sigma/v})G(0,i)$ is contained in $\rho(R_\xi)[v]$ by the nature of $\mathfrak{d}_{\sigma/v}$. If $ord_\xi(\mathfrak{d}_{\sigma/v}G(0,i)) = d' > d$ then $\mathfrak{d}_{\sigma/v}G(0,i) \subset \rho(R_\xi)[v] + (max(R_\xi))^{d'}$ against the definition of d. □

Remark 4.2. Let us examine the dependence of $G(1,i,\xi,\sigma/v)$ on the choice of σ/v. We may work in the completion $\hat{R}_\xi = K[[\sigma, v]]$ with an algebraic extension K of $\mathbb{K}$ if we need to show the changes explicitly although the changes actually take place as automorphisms of R_ξ. If we fix v and change σ then the change of $G(1,i,\xi,\sigma/v)$ is nothing more than addition of an element ψ of $\rho(R_\xi)[v]$ to it. Moreover $ord_\xi(\psi) \geq \mathbf{d}(1,i,\xi)$. On the other hand if we fix σ and change v into v' then $G(1,i,\xi,\sigma/v)$ is changed by replacing v into v' in the algebra $\rho(R_\xi)[v] \subset K[[\rho(\sigma), v]]$. Recall that we are allowing only the kind of change, say from v to v', where v'_j is $u_j v_j$ with a unit $u_j \in R_\xi$ for each j. This means that the change in the initial $in_\xi(G(1,i,\xi,\sigma/v)$ is constant multiples in $in_\xi(v)$ component by component.

Lemma 4.2. *Locally at $\xi \in \varpi(i)$ such that $\mathbf{d}(1,i,\xi) = \mathbf{d}(1,i)$, let us pick any (v,ξ) from among those which gives the maximum $\mathbf{d}(1,i)$ of Def.(4.13). We then pick any σ of Lem.(4.1) and define the following ideal.*

$$\text{(4.16)}\qquad H(1,i,\xi,\sigma/v) \;=\; \big(id - \mathfrak{d}_{\sigma/v}\big)G(0,i) \;\subset\; \rho(R_\xi)[v]$$

where the inclusion is by the definition of the idempotent operator. We then claim that the following ideal in R_ξ

$$H(1,i,\xi,\sigma/v) \ + \ max(R_\xi)^{\mathbf{d}(1,i,\xi)}$$
and equivalently
$$H(1,i,\xi,\sigma/v) \ mod \ max(R_\xi)^{\mathbf{d}(1,i,\xi)}$$

is independent of the choice of σ/v.

Proof. Pick any other (σ'/v'). For any element $f \in G(0,i,\xi)$ we have

$$\begin{aligned} &\big(id - \mathfrak{d}_{\sigma/v}\big)f - \big(id - \mathfrak{d}_{\sigma'/v'}\big)f \\ = \ &\big(\mathfrak{d}_{\sigma'/v'}\big)f \ - \ \big(\mathfrak{d}_{\sigma/v}\big)f \ \in \ max(R_\xi)^{\mathbf{d}(1,i,\xi)} \end{aligned}$$

The claim of the lemma follows by Lem.(4.1). □

Incidentally this lemma is significant only if $\mathbf{d}(1,i) > \mathbf{d}(0,i)$. It is trivially true if these are equal.

Lemma 4.3. *Locally at $\xi \in \varpi(i)$ such that $\mathbf{d}(1,i,\xi) = \mathbf{d}(1,i)$, we have*

$$\begin{aligned} ord_\xi(G(1,i,\xi,\sigma/v)) = \mathbf{d}(1,i,\xi) \ \textit{and} \\ G(1,i,\xi,\sigma/v) \not\subset \rho(R_\xi)[v] + max(R_\xi)^{\mathbf{d}(1,i,\xi)+1} \end{aligned} \tag{4.17}$$

Moreover the following $\mathbb{K}$-module

$$G(1,i,\xi,\sigma/v) + (v)^{\mathbf{d}(1,i,\xi)}\rho(R_\xi)[v] + max(R_\xi)^{\mathbf{d}(1,i,\xi)+1} \tag{4.18}$$
and equivalently
$$in_\xi(G(1,i,\xi,\sigma/v)) \ mod \ (in_\xi(v))^{\mathbf{d}(1,i,\xi)}\mathbb{K}$$

is independent of the choice of (σ,v). Here the symbol of the initial $in_\xi(J)$ for an ideal J means $\big(J + max(R_\xi)^{d+1}\big)/max(R_\xi)^{d+1}$ with $d = ord_\xi(J)$.

Proof. We have $ord_\xi(G(1,i,\xi,\sigma/v)) = \mathbf{d}(1,i,\xi)$. If $G(1,i,\xi,\sigma/v)$ is contained in $\rho(R_\xi)[v] + max(R_\xi)^{\mathbf{d}(1,i,\xi)+1}$, then $ord_\xi(\mathfrak{d}_{\sigma/v}G(0,i)) = d' > d$ against Lem.(4.1). Thus Eq.(4.17) is proven. Next if we fix v then the change of σ causes a change of $G(1,i,\xi,\sigma/v)$ by addition of elements of $\rho(R_\xi)[v]$ which proves Eq.(4.18) in this case. If we fix σ then the change of v is only unit multiplications on the components of v which make a change of $v^{\gamma(0,i)}$ by a constant multiple and a change of $in_\xi(v^{-\gamma(0,i)}G(1,i,\xi,\sigma/v))$ by nonzero constant multiples on the components of the initial $in_\xi(v)$. this proves Eq.(4.18) in all cases. □

With $G(1,i,\xi,\sigma/v)$ of Lem.(4.3) we define

$$\begin{aligned} (4.19) \qquad \mathcal{D}(1,i,\xi) &= Diff_{\varpi(i)}^{(\mathbf{d}(1,i)-2)} Der_{\varpi(i)} G(1,i,\xi,\sigma/v) \\ & \textit{and} \ \ \tau(1,i) \subset T_{\varpi(i)} \ \ \textit{defined by} \\ I(1,i,\xi) &= \Big((v)R_\xi + \mathcal{D}(1,i,\xi) \quad \mathrm{mod} \ \ (max(\mathcal{O}_{\varpi(i)}))^2 \Big) \end{aligned}$$

for every closed point $P = (i,\xi)$. The $I(1,i,\xi)$ is independent of the choice of (σ,v) by Lem.(4.3). The $I(1,i,\xi)$ is natural image at ξ of the coherent ideal $\mathcal{D}(1,i)$ into the cotangent module of $\varpi(i)$ and it defines a coherent fiber subscheme of the tangent bundle $T_{\varpi(i)}$. We will be primarily interested in the restriction $\tau(1,i)|_{S(1,i)}$ to the singular locus $S(1,i) \subset \varpi(i)$ of Def.(**??**). It should be noted that we obtain a "global coherent ideal" $\mathcal{D}(1) \subset \mathcal{O}_{\mathcal{Z}}$ on the A-site $\mathfrak{Z}$ such that

$$(4.20) \qquad \mathcal{D}(1,i) = \Big(\mathcal{D}(1)\Big)(i) \ \ \textit{for all} \ \ i \in \mathcal{I}$$

Definition 4.1. With reference to the singular locus $S(\mathcal{H}(0,i))$ of Def.(**??**) we now define

$$S(\mathcal{H}(1,i)) = \{\xi \in S(\mathcal{H}(0,i)) \big| \, ord_\xi(\mathcal{H}(1,i))\}$$

where $\mathcal{H}(1,i) = ()$.

We then define the next "derivative image" to be

$$\begin{aligned} (4.21) \qquad \mathcal{D}(1,i) &= Diff_{\varpi(i)}^{(\mathbf{d}(1,i)-2)} Der_{\varpi(i)} F(1,i) \\ \textit{and} \ \ \tau(0,i) &= \Big(\mathcal{D}(0,i) \quad \mathrm{mod} \ \ (max(\mathcal{O}_{\varpi(i)}))^2 \Big) \end{aligned}$$

where *mod* $(max\mathcal{O}_{\varpi(i)})^2$ means to take modulo $max(R_\xi)^2$ for each ξ of $P = (i,\xi)$. As for $\tau(0,i)$ we will be only interested in its restriction to the singular locus $S(\mathcal{H}(0,i))$ of Def.(**??**). It is important that the module of derivations $Der_{\varpi(i)}$ kills all the differences from $\rho(\mathcal{O}_{\varpi(i)})$ so that we obtain a "global coherent ideal" $\mathcal{D}(0) \subset \mathcal{O}_{\mathcal{Z}}$ on the A-site $\mathfrak{Z}$ which consists of

$$(4.22) \qquad \mathcal{D}(0,i) = \Big(\mathcal{D}(0)\Big)(i) \ \ \textit{for all} \ \ i \in \mathcal{I}$$

We also obtain a "global coherent fiber subspace" $\tau(0)$ of the cotangent bundle $T_{\mathfrak{Z}}$ of the the A-site $\mathfrak{Z}$ such that

$$(4.23) \quad \tau(0,i) \ \textit{is the subbundle of} \ \tau(0)|_{S(\mathcal{H}(0,i))} \ \textit{for all} \ i \in \mathcal{I}$$

There then are the following two cases:

$$(4.24) \qquad \mathcal{D}(0,i) = \begin{cases} \subset \ Span\,(derivatives \ of \ v) \\ \not\subset \ Span\,(derivatives \ of \ v) \end{cases}$$

(1) In the first case,

$$F(0,i) \ \subset \ v^{\gamma(0,i)}\rho(\mathcal{O}_{\varpi(i)}) + (max(\mathcal{O}_{\varpi(i)})^{\mathbf{d}(0,i)+1}$$

(2) In the second case there exists a system of $\sharp$-key parameters of $\mathcal{F}$ at $P = (i,\xi)$

$$\zeta(0,i) = (\zeta(0,i)_1, \cdots, \zeta(i,0)_{s(i,\xi)})$$

such that $(in_\xi(v(i,\xi), \zeta(0,i))$ is a free base of the stalk $\mathcal{D}(0,i)_\xi$. From now on we oten write $v(i)$ or even v for $v(i,\xi)$.

Still staying in the local situation at $P = (i,\xi)$, let us pick a system of parameters $\sigma = (\sigma_1, \cdots, \sigma_s)$ such that $\eta = (\sigma, v)$ is a regular system of parameters at P. Then consider the idempotent differential operators

$$\mathfrak{d}^{(a)} \in Diff^*_{R_\xi/\rho(R_\xi)}, a \in \epsilon^n(q),$$

where $\epsilon^n(q) = \{a \in \mathbb{Z}^n \,|\, 0 \leq a_j < q, \forall j\}$. For the sake of notational convenience we will mean $\mathfrak{d}^{(b)} = 0$ for all $b \in \mathbb{Z}^n \setminus \epsilon^n(q)$. Here the idempo is defined with respect to the pamameters η. We denote by $*\mathfrak{d}(\eta/v)$ the $*$-full idempotent operator in

$$Diff^*_{\rho(R_\xi)[\eta]/\rho(R_\xi)[v]} \textit{ with respect to } \eta = (\sigma, v)$$

Then let us define

$$\begin{aligned} F(0,i,1) &= F(0,i) - \big(*\mathfrak{d}(\eta/v)\big)F(0,i) \\ &= \Big(\sum_{\beta \in \gamma(i,0) + \mathbb{Z}_0{}^{n-s}} \mathfrak{d}^{((0),\beta)} \Big) F(0,i) \end{aligned} \tag{4.25}$$

which is an element of $v^{\gamma(0,i)}\rho(R_\xi)[v]$. Write $F(0,i,1)$ in the form $v^{\gamma(0,i)}\mathbf{j}(0,i,1)$ and we have

$$ord_\xi(\mathbf{j}(0,i,1) = ord_\xi(F(i,0)) - |\gamma(0,i)|.$$

where $ord_\xi(F(i,0)) = ord_\xi(\mathcal{F}(0,i))$.

It is important to note that if we change σ by another system as above then $F(i,0,1)$ is changed by addition of a certain element of $\rho(R_\xi)[v]$. On the other hand if σ is fixed and v is replaced by another such system v° (each component v_j is replaced only by a unit multiple v_j° in R_ξ) then $F(0,i,1)$ is transformed by the $\rho(R_\xi)$-algebra isomorphism from $\rho(R_\xi)[v]$ to $\rho(R_\xi)[v^\circ]$ which maps v to v°. However when v is replaced by v° the difference $F(0,i) - F(0,i,1)$ can even change its order at ξ.

Let us then choose σ and v such that $ord_\xi(F(0,i)-F(0,i,1))$ attains its maximum which we call $D(0,i,1)$. We have always

$$D(0,i,1) \;\geq\; \mathbf{d}(0,i,1) \tag{4.26}$$

where

$$\mathbf{d}(0,i,1) = ord_\xi(F(0,i))$$

From now on we will consider only those (σ, v) which have $ord_\xi(F(0,i)-F(0,i,1)) = D(0,i,1)$.

Let $\Delta(0,i,1) = F(0,i) - F(0,i,1)$.

Lemma 4.4. *If* $ord_\xi(\Delta(0,i,1)) = D(0,i,1)$*, say* $= D$*, then*

$$Diff_{\varpi(i)}^{(D-2)} Der_{\varpi(i)}(\Delta(0,i,1)) \not\subset max(R_\xi)^2 + (v)R_\xi$$

Note that the following cotangential module

$$\left(Diff_{\varpi(i)}^{(D-2)} Der_{\varpi(i)}(\Delta(0,i,1)) + (v)R_\xi\right) \mod max(R_\xi)^2 \tag{4.27}$$

is an invariant of $(\mathcal{G}, \Gamma)$ at $P = (i, \xi)$.

5. Primary stability conditions

Let $\mathcal{G} = (z^{q\beta} v^{\gamma} f \| /^{q})$ with q-cofactor v^{γ} and with a residual factor f in the sense of Def.(**??**) and Def.(**??**) at a closed point $\xi \in \mathit{Sing}(\mathcal{G})$. We let $\mathit{resord}_{\xi}(\mathcal{G}) = \mathit{ord}_{\xi}(f) = d$. We have a regular system of parameters $x = (z, \omega)$ of R_{ξ} with $z = (v, w)$ in the manner of Eq.(**??**).

Remark 5.1. Assume that we are given a member
Let us then define

$$\mathcal{G}_i \;=\; (\, z^{q\beta+\gamma} f_i \, \| \, /^{q}) \;\; \text{for} \;\; i \;=\; 1, 2 \tag{5.1}$$

having the same q-factor and q-cofactor as $\mathcal{G}$.

(1) $\mathit{ord}_{\xi}(f_1) = \mathit{ord}_{\xi}(\zeta f_2) = \mathit{ord}_{\xi}(f) = \mathit{resord}_{\xi}(\mathcal{G}) = d$
(2) f_1 is a residual factor of $\mathcal{G}_1$
(3) ζ is a member of a regular system of parameters x of R_{ξ} and $\mathit{in}_{\xi}(\zeta)$ is not in the expression of $\mathit{in}_{\xi}(f_1)$ in $\kappa_{\xi}[\mathit{in}_{\xi}(x)]$.

Take any fitted permissible blowup $\pi : Z' \longrightarrow Z$ with center D for both $\mathcal{G}_i, i = 1, 2$, and pick any closed point

$$\xi' \;\in\; \pi^{-1}(\xi) \;\cap\; \bigcap_{i=1,2} \mathit{Sing}(\mathcal{G}'_i)$$

where $\mathcal{G}'_i$ denotes the transform of $\mathcal{G}_i$ by π for each i.

Theorem 5.1. *Under the conditions of Rem.(5.1) let us assume*

$$\mathit{in}_{\xi}(f_1) \;\notin\; \rho^{e}(\mathit{gr}_{\xi}(R_{\xi})) \;\; \text{where} \;\; q = p^{e}. \tag{5.2}$$

Then for any π of Rem.(5.1) there cannot exist any ξ' which is metastable of $\mathcal{G}$ provided that any one of the following conditions is satisfied:

(1) $\zeta \notin L_{q-max}(\mathcal{G})^{cofa}$.
(2) $\mathfrak{M}(\mathcal{G}_1) \;=\; \emptyset$.
(3) $\delta^{0}_{v,\zeta}(\zeta\, f_2) \;\neq\; 0$.
(4) $\mathit{in}_{\xi}(\zeta f_2) \notin \kappa_{\xi}[\mathit{in}_{\xi}(x \setminus \zeta), \mathit{in}_{\xi}(\zeta)^{q}]$

Theorem 5.2. *If the primary m-scheme of $\mathcal{G}$ at a closed point $\xi \in \mathit{Sing}(\mathcal{G})$ then there exists no metastable points appears unless once the residual order drops.*

6. Local analysis of global resolution program

7. Reduction from $/^q$ to $/^p$

8. $/^p$-decomposition, $\sharp$ vs $\flat$

9. Cotangential flags local-global relation

10. sum-division, stable part and *sharp*-flag part

In the positive characteristic case, new kind of hard questions in $p > 0$ are mostly of "local" nature. But many of those "local" questions should be handled "open-locally" in Zariski topology. There then more special care is required than the cases of "wedge-local (or micro-local)" which are usually used in the "local" uniformization techniques. The "Zariski-open-local obsession" is indeed the heart of our endeavor in this paper.

We now begin with any given ideal exponent $E = (J, b)$ in a smooth irreducible scheme Z over a perfect base field $\mathbb{K}$ of characteristic $p > 0$. We exclude the trivial case of $J = (0)$. We have the theory of infinitely near singularities $\mathfrak{S}$ of E which determines the geometric characteristic algebra $\wp_{geo}(E)$ of Eq.(**??**). This then coincides with the algebraic one $\wp_{alg}(E)$ of Eq.(**??**) because $\mathbb{K}$ is perfect. We thus denote them by $\wp(E)$. This graded $\mathcal{O}_Z$-algebra is finitely presented and gives us its *edge data* at every closed point $\xi \in Sing(E)$ which consists of *edge parameters* $y = (y_1, \cdots, y_r)$ and *edge generators* $g = (g_1, \cdots, g_r)$ such that we have the equivalence in terms of infinitely near singularities expressed as

$$E \ \dagger \ \Big(\bigcap_{i=1}^{r} E_i \Big) \bigcap F \quad \textit{with remainder} \quad F$$

where $E_i = (g_i \mathcal{O}_Z,\, q_i)$ with $g_i = y_i^{q_i} + \epsilon_i$ and $F = (I, c), ord_\xi(I) > c, \forall \xi$. The edge data are nontrivial (i.e., $r > 0$) if and only if $ord_\xi(J) = b$. In general we have

$$\max_{xi \in Sing(E)} ord_\xi(J) \ = \ \mathbf{d} \ \geq b$$

and if $\mathbf{d} > b$ then we make base-lift from b to $\mathbf{d}$. Namely we apply the theory of infinitely near singulaities to $E^* = (J, \mathbf{d})$. The only difference between the permissible transforms of E^* and E is a NC-monomial factor and therefore by a certain simple canonical procedure we can reduce the desingularization problem of E to that of E^*.

Thus from now on we consider only the case of E with $\mathbf{d} = b$. We then extract the edge data as above from the characteristic algebra $\wp(E) = \sum_{i \in \mathbb{Z}_0} \wp(E)_i$ where $\wp(E)_i$ denotes the homogeneous part of degree i. We pay our special attention to the smallest exponent q_1 of the edge data of $\wp(E)$ at every closed point ξ.

(1) We define the number $q_\ell(E)$ which is the smallest number among the lowest edge exponents $q_1(\xi)$ for all closed points $\xi \in Sing(E)$. Namely

$$q_\ell(E) \;=\; \min_{\xi \in Z_{cl}} \Big(\min\{i \in \mathbb{Z}_0 \big| \, ord_\xi(\wp(E)_i) = i\} \Big)$$

which is of the form p^e with $e \in \mathbb{Z}_0$.

(2) We always have $q_\ell(E) \geq 1$ by the definition of $Sing(E)$. When $q_\ell(E) = 1$ we make use of "ambient reduction theorem" Th.(**??**) and reduce the problem in the case of lower effetive ambient dimension in the global induction setup.

(3) From now on we assume $q_\ell(E) > 1$ so that $q_\ell(E) \geq p$. Given a closed point $\xi \in Sing(E)$ we choose a local system of parameters $x = (z, \omega)$ with z consisting of those defining the members of Γ passing through ξ. With respect to such an x we take the $*$-full idempotent operator $\partial^\circ \in Diff_{\mathcal{O}_Z/\rho(\mathcal{O}_Z)}$ and with $q = q_\ell(E)$ we define

$$\mathcal{E}^\circ \;=\; (J^\circ \,\|\, /^p) \quad with \quad J^\circ = \partial^\circ(\wp(E)_q) \tag{10.1}$$

Here J° is viewed as a $\rho\mathcal{O}_Z$-module in accord with Def.(**??**).

Theorem 10.1. *Assume $q_\ell = q_\ell(E) \geq p$. Firstly we see that*

$$q_\ell(E) = \max_{\eta \in Z}(ord_\eta(\mathcal{E}^\circ)) = \max_{\xi \in Sing(E)}\{ord_\xi(J^\circ)\}$$

The $/^p$-exponent $\mathcal{E}^\circ$ is globally well defined as such. When $q_\ell = p$ we have $\mathcal{E}^\circ = (\wp(E)_p \,\|\, /^p)$. In general every permissible blowup for E is also permissible for $\mathcal{E}^\circ$. For the transforms E' of E and $\mathcal{E}^{\circ\prime}$ of $\mathcal{E}^\circ$ by any such blowup, only possible differences between $\mathcal{E}^{\circ\prime}$ and E'° is a Γ-monomial factor. To be precise let $\pi : Z' \to Z$ be the blowup with center D and let L be the ideal of the exceptional divisor $\pi^{-1}(D)$. Then the ideal of $\mathcal{E}^{\circ\prime}$ is the product of $L^{(p^{e-1}-1)p}$ and the ideal of E'°.

(4) We are given a global NC-data $\Gamma = (\Gamma_1, \cdots, \Gamma_s)$ in the sense of the paragraph of Def.(**??**) and Def.(**??**). Then thanks to Th.(**??**) and Rem.(**??**), we have the well-defined global divisorial ideal in $\mathcal{O}_Z$, say

$$\mathfrak{D}(\mathcal{E}^\circ) \;=\; \mathfrak{z}^{\mathbf{a}} \;=\; \prod_{1 \leq i \leq s} \mathfrak{z}_i^{\mathbf{a}_i} \tag{10.2}$$

where $\mathfrak{z} = (\mathfrak{z}_1, \cdots, \mathfrak{z}_s)$ and $\mathfrak{z}_i$ is the ideal of $\Gamma_i \subset Z$. It is by Th.(**??**) that $\mathfrak{D}(\mathcal{E}^\circ)$ induces the local monomial factor $z^{\mathbf{a}}$ of $\mathcal{E}^\circ$

in the sense of the *abc*-expression of Def.(**??**) at every closed point of $Sing(\mathcal{E}^\circ)$. Accordingly we define the global factorization

$$\mathfrak{z}^{\mathbf{a}} = \mathfrak{z}^{p\mathbf{b}}\mathfrak{v}^{\mathbf{c}} \quad with \quad \mathfrak{v} \subset \mathfrak{z} \tag{10.3}$$

with $0 < \mathbf{c}_j < p, \forall j$, in the accord with the local *abc*-expressions.

(5) We then define the following ideal exponent.

$$\mathcal{E}(\circ) \;=\; (J(\circ) \| /^p) \quad with \quad J(\circ) = \mathfrak{z}^{-p\mathbf{b}} J^\circ$$

(6) Define the following number.

$$d(\circ) \;=\; d(\circ)(E) \;=\; \max_{\xi \in \mathbf{S}} \Big(ord_\xi(\mathcal{E}(\circ)) \Big) \;=\; \max_{\xi \in \mathbf{S}_{cl}} \Big(ord_\xi(\mathcal{E}(\circ)) \Big)$$

where $\mathbf{S} = Sing(\mathcal{E}^\circ) \cap Sing(E)$.

(7) We then define the ideal exponent $E_\circ = (J_\circ \| /^1)$ with $J_\circ = Diff_Z^{*(d(\circ)-1)} J(\circ)$. It should be noted that $E_\circ$ is a globally well defined as "$/^1$-exponent" because $Diff_Z^{*(d(\circ)-1)}$ maps $\rho(\mathcal{O}_Z)$ into itself.

Theorem 10.2. *The ideal $J_\circ$ of $\mathcal{E}^\circ$) contains every $\sharp$-key parameters of $\mathcal{E}(\circ)$, and equivalently such of $\mathcal{E}^\circ$, at every closed point of $Sing(\mathcal{E}^\circ)$. Moreover $\mathcal{E}^\circ$ defines a unique vector subbundle of the cotangent bundle of Z resticted to $Sing(\mathcal{E}^\circ)$. In other words at every closed point $\xi \in Sing(\mathcal{E}^\circ)$, the natsural image of $J_\circ$ into M_ξ/M_ξ^2 is independent of the choice of abc-expressions of $\mathcal{E}^\circ$) at ξ.*

(8) We then introduce the following $/^p$-exponent.

$$\mathcal{E}^\dagger \;=\; (J^\dagger \| /^{q(E)}) \quad with \quad J^\dagger = \mathfrak{z}^{p\mathbf{b}} {J_\circ}^{d(\circ)} \cap \mathfrak{v}^{\mathbf{c}} \mathcal{O}_Z \tag{10.4}$$

Refer to Eq.(10.2) and Eq.(10.3).

Theorem 10.3. *$\mathcal{E}^\dagger$ is a global $/^q$-exponent which is uniquely determined by E. Its ideal $J^\dagger$ contains all those monomials of the key parameters of E° with degrees $\leq q(E)$.*

Immediately after a metastable jump has occurred at a closed point $\xi' \in \pi^{-1}(\xi) \cap Sing(\mathcal{G}')$, the ambient reduction to any center $D' \subset Z'$ contained in $\{v_1 = T_j = 0, \forall j\}$ is idealistic because we have $f' \in (v_1 = T_j = 0, \forall j)R_{\xi'}$.

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