RESOLUTION OF SINGULARITIES
PART (4)
INDUCTIVE PROCEDURE FOR GLOBAL DESINGULARIZATION

1. SITE AND SUBSITE

Definition 1.1. Define a *site* (precisely *subetale site*), denoted by

$$\mathfrak{Z} \;=\; (\mathcal{I}, \mathcal{Z}, \varpi)$$

to mean a pair of categories $\mathcal{I}$ and $\mathcal{Z}$ combined with a covariant functor $\varpi : \mathcal{I} \to \mathcal{Z}$ where

(1) $\mathcal{I}$ is an abstract category called the *index category* of the site,
(2) $\mathcal{Z}$ is a category of affine schemes of finite type over a fixed *base ring* or *base field*, denoted by $\mathbb{K}$. (eg. any finite field.)
(3) and $\varpi(m) : \varpi(i) \to \varpi(j)$ is an *subetale* morphism in $\mathcal{Z}$ for every arrow $(m : i \to j) \in mor(\mathcal{I})$.

The word *subetale* of $\varpi(m)$ means that at every closed point $\eta \in \varpi(i)$ with its image $\xi = \varpi(m)(\eta)$, $\varpi(m)$ is a composition of an local etale extension $\mathcal{O}_{\varpi(j),\xi} \to R$ and a surjective homoimorphism $R \to \mathcal{O}_{\varpi(i),\eta}$. In other word *subetale* morphism is the one extendable to *etale* morphism.

We will often write $\mathfrak{Z}(i)$ or $\mathcal{Z}(i)$ meaning $\varpi(i)$ when it is convenient and understandable.

In our theory the category $\mathcal{Z}$ may be chosen to be the one of all affine algebraic $\mathbb{K}$-schemes and all subetale morphisms. At any rate $\mathcal{Z}$ should be plentiful as we later need to enlarge $\mathcal{I}$ many times in terms of *satulations* of the given site with respect to singularity data, given or newly created.

Definition 1.2. A *subsite* of a site $\mathfrak{Z} = (\mathcal{I}, \mathcal{Z}, \varpi)$ means $D = (\mathcal{J}, \mathcal{D}, \varsigma)$ where $\mathcal{J}$ is a subcategory of $\mathcal{I}$, ς is induced by ϖ and

(1) $D(i)$ is a closed subscheme of $\mathcal{Z}(i)$ for every $i \in ob(\mathcal{J}) \subset ob(\mathcal{I})$
(2) and $D(i) = \varpi(m)^{-1}(D(j))$ for every $(m : i \to j) \in mor(\mathcal{J}) \subset mor(\mathcal{I})$.

Definition 1.3. Consider a subsite of a site in the sense of Def.(1.2). A *site-point* (or *point* for short) of D means a pair $P = (i, \xi)$ with $i \in ob(\mathcal{J})$ and $\xi \in D(i)$. For a point $P = (i, \xi)$, the *dimension* of D at P means $dim_\xi D(i)$ within a sufficiently small neighborhood of

$\xi \in D(i) = \varsigma(i)$ and it is denoted by $dim_P(D)$. We let $dim(D)$ denote the maximum of the $dim_P(D)$ for all points P of D. We write $trd(P)$ for the transcendence degree of the residue field of $\mathcal{O}_{D(i),\xi}$ over the $\mathbf{K}$.

Definition 1.4. A point $P = (i, \xi) \in D$ is said to be D-connected to another point $P^\dagger = (i^\dagger, \xi^\dagger) \in D$ if there exists a finite chain of morphisms m_k's in $mor(\mathcal{J})$, starting from i forward and/or backward ending with $i^\dagger$, such that

1. each $\varsigma(m_k)$ has non-empty source and target,
2. the connected component of $D(i)$ containing ξ and that of $D(i^\dagger)$ containing $\xi^\dagger$ are connected by a chain of connected components of the $D(k)$'s, one for each k suitable selected, where $D(k)$ denotes the successive joint scheme, "source=target" and/or "source=source" and/or "target=target" of $\varsigma(m_k)$'s.

We say that D is *connected* if every pair of points in D are *D-connected*. We say that P is *D-equivalent* to $P^\dagger$ if there exists a chain as above such that ξ is mapped to $\xi^\dagger$ by the string of morphisms $\varsigma(m_k)$'s.

Lemma 1.1. *If a site point P of $\mathfrak{Z}$ is $\mathfrak{Z}$-equivalent to another point Q we then have $trd(P) = trd(Q)$ fn the sense of the transcendence degree trd of Def.(1.3).*

Proof. Since every morphism in $\mathcal{Z}$ of $\mathfrak{Z}$ are subetale and hence the residue fields of $\mathfrak{Z}$-equivalent points have the same algebraic closure and their trd's are the same. □

Definition 1.5. We define $maxdim_P(D)$ to be the maximum of $dim_Q(D)$ for all Q which are D-equivalent to P. A point P of D) is called a *generic point* of D if $trd(P) = maxdim_P(D)$.

Definition 1.6. Consider a subsite D of a site $\mathfrak{Z}$. A finite subset $S \subset ob(\mathcal{J})$ will be called an *equidimensional net* in D, or *equinet* in D for short, if we have

1. every generic point $P = (i, \xi) \in D$ with $i \in S$ is D-equivalent to a point $P^\dagger = (i^\dagger, \xi^\dagger) \in D$ for every $i^\dagger \in S$
2. and $dim_Q(D)(= dim_\eta(\varsigma(j))$ is a constant for all $Q = (j, \eta) \in D$ with $j \in S$.

Therefore an equinet S in D has a unique constant dimension, denoted by $dim_S(D) = dim_Q(D)$, at all points Q and a unique transcendent degree, denoted by $trd_S(D) = dim_S(D)$, for all generic points P as above. An equinet S is said to be *conjoint* in D if there exists an index $\ell \in \mathcal{J}$ such that

(1) there exist $(m_i : \ell \to i) \in mor(\mathcal{I})$ for every $i \in S$ such that $n_i m_i : \ell \to k$ is independent of i as long as $(n_i : i \to k) \in mor(\mathcal{J})$ have the same $k \in ob(\mathcal{J})$, and
(2) $S \cup \{\ell\}$ is also an equinet in D so that $dim(\varsigma(\ell)) = dim_S(D)$.

We say that D is site-conjoint (conjoint, for short) if $dim(D) < \infty$ and every equinet S with $dim_S(D) = dim(D)$ is conjoint in D.

Definition 1.7. We say that D is quasicompact if there exists a finite subset S of $ob(\mathcal{I})$ such that every point $P = (i, \xi) \in D$ is D-equivalent to a point $P^\dagger = (i^\dagger, \xi^\dagger)$ with $i^\dagger \in S$. Such a set S will be called a *generating index set* of D. We say that D is *irreducible* if there exists an equinet S satisfying the following two conditions.
(1) $dim_S(D) = dim(D)$ and S is conjoint in D
(2) S is a generating index set of D so that D is quasicompact.

Note that every generic point of D is D-equivalent to a generic point $Q = (j, \eta)$ with $j \in S$.

Remark 1.1. Even if D is irreducible (hence quasicompact) its categories $\mathcal{J}$ and $\mathcal{D}$ can have infinitely many objects and morphisms. It can have a subsite which is not quasicompact. This allowance is sometimes technically useful or just convenient.

In our work with a fixed $\mathfrak{Z}$ we will be mostly interested in those subsites having the same index category.

Definition 1.8. When a subsite D of Def.(1.2) has the same index category $\mathcal{J} = \mathcal{I}$ of $\mathfrak{Z}$, we call D an *I-subsite* to express its special nature. Incidentally then a pair of points of D are $\mathfrak{Z}$-equivalent if an only if they are D-equivalent.

2. Sites in charcteristic $p > 0$

From now on we assume a fixed base field $\mathbb{K}$ which is perfect of characteristic $p > 0$. Moreover we will be working within "ambient sites" defined as follows.

Definition 2.1. An *ambient smooth site*, or *A-site* for short, means a site $\mathfrak{Z}$ in the sense of Def.(1.1) having the following properties.
(1) $\mathcal{Z}$ is a category of smooth irreducible affine $\mathbb{K}$-scheme of finite type,
(2) $\mathfrak{Z}$ is *irreducible* as a site in the sense of Def.(1.7), and

(3) for every point P of $\mathfrak{Z}$ we have

$$\infty > dim(\mathfrak{Z}) = maxdim_P(\mathfrak{Z}) \tag{2.1}$$

We next want to globalize the notion of exponents, ideal (in another term idealistic) or $/^q$ or mixed, to an A-site $\mathfrak{Z}$ in the sense of Def.(1.1).

Incidentally, a "mixed exponent" in a smooth scheme Z means an "intersection" of a finite number of ideal exponents and $/^q$-exponents in the sense of the symbol $\mathfrak{S}$ from the theory of infinitely near singularities. Recall that $\mathfrak{S}$ denotes the totality of the infinitely near singularities.

Note that given any mixed exponent $\mathfrak{M}$ we can choose an ideal exponent E and a $/^q$-exponent $\mathcal{G}$ in Z such that

$$\mathfrak{S}(\mathfrak{M}) = \mathfrak{S}(E \cap \mathcal{G}) = \mathfrak{S}(E) \cap \mathfrak{S}(\mathcal{G})$$

where the last equality is the definition of $E \cap \mathcal{G}$. Here of course it is possible that one of the E and $\mathcal{G}$ is irrelevant for the intersection so that $\mathfrak{M}$ is either an ideal exponent or a $/^q$-exponent.

Definition 2.2. An *exponent* E in an A-site $\mathfrak{Z}$ will mean a collection of exponents $\{E(i)\ in\ \varpi(i),\ i \in ob(\mathcal{I})\}$ which satisfy the follwing condition for every morphism $\varpi(m)$ with $(m : i \to j)$ in $mor(\mathcal{I})$.

(1) $E(i)$ is the $\mathfrak{S}$-equivalent ambient reduction of $E(j)$ by $\varpi(m)$, and
(2) $\varpi(m)$ is local $\mathfrak{S}$-equivalence everywhere in $\varpi(i)$.

To be precise the "local $\mathfrak{S}$-equivalence" by $\varpi(m)$, say at a closed point $\xi \in \varpi(i)$, means to satisfy the following condition. We can choose open neighborhoods B of $\eta = \varpi(m)(\xi) \in \varpi(j)$ and A of $\xi \in \varpi(i)$ such that

(1) there exists a closed imbedding $\iota : A \subset U$ and a surjective etale morphism $\epsilon : U \to B$ with $\varpi(m)|_A = \epsilon\iota$.
(2) Pick any element $\theta \in \mathfrak{S}(E(j)|_B)$ which is written as a finite sequence of blowups over $B[w] = B \times Spec(\mathbb{K}[w])$ where w is a finite system of indeterminates. The centers of the pullback of θ to $U[w]$ by $\epsilon \times id_w$ are all contained in the strict transforms of $A[w]$ and hence θ induces a sequence of blowups over $A[w]$. Moreover this induced sequence belongs to $\mathfrak{S}(E(i)|_A)$, which we denote $\theta|_A$.
(3) Every element of $\mathfrak{S}(E(1)|_A)$ is oftained as $\theta|_A$ with some $\theta \in \mathfrak{S}(E(j)|_B)$.

Remark 2.1. The "local $\mathfrak{S}$-equivalence" relation between $E(i)|_A$ and $E(j)|_B$ described above can be rephrased as follows: By choosing $A \subset U$ and B sufficiently small, we pick any exponent F on U such that $E(i)|_A$ is the ambient reduction (not necessarily equivalent) of F from

U to A. For example if there exists a retraction $r : A \subset U \to A$ then the simple pullback F of $E(i)|_A$ will do the job. Then the $\mathfrak{S}$-equivalence means

$$\mathfrak{S}(E(j)|_B) \;=\; \mathfrak{S}(F \cap \mathbf{I}(A \subset U)) \;=\; \mathfrak{S}(F) \cap \mathfrak{S}(\mathbf{I}(A \subset U))$$

where $\mathbf{I}(A \subset U)$ denotes the ideal exponent $(I, 1)$ in U with the ideal I defining $A \subset U$.

Definition 2.3. Let E in $\mathfrak{Z}$ be of Def.(2.2). Then the order of E at a site-point $P = (i, \xi)$, denoted by $ord_P(E)$, is defined to be $ord_\xi(E(i))$.

Remark 2.2. It should be noted that when $dim_P(\mathfrak{Z}) \neq dim_{P^\dagger}(\mathfrak{Z})$ for a pair of closed points $(P, P^\dagger)$ of $\mathfrak{Z}$, we may not have the equality $ord_P(E) = ord_{P^\dagger}(E)$ even when $P^\dagger$ is $\mathfrak{Z}$-equivalent to P. For instance consider the case of Rem.(2.1) after Def.(2.2) in which $dim_\xi(\varpi(i)) < dim_\eta(\varpi(j)$ with $\eta = \varpi(m)(\xi)$. Then $ord_\eta(E(j)) \leq 1$ because of the presence of $\mathbf{I}(A \subset U)$ independent of F and $E(i)$ while $ord_\xi(E(i))$ could be any positive number.

Definition 2.4. Recall Def.(1.2) and Def.(1.8). An I-subsite D of an A-site $\mathfrak{Z}$ is called *hypersurface* if it satisfies the following conditions.

(1) $D(i) \subset \varpi(i)$ is defined by an ideal, coherent, which is either zero or invertible as $\mathcal{O}_{\varpi(i)}$-module for every $i \in \mathcal{I}$.
(2) D has only a finite number of generic points up to D-equivalence.
(3) For every generic point P of D we have $trd(P) = dim((Z)) - 1$.

An I-subsite is said *smooth* if $D(i)$ is so for every $\in ob(\mathcal{I})$. An I-subside is called a *subscheme* if the natural morphism $D(i) \to D(j)$ by $\varsigma(m)$ is etale for every $(m : i \to j) \in mor(\mathcal{I})$. It should be noted that a hypersurface may not be a subscheme in A-site $\mathfrak{Z}$.

Definition 2.5. Let $D = (\mathcal{I}, \mathcal{D}, \varsigma)$ be an I-subsite of $\mathfrak{Z}$. Given a site-point $P = (i, \xi) \in D$, the connected component $C \ni P$ of D means the I-subsite $C = (\mathcal{I}, \mathcal{C}, \vartheta)$ of D such that for $j \in ob(\mathcal{I})$ we have

(1) $\vartheta(j) = \varsigma(j)$ if there exists $\eta \in \varsigma(j)$ such that $Q = (j, \eta)$ is D-equivalent to P
(2) while if there exists no such Q then $\vartheta(j) = \emptyset$.

Lemma 2.1. *Every connected component of a subscheme D of an A-site $\mathfrak{Z}$ is a suscheme of $\mathfrak{Z}$.*

Definition 2.6. A finite system of smooth hypersurfaces in an A-site $\mathfrak{Z}$, say $\Gamma = (\Gamma_1, \cdots, \Gamma_t)$, is called NC-data (normal crossing data) if the following conditions are satsified.

(1) Each $\Gamma_k, 1 \leq k \leq k$, is an A-site by itself in the sense of Def.(2.1) so that it is connected, generically conjoint and quasicompact.

(2) Each $\Gamma_k(i), i \in ob(\mathcal{I})$, is a smooth closed subscheme of $\varpi(i)$ where we do not exclude the cases in which some of them may be either empty or the whole scheme $\varpi(i)$.
(3) For every closed point $P = (i, \xi)$ of $\mathfrak{Z}$ we define the symbol $\Gamma(P)$, to be the subsystem of those hypersurfaces $\Gamma_k(i)$ which contain ξ but $\neq \mathcal{Z}(i)$. There the hypersurfaces of the subsystem $\Gamma(P)$ are mutually distinct and have normal crossings in a neighborhood of $\xi \in \varpi(i)$.
(4) The schematic intersection of every finite subsystem, say $\Gamma_B = \cap_{k \in B} \Gamma_k$, is smooth and a finite union of disjoint I-subsites which are themselves A-sites.

Lemma 2.2. *Consider a subscheme D of an A-site $\mathfrak{Z}$ in the sense of Def.(2.4). Since D-connectedness for a pair of points in D is an equivalence relation by Def.(1.4), D becomes a disjoint union of connected components $D = \cup_{c \in \nabla} D_c$. For every $c \in \nabla$ we claim that D_c is a connected quasi-compact I-subsite of $\mathfrak{Z}$ and it is a subscheme of $\mathfrak{Z}$ by itself in the sense of Def.(2.4). Moreover we claim the number of connected components $|\nabla|$ is finite.*

Proof. We have a *generating index set* S of the A-site in the sense of Def.(2.1) after Def.(1.7). For every point $P = (i, \xi) \in D$ there exists a point $Q = (k, \eta) \in \mathfrak{Z}$ with $k \in S$ such that Q is $\mathfrak{Z}$-equivalent to P. Let $\nabla \subset S$ denote the set of those k for which there exist Q and P as above. Let us first recall the definition of "subscheme" which implies that all the arrows of an $\mathfrak{Z}$-equvalence chain for P and Q induce etale morphisms in D. Any sequence of etale morphisms is generocally surjective and finite to finite in the correspondence of points. Therfore the $\mathfrak{Z}$-equvalence chain as above is in fact a D-equivalence chain. Now for each index $k \in \nabla$ let $D(k)$ be the closed subscheme $\varsigma(k)$ of $\varpi(k)$. Then every point $P \in D$ is D-equivalent to a point of at least one of the $D(k), k \in \nabla$. Moreover for each $k \in \nabla$ the number of the conected components of $D(k)$ is finite and $D_c(k)$ is a union of some of these connectd components for each $c \in \nabla$. The lemma follows. □

Remark 2.3. Assume that an exponent E and an NC-data Γ in an A-site $\mathfrak{Z}$ are given in the sense of Def.(2.2) and Def.(2.6). Consider an index $i \in ob(\mathcal{I})$ and a subetale morphism $f : V \to \varpi(i)$ with a smooth irreducible scheme V furnished with an exponent F and an NC-data Δ in it. We then ask for a possibility of a (E, Γ)-*augumentation*, that is to enlarge the given A-site $\mathfrak{Z}$ so as to include (f, V, F, Δ) in a categorically natural manner. There are two cases as follows.

(1) The first is the case in which such an augumention is *unnecessary.* Namely there already exist $k \in ob(\mathcal{I})$ and $(\mu : k \to i) \in mor(\mathcal{I})$ such that $\varpi(\mu) = fh$ with an isomorphism $h : \varpi(k)\tilde{\to}V$.
(2) Next is the case when we want to augument the A-site $\mathfrak{Z}$ by addting a new object, say $*$, to $ob(\mathfrak{I})$ and a new map $\nu : * \to i$ to $mor(\mathfrak{I})$ in such a way that $\varpi(\nu) = f$.

This second case is to define the (E, Γ)-augumentation by means of (f, V, F, Δ) if it is "allowed" at all. We will call "EN-conditions" on (f, V, F, Δ) under which it is "allowed".

(1) F is a locally equivalent ambient reduction of $E(i)$ from $\varpi(i)$ to V with respect to f,
(2) V has normal crossing with Γ and we have $\Delta = f^{-1}(\Gamma(i))$. Here we discard any member $\Gamma(i)_k$ of $\Gamma(i)$ such that $f^{-1}(\Gamma(i)_k)$ is either empty or the whole V.

Needless to say, in order to justify the categorial property of the augumented A-site we must add all the morphisms newly created by compositions of ν and id_V with old members of $mor(\mathcal{I})$.

Definition 2.7. We say the (E, Γ)-augumentation of $\mathfrak{Z}$ (for instance by (f, V, F, Δ) as above) is *allowed* if the conditions of Rem.(2.3) are satisfied. An (E, Γ)-augumentation is called E-augumentation when Γ is not given or when we intentionally ignore the given Γ. In any case any such (E, Γ)-augumentation does not change $dim(\mathfrak{Z})$ and maintains $\mathfrak{Z}$ to be an A-site.

The following example of E-augumentation, is useful when a site $\mathfrak{Z}$ is not conjoint in the sense of Def.(1.6). The conjoining is always "allowed" in the sense of Def.(2.7).

Example 2.1. Let S be an equinet in an A-site $\mathfrak{Z}$ of Def.(1.6). We define

$$V \subset \left(\prod_{i \in S} \varpi(i)\right) \ \textit{which is defined by} \tag{2.2}$$
$$X = (X_i, i \in S) \in V \ \ \textit{if and only if}$$
$$\varpi(m)(X_j) = X_k \ \ \textit{for } \forall m = (j \to k) \in mor(\mathcal{I}) \ \textit{with } (jk) \in S^2$$

Then V is a smooth closed subscheme of $\prod_{i \in S} \varpi(i)$. Its dimension is $dim_S(\mathfrak{Z})$ of Def.(1.6). We let $f_i : V \to \varpi(i)$ be the projection for every $i \in S$. Add $*$ to $ob(\mathcal{I})$ and define $\varpi(*) = V$. We are then allowed to perform the augumentations for all $i \in S$ by

$$\left(f_i,\, V,\, f_i^{-1}(E(i)),\, f_i^{-1}(\Gamma(i)) \right) \ \ \textit{for} \ \ \forall i \in S$$

When exponent E and NC-data Γ are given in $\mathfrak{Z}$, the "EN-conditions" of Def.(2.3) are all satisfied.

This type of (E,Γ)-augumentation will be called "conjoining" by the given equinet. The result of conjoining is again an A-site.

3. Furnished saturation

Remark 3.1. Assume that we are given an exponent E and an NC-data Γ in an A-site $\mathfrak{Z}$. We then make all possible site-augumentations, allowable respect with to the given (E,Γ) and/or to E, For instance conjoinings of Ex.(2.1), those of Def.(2.7) in the manner of Rem.(2.3), subtale morphism of smooth affines into $\varpi(i)$ with $i \in ob(\mathcal{I})$ and embeddings of $\varpi(j)$ into smooth affines with dimensions $\leq dim(\mathfrak{Z})$, so long as they are "allowable" with respect to the extendability of (E,Γ) and/or to E and so long as the augumentations are "significant" (i.e. there is no isomorphic one already there). Moreover a new stale morphism from from $\varpi(i)$ into $\varpi(j)$ can be added provided it maps (E,Γ) and/or to E into itself,

It should be kept in mind that the following conditions must always honored.

$$\text{local } \mathfrak{S}\text{-equivalent ambient reduction of } E \\ \text{and normal crossings of and with } \Gamma \tag{3.1}$$

cf. Rem.(2.3), Def.(2.7) and Eq.(3.1).

Definition 3.1. We say that $\mathfrak{Z}$ is (E,Γ)-saturated (resp. E-saturated) if any allowed (E,Γ)-augumentation (resp. E-augumentation) and necessary morphism additions are "unnecessary" in the sense of Rm.(3.1). We can always create an (E,Γ)-saturated (respectively an E-saturated) A-site anew which contains any given A-site with E and Γ. The result will be called (E,Γ)-saturation (respectively E-saturation) of $\mathfrak{Z}$.

Theorem 3.1. *Consider an exponent E in an A-site $\mathfrak{Z}$. There then exists a closed subscheme of $\mathfrak{Z}$ in the sense of Def.(2.4), denoted by $Sing(E)$, such that $Sing(E)(i)$ is the singular locus of $E(i)$ for every $i \in \mathcal{I}$. We call $Sing(E)$ the singular locus of E.*

Proof. The following are all we need to prove for every $(m : i \to j) \in mor(\mathcal{I})$.

1. We have $\varpi(m)^{-1}(Sing(E)(j)) = Sing(E)(i)$. This equality is immediate from the definition of an ambient reduction.

(2) We should recall that $Sing()$ is a Zariski closed subset not only for ideal exponent but also for $/^q$-exponent and hence for every mixed exponent.

$\square$

Lemma 3.2. *Let $f : V \to W$ be a subetale morphism from smooth V to smooth W. Given any closed point $\xi \in V$ we can find an etale morphism $h : W' \to W$ and an embedding $V' \subset W'$ with a neighborhood V' of $\xi \in V$ such that $h\big|_{V'} = f\big|_{V'}$.*

Ths is straight forward by the well-known local etale criterion.

Lemma 3.3. *Consider an exponent F in V and E in W. Assume we have a retraction $r : V \subset W \to V$ inducing the identity in V. Then F is an $\mathfrak{S}$-equivalent ambient reduction of E if and only if*

$$\mathfrak{S}(E) = \mathfrak{S}\Big(r^{-1}(F) \cap \mathbf{I}(V \subset W)\Big) = \mathfrak{S}\big(r^{-1}(F)\big) \cap \mathfrak{S}\big(\mathbf{I}(V \subset W)\big)$$

where $\mathbf{I}(V \subset W) = (I(V \subset W), 1)$.

Proof. The presence of $\mathbf{I}(V \subset W)$ means that all the centers of permissible blowups are mapped down into V. Thus the lemma is straight from the deffinition of the equivalent ambient reduction. $\square$

Lemma 3.4. *Consider the case of $\varpi(m) : \varpi(i) \to \varpi(j)$ where we have $dim_\xi(\varpi(i)) < dim_\xi(\varpi(j))$ with a closed point $\xi \in Sing(E)(i) \subset \varpi(i)$. We then have $ord_{\varpi(m)(\xi)}(E(j)) = 1$.*

Proof. Immediate from Lem.(3.3) by Lem.(3.2). $\square$

Proposition 3.5. *Given an exponent E in a A-site $\mathfrak{Z}$ we consider a pair of morphisms*

$$\varpi(m) : \varpi(i) \to \varpi(k) \leftarrow \varpi(j) : \varpi(n)$$

with $\varpi(m)$ and $\varpi(n)$ having the same target $\varpi(k)$. Pick a pair of closed points $\xi \in Sing(E)(i)$ and $\eta \in Sing(E)(j)$ such that $\varpi(m)(\xi) = \varpi(n)(\eta)$, called $\zeta \in \varpi(k)$. Assume $dim_\xi(\varpi(i)) \leq dim_\eta(\varpi(j)$. Then we have an etale morphism $f : W \to \varpi(k)$ with a point $\theta \in W$ and a pair of etale morphisms from subschemes $(W(i), W(j))$ as follows.

$$f(i) : \varpi(i) \leftarrow W(i) \subset W \supset W(j) \to \varpi(j) : f(j)$$

such that

(1) W is a smooth irreducible scheme having a pair of closed smooth irreducible subschemes $W(i)$ and $W(j)$,
(2) $\theta \in W(i) \cap W(j)$ with $\xi = f(i)(\theta)$ and $\eta = f(j)(\theta)$,
(3) $\varpi(m)f(i)$ and $\varpi(n)f(j)$ are restrictions of f respectively,

(4) *and moreover there exists an automorphism* ϕ *of* W *which maps* $W(i)$ *into* $W(j)$ *and* $\phi(f(i)^{-1}(E(i))$ *is the* $\mathfrak{S}$*-equivalent ambient reduction of* $f(j)^{-1}(E(j))$.

Proof. Let $r = dim(\varpi(k)) - dim(\varpi(i))$ and let $\mathbb{A}^r$ be the affine space of dimension r over the base field $\mathbb{K}$. Choose a local projection $\pi : \varpi(k) \to \mathbb{A}^r$ which is transversal both to the local images of $\varpi(i)$ and $\varpi(j)$ at ζ and take the fiber product of $\varpi(i)$ and $\varpi(j)$ over $\mathbb{A}^r$ by means of the projection π. Pick the point $\theta = \xi \times \eta$ in the fiber product. Then restrict the so-obtained diagram of morphisms to suitable neighborhoods of θ, ξ and η. □

Lemma 3.6. *Consider the case of* $dim_\xi(\varpi(i)) < dim_\eta(\varpi(j))$ *which is* $\leq dim_\zeta(\varpi(k))$*. Then there exists a neighborhood* U *of* $\zeta \in \varpi(k)$ *and a smooth irreducible closed subscheme* V *of* $U \cap Im(\varpi(n))$ *such that* $dim_\zeta(V) = dim_\xi(\varpi(i))$ *and there exists an equivalent ambient reduction* $F(j)$ *of* $E(j)$ *from* $\varpi(n)^{-1}(U)$ *to* $\varpi(n)^{-1}(V)$.

Proof. By the smoothness of the schemes and etaleness of the morphisms of the lemma we can find a regular system of parameters $x = (v, w, y)$ of $\mathcal{O}_{\varpi(k),\zeta}$ such that v generates the ideal of $Im(n)$ and w is contained in the ideal of $Im(m)$ while (v, w) defines a local smooth subscheme V of dimension equal to $dim_\xi(\varpi(i))$ in $Im(\varpi(n))$. The existence of equivalent ambient reduction of $F(j)$ follows by the criterion of Lem.(3.3). □

Lemma 3.7. *Resume* E *together with* $\varpi(m)$ *and* $\varpi(n)$ *such that* $\varpi(m)(\xi) = \varpi(n)(\eta) = \zeta \in \varpi(k)$ *in accord with Lem.(3.6). This time we consider the case with* $dim_\xi(\varpi(i)) = dim_\eta(\varpi(j))$*, say* $= r$*. Then there exists a smooth irreducible affine scheme* $U \in ob(\mathcal{Z})$ *with an exponent* F *in it and etale morphisms* $m^\circ : U \to \varpi(i)$ *and* $n^\circ : U \to \varpi(j)$ *such that*

(1) *the images of both* mm° *and* nn° *contain a neighborhood of* ζ *in* $\varpi(k)$*,*
(2) *the pullbacks* $(mm^\circ)^{-1}(E(i))$ *and* $(nn^\circ)^{-1}(E(j))$ *are both* $\mathfrak{S}$*-equivalent to* F *in* U
(3) F *is a locally equivalent ambient reduction of* $E(k)$ *with respect to* mm° *as well as to* nn°*.*

Proof. We choose an etale morphism $h; W \to \varpi(k)$ with closed smooth subschemes V_1 and V_2 such that $h\big|_{V_1}$ is factored by $\varpi(m)$ and $h\big|_{V_2}$ is by $\varpi(n)$. Here the existence is by a repeated aplication of Lem.(3.2). Then pick a point $\xi' \in V_1$ mapped to ξ and $\eta' \in V_2$ mapped to η. Clearly it is enough to prove the lemma for V_1 and V_2 instead of $\varpi(i)$ and ϖ_2. In other words the proof is reduced to the special case in which $\varpi(m)$

and $\varpi(n)$ are closed embddings. After restricting the given data to a neighborhood of $\zeta \in \varpi(k)$, we can have a projection $f : \varpi(k) \to \mathbb{A}^r$ which is transversal to $\varpi(i)$ as well as to $\varpi(j)$ at $\zeta(= \xi = \eta)$. Then we take the base change from $\mathbb{A}^r$ to a neiborhood of the point (ξ, η) in $\varpi(i) \times_{\mathbb{A}^r} \varpi(j)$, which corresponds to ζ, say B. We then see that our problem is reduced to the case in which $\varpi(i)$ and $\varpi(j)$ become two sections of the projection after the base change, In other words we have two retractions in the sense of Lem.(3.3) as follows.

$$f : \varpi(i) \subset \varpi(k) \to \varpi(i) \quad \textit{and} \quad f : \varpi(j) \subset \varpi(k) \to \varpi(j)$$

with the same f and with the same target, say B, which is isomorphic to both $\varpi(i)$ and $\varpi(j)$ by the projection f. Write $e_i : \varpi(i) \to B)$ and $e_j : \varpi(j) \to B)$ for the isomorphisms induced by f. We then apply the $\mathfrak{S}$-equivalence of Lem.(3.3) which asserts

$$\begin{aligned} &\mathfrak{S}\big(f^{-1}(e_i(E(i)))\big) \cap \mathfrak{S}\big(\mathbf{I}(\varpi(i) \subset \varpi(k))\big) \\ &= \mathfrak{S}\big(f^{-1}(e_j(E(j)))\big) \cap \mathfrak{S}\big(\mathbf{I}(\varpi(j) \subset \varpi(k))\big) \end{aligned} \tag{3.2}$$

Writing $\mathbf{I}_i = \mathbf{I}(\varpi(i) \subset \varpi(k))$ and $\mathbf{I}_j = \mathbf{I}(\varpi(j) \subset \varpi(k))$ we have that Eq.(3.2) becomes equal to

$$\mathfrak{S}\Big(f^{-1}(e_i(E(i))) \cap f^{-1}(e_j(E(j)) \cap \mathbf{I}_i \cap \mathbf{I}_j\Big)$$

Now we let

$$F \;=\; f^{-1}(e_i(E(i))) \cap f^{-1}(e_j(E(j)) \cap \mathbf{I}_i \cap \mathbf{I}_j \tag{3.3}$$

which is symmetric for (i, j) and $\mathfrak{S}$-equivalent to $E(k)$ in $\varpi(k)$. Now the lemma is straight forward. □

Lemma 3.8. *Under the assumption of Lem.(3.7) we have* $ord_\xi(E(i)) = ord_\eta(E(j))$.

Proof. The coefficients in the required divisor are defined directly at the generic points of the $\Gamma \cap \varpi(i)$ □

A proof of Th.(3.9) is staight forward from the lemmas and a proposition above.

Theorem 3.9. *Let us recall the general assumption of Rem(1.1). We then consider a pair of closed points* $P = (i, \xi)$ *and* $P^\dagger = (i^\dagger, \xi^\dagger)$ *of the singular locus* $Sing(E)$ *of Th.(3.1). Assume that* P *is* $\mathfrak{Z}$*-equivalent to* $P^\dagger$ *and moreover* $dim(\varpi(i)) = dim(\varpi(i^\dagger))$. *We then have* $ord_\xi(E(i)) = ord_{\xi^\dagger}(E(i^\dagger))$, *i.e,* $ord_P(E) = ord_{P^\dagger}(E)$. *It should be noted that this order equality can fail if the dimension equality is not assumed.*

Proof. Since P is $\mathfrak{Z}$-equivalent to $P^{\dagger}$, we have a correspondence chain C of morphisms in $\mathcal{I}$ according to Def.(1.4). Our question is local about the successively corresponding points in C which define $\mathfrak{Z}$-equivalence from P to $P^{\dagger}$. Therefore we may replace the ambient schemes by neighborhoods of those points. At the same time we may then shorten the chain C as much as possible. For instance if C contains a string of arrows in the same direction we can replace it by its composition. Moreover if we find any equinet contained in C we would shorten C by means of equinet-conjoining, for $\mathfrak{Z}$ is assumed to be conjoint in the sense of Def.(1.6). Therefore we may assume that the arrows in C alterate directions and the target of every arrow has a strictly higher dimension than the source. In other words C may be viewed as a string of triplets of the form

$$\begin{aligned} &\varpi(k-1) \to \varpi(k) \leftarrow \varpi(k+1) \\ &\textit{with}\ \ dim(\varpi(k)) > max\{dim(\varpi(k-1)), dim(\varpi(k+1))\} \end{aligned} \tag{3.4}$$

except for the cases in which either $P \in \varpi(k-1)$ or $P^{\dagger} \in \varpi(k+1)$. If one of the couple $\{dim(\varpi(k)), dim(\varpi(k+1))\}$ is bigger than the other, we apply Lem.(3.6) to it and replace it by the new one (V of Lem.(3.6)) having the smaller dimension of the couple. Such a replacement is possible for C within the given site $\mathfrak{Z}$ which is assumed to be E-saturated. Now keep repeated application of Lem.(3.6) in this manner and reach the state in which we have $dim(\varpi(k)) = dim(\varpi(k+1))$ for every one of the triplets except for the ones having either $P \in \varpi(k-1)$ or $P^{\dagger} \in \varpi(k+1)$. This repetition is finished by a finite number of times. Now then if the dimension at P and $P^{\dagger}$ is bigger than the middle lower dimension of the chain C then we have $ord_P(E) = ord_{P^{\dagger}}(E) = 1$ by Lem.(3.4). If otherwise we must have the middle lower dimension equal to the dimesnion at P and $P^{\dagger}$ because these points must then belong to the group of the triplets. We then conclude $ord_P(E) = ord_{P^{\dagger}}(E)$ by Lem.(3.6). Th.(3.1) is now proven. □

Theorem 3.10. *Let $\tilde{\mathfrak{Z}}$ be an E-saturation of $\mathfrak{Z}$. Then we have the following are true.*

1. *There exists a unique exponent $\tilde{E}$ in $\tilde{\mathfrak{Z}}$ which contains E.*
2. *$Sing(\tilde{E})$ is the unique subscheme of $\tilde{\mathfrak{Z}}$ which contains $Sing(E)$.*
3. *Every connected component of $Sing(\tilde{E})$ is irreducible as a site.*
4. *The connected components $\{C\}$ of $Sing(E)$ are in a one-to-one correspondence with the connected components $\{\tilde{E}\}$ of $Sing(\tilde{E})$ in such a way that $\tilde{E}$ is the unique subscheme of $\tilde{\mathfrak{Z}}$ which contains C.*

(5) *Let* $C \subset \tilde{C}$ *be a corresponding pair of cnnected components as above. Then a* $\tilde{E}|_{\tilde{C}}$*-saturation of* $\tilde{\mathfrak{Z}}$ *is a* $E|_C$*-saturation of* $\mathfrak{Z}$.

(6) *Let* $P = (i, \xi)$ *and* $Q = (j, \eta)$ *be points of* $Sing(\tilde{E})$*. If* P *is* $\tilde{E}$*-equivalent and* $dim_P(\tilde{E}) = dim_Q(\tilde{E})$*, say* $= d$*, then there exists a point* $R = (k, \zeta)$ *of* $\tilde{E}$ *with* $(m : k \to i) \in mor(\tilde{\mathcal{I}})$ *and* $(n : k \to j) \in mor(\tilde{\mathcal{I}})$ *such that* $\tilde{\varpi}(m)(R) = P$ *and* $\tilde{\varpi}(n)(R) = Q$*. We can choose such an* R *having* $dim_R(\tilde{E}) = d$*. Moreover there exists an automorphism* $\ell : k \to k) \in mor(\tilde{\mathcal{I}})$ *such that* $n\ell = m$.

4. DIMENSION LEVEL FOCUSING

We now consider an arbitrary A-site $\mathfrak{Z}$ and comsider its subsite with each choice of dimension level. We will choose a suitable dimension level of ambient schemes for embedding of the given singular data in order to show the structural feature more explicitly. It is useful for instance to exhibit Γ-divisorial factors of a given (E, Γ) inside those chosen ambient schemes.

Definition 4.1. We define the subsite of *dimension level* d of $\mathfrak{Z}$, denoted by $\mathfrak{Z}[d]$, for each positive integer d as follows.

$$\mathfrak{Z}[d] = (\mathcal{I}[d], \mathcal{Z}[d], \varpi[d]) \tag{4.1}$$
$$ob(\mathcal{I}[d]) \;=\; \{i \in ob(\mathcal{I}) \big| dim(\varpi(i)) = d\}$$
$$mor(\mathcal{I}[d]) = mor(\mathcal{I})|_{ob(\mathcal{I}[d])},\ \mathcal{Z}[d] = \mathcal{Z}|_{ob(\mathcal{I}[d])},\ \varpi[d] = \varpi|_{ob(\mathcal{I}[d])}$$

When E and/or Γ are given we define $E[d]$ and/or $\Gamma[d]$ to be their restrictions to $\mathfrak{Z}[d]$. We may write $\varpi(i)$, $E(i)$ and $\Gamma(i)$ instead of $\varpi[d](i)$, $E[d](i)$ and $\Gamma[d](i)$ respectively for $i \in ob(\mathcal{I}[d])$. We will be mainly interested in the case in which $\mathfrak{Z}$ is (E, Γ)-saturated and the bound dimension d is $\leq\ embd_{\mathfrak{Z}}(E, \Gamma)$ of Def.(4.3).

Definition 4.2. We define the subsite of *dimension bounded by* d of $\mathfrak{Z}$, denoted by $\mathfrak{Z}[d\triangleright]$, for each positive integer d as follows.

$$\mathfrak{Z}[d\triangleright] = (\mathcal{I}[d\triangleright], \mathcal{Z}[d\triangleright], \varpi[d\triangleright]) \tag{4.2}$$
$$ob(\mathcal{I}[d\triangleright]) \;=\; \{i \in ob(\mathcal{I}) \big| dim(\varpi(i)) \leq d\}$$
$$mor(\mathcal{I}[d\triangleright]) = mor(\mathcal{I})|_{ob(\mathcal{I}[d\triangleright])},\ \mathcal{Z}[d\triangleright] = \mathcal{Z}|_{ob(\mathcal{I}[d\triangleright])},\ \varpi[d\triangleright] = \varpi|_{ob(\mathcal{I}[d\triangleright])}$$

For E and Γ, let $E[d\triangleright]$ and $\Gamma[d\triangleright]$ denote their restrictions to $\mathfrak{Z}[d\triangleright]$.

We next define the "embedding dimension" of an exponent E in an A-site $\mathfrak{Z}$ after replacing E by its extension into the E-saturation of $\mathfrak{Z}$. The "embedding dimensions", local and/or global, become most significant "dimension levels" in the study of the given singular data.

Definition 4.3. Recall that $\mathfrak{Z}$ is irreducible (connected, generically conjoint and quasicompact) and let us assume that $\mathfrak{Z}$ is (E, Γ)-saturated. For any given point $P = (i, \xi) \in Sing(E)$ we define the (E, Γ)-*embedding dimension* in $\mathfrak{Z}$ or the *embedding dimension* of (E, Γ) in $\mathfrak{Z}$ at P, denoted by $embd_{\mathfrak{Z},P}(E, \Gamma)$ as follows.

$$\begin{aligned} embd_P(E,\Gamma) &= embd_{\mathfrak{Z},P}(E,\Gamma) \\ &= min\{dim_{P^\dagger}(\mathfrak{Z}) \,|\, all\ P^\dagger \sim P\} \\ &\text{where } \sim \text{ denotes } Sing(E)\text{-equivalence} \end{aligned} \tag{4.3}$$

Here $Sing(E)$-equivalence is the same as $\mathfrak{Z}$-equivarence for $Sing(E)$ is an I-subside of $\mathfrak{Z}$ in the sense of Def.(1.8). Globally we define

$$\begin{aligned} embd(E,\Gamma) &= embd_{\mathfrak{Z}}(E,\Gamma) \\ &= max_P(embd_{\mathfrak{Z},P}(E,\Gamma)) \\ &\text{where } max_P \text{ is for all } P \in Sing(E). \end{aligned} \tag{4.4}$$

Definition 4.4. We define and will make use of E-saturation $\tilde{\mathfrak{Z}}$ of $\mathfrak{Z}$ and the unique extension $\tilde{E} \supset E$ into $\tilde{\mathfrak{Z}}$, ignoring the given NC-data Γ. When Γ is not empty the E-staturation could be bigger than (E, Γ)-saturation and we always have the former to contain the latter. Moreover the embedding dimension of the former could be smaller than that of the latter. As for the E-stauration, we drop Γ from the notation of Def.(4.3) and write $embd_P(E)$, $embd_{P,\mathfrak{Z}}(E)$, $embd_{\mathfrak{Z}}(E)$ and $embd(E)$ in order to distinguish them from those having Γ in their symbols.

Theorem 4.1. *Given E in $\mathfrak{Z}$ as above, let $\tilde{\mathfrak{Z}}$ be the E-saturation of E and $\tilde{E} \supset E$ be the unique extention into $\tilde{\mathfrak{Z}}$. Pick any positive integer $d \geq embd(\tilde{E})$. We then have the following:*

(1) $\tilde{\mathfrak{Z}}[d\triangleright]$ is the $\tilde{E}[d]$-saturation of $\tilde{\mathfrak{Z}}[d]$.
(2) $\tilde{E}[d\triangleright] \supset \tilde{E}[d]$ is the unique extension into the saturation $\tilde{\mathfrak{Z}}[d\triangleright]$.
(3) Pick a connected component C of $Sing(\tilde{E}[d])$ then there exists a unique extension $\tilde{C} \supset C$ of C into $\tilde{\mathfrak{Z}}$. The $\tilde{C}$ is necessarily a connected component of $Sing(\tilde{E})$. The $\tilde{C}[d\triangleright]$ is the unique extension of C into $\tilde{\mathfrak{Z}}[d\triangleright]$.

Remark 4.1. Given an A-site $\mathfrak{Z}$ with (E, Γ) as above, we assume that $\mathfrak{Z}$ is (E, Γ)-saturated and we fix an E-saturation $\tilde{\mathfrak{Z}}$ of $\mathfrak{Z}$. We then have unique extensions ($\tilde{E} \supset E$) and $Sing(\tilde{E}) \supset Sing(E)$ in $\mathfrak{Z}$. We choose the following pair of embedding dimensions.

$$d = d(E, G) = embd(E) \quad and \quad \tilde{d} = d(\tilde{E}) = embd(\tilde{E}) \tag{4.5}$$

$$In\ general\ we\ have\ \ \tilde{d} \ \leq \ d$$

This is in accord with Eq.(4.4).

5. NC-divisors, local and global

Let $\mathfrak{z}_k, 1 \leq k \leq t$, denote the site-ideal in $\mathfrak{Z}$ defining the hypersurface Γ_k in the sense of Def.(2.4) and write

$$\mathfrak{z} = (\mathfrak{z}_1, \cdots, \mathfrak{z}_t) \tag{5.1}$$

Remark 5.1. In the following theorem and later on, we will often use symbol of the form J^a with an ideal J and a rational number $a > 0$. Then J^a should be understood as the equivalence class of pairs (m, J^{ma}) for all those integers $m > 0$ such that ma is an integer.

Theorem 5.1. *Let us consider the case in which $\mathfrak{Z}$ is not only (E, Γ)-saturated but also E-saturated so that we have $d = \tilde{d}$ for the dimensions of Eq.(4.5). Let us pick a connected component C of $Sing(E[d])$ which has its unique extension to a connected component $C\triangleright$ of $Sing(E[d\triangleright])$. We then have a Γ-divisor $Div(E, \Gamma, C)$ uniquely determined by (E, Γ, C) as follows.*

$$Div(E, \Gamma, C) \ = \ \mathfrak{z}^{\alpha(E,\Gamma,C)} \ = \ \textstyle\prod_{k=1}^t \mathfrak{z}_k^{\alpha(E,\Gamma,C,k)} \tag{5.2}$$

with $\mathfrak{z}$ of Eq.(5.1) and nonnegative rational numbers $\alpha(E, \Gamma, C, k)$, which have the following properties. For every closed point $Q = (j, \eta) \in Sing(\tilde{E})$, $Div(E, \Gamma, C)$ induces an ideal generated by $\Gamma(Q)$-divisorial element $div_Q(E, \Gamma, C)$ in the local ring $R_Q = \mathcal{O}_{\tilde{\varpi}(j),\eta}$ such that

$$\begin{aligned} div_Q(E, \Gamma, C) \ &= \ u(Q) z(j)^{\alpha(E,\Gamma,C)} \quad where \\ z(j)^{\alpha(E,\Gamma,C)} \ &= \ \textstyle\prod_{k \in \nabla(Q)} z(j)_k^{\alpha(E,\Gamma,C,k)} \\ Sing(\tilde{E}) \cap \Gamma_k = \emptyset \ &\Rightarrow \ \alpha(E, \Gamma, C, k) = 0. \end{aligned} \tag{5.3}$$

where

(1) $\nabla(Q) \subset [1, t]$ is the subset of those k having $\eta \in \Gamma_k(j) \neq \tilde{\varpi}(j)$,
(2) $u(Q)$ is a unit in R_Q and $z(j)_k$ generates the ideal $\mathfrak{z}(j)_k$ in R_Q,
(3) $div_Q(E, \Gamma, C)$ is the maximal $\Gamma(j)$-factor of $\tilde{E}(j)$ at η.

It should be noted that the rational numbers $\alpha(E,\Gamma,C,k)$ are independent of Q for each choice of (C,k) for the given (E,Γ).

The proof of the theorem is done as the combination of the following two kinds of direct connections,

Lemma 5.2. *Pick a pair of "closed" points $P=(i,\xi)$ and $Q=(j,\eta)$ contained in a connected component C of $Sing(E)[d]$ where $dim_\eta(\varpi(j) = dim_\zeta(\varpi(k)) = d$. Assume that P is $\mathfrak{Z}[d]$-equivalent to Q. Then there exists a closed point $R=(k,\zeta)\in Sing(E)[d]$ together with $(m: k\to i)\in mor(\mathcal{I})$ and $(n: k\to j)\in mor(\mathcal{I})$ such that $(\varsigma[d](m))(R) = P$ and $(\varsigma[d](n))(R)=Q$.*

Proof. Easy proof. □

Lemma 5.3. *Consider the case in which $P=(i,\xi)$ and $Q=(i,\eta)$ have the same i while ξ and η are two closed points of of the same connected component $C(i)\subset Sing(E[d])(i)\subset\varsigma[d](i)$. Then there exists $(\ell: k\to k)\in mor(\mathcal{I})$ such that $\varsigma[d](\ell)$ is an automorphism of $\varsigma[d](\ell)$ such that $\varsigma[d](\ell)(\xi)=\eta$. $\varsigma[d](\ell)$ must then map every component $\Gamma_k(i)$ of $\Gamma(i)$ into itself for every k.*

Let us restrict our interest to the special case in which E is a $/^q$-exponent in the A-site $\mathfrak{Z}$, locally expressed as $E_Q = (\mathbf{g}\,\|\,/^q)$ at a given closed point $Q=(i,\xi)$ in accord with the Th.(5.1). This is, however, the case in which we need especially detailed investigation on the structure of the monomial $div_Q(E,\Gamma,C)$, or the rational vector $\mathbf{a}(E,\Gamma,C)$. Letting

$$\mathbf{a}(E,\Gamma,C) \;=\; q\alpha(E,\Gamma,C) \text{ refering to Eq.(5.3).} \tag{5.4}$$

It should be noted that all the components of $\mathbf{a}(E,\Gamma,C)$ are integers.

$$\mathfrak{z}^{\,\mathbf{a}(E,\Gamma,C)} \;=\; \mathfrak{z}^{\,q\mathbf{b}(E,\Gamma,C)}\mathfrak{z}^{\mathbf{c}(E,\Gamma,C)} \tag{5.5}$$

globally along $\tilde{C}$ where $\mathbf{b}(E,\Gamma,C)\in\mathbb{Z}^t$ and $0\le\mathbf{c}(E,\Gamma,C,k)<q,\forall k$. Then locally at $Q=(j,\eta)$ we write it as follows.

$$z(j)^{\,\mathbf{a}(E,\Gamma,C)} \;=\; z(j)^{\,\mathbf{b}(E,\Gamma,C)}v(Q)^{\mathbf{c}(E,\Gamma,C)} \tag{5.6}$$

where $v(Q)$ is the subsystem of $z(j)$ consisting of those $z(j)_k$ with those k such that the k-th component of $z(j)^{\,\mathbf{a}(E,\Gamma,C)}$ is not divisible by q.

6. PERMISSIBLE BLOWUP

Conisder an A-site $\mathfrak{Z}$ with (E, Γ) as above.

Definition 6.1. A subscheme D of $\mathfrak{Z}$ is called a permissible center for (E, Γ) if it has the following properties.

(1) D is smooth irreducible and contained in $SingE$,
(2) D has normal crossing with Γ, i.e., for every point $P = (i, \xi)$ there exists a regular system of parameters $x = (x_1, \cdots, x_m)$ of $R_P = \mathcal{O}_{\varpi(i),\xi}$ such that
  (a) if $\xi \in \Gamma_k(i)$ then the ideal of $\Gamma_k(i)$ in R_P is generated by one of the x_j's for every member Γ_k of Γ and
  (b) if $P \in D$ then the ideal of $D(i)$ in R_P is generated by a subsystem of x.
(3) $ord_\xi(E(i))$ is constant for every $P \in D_{cl}$.

It should be noted that we are taking only closed points in the last condition. This is important because of possibility of "generic down" phenomena when a $/^q$-exponent is involed with E.

Definition 6.2. Given a permissible center D for (E, Γ) in $\mathfrak{Z}$ we define the blowup

$$\pi: \ \mathfrak{Z}' = (\mathcal{I}', \mathcal{Z}', \varpi') \ \rightarrow \ \mathfrak{Z} = (\mathcal{I}, \mathcal{Z}, \varpi) \ \ \textit{with center} \ \ D \tag{6.1}$$

as follows. Pick any index $i \in I = \mathcal{I}$ and any element a in the ideal I of $D(i) \subset \varpi(i) = Spec(A)$, and define $\varpi'(i, a) = Spec(A[a^{-1}I])$ which is a smooth affine scheme. We define $ob(\mathcal{I}')$ to be the set of all pairs (i, a) as above and $mor(\mathcal{I}')$ to be the set of all pairs $n = (m, f)$ with $(m : i \to j) \in mor(\mathcal{I})$ and with a morphism of affine schemes $f : \varpi'(i, a) \to \varpi'(j, b)$ which is compatible with $\varpi(m)$. The f is named $\varpi'(n)$. Thus we have defined $\mathcal{I}'$ and ϖ'. The category $\mathcal{Z}'$ is composed of all those schemes $\varpi'(i, a)$ and all those morphisms f as above. The π of Eq.(6.1) signifies the index projections $(i, a) \to i$ and the scheme morphisms $\varpi'(i, a) \to \varpi(i)$ associated with the natural $A \to A[a^{-1}I]$. We have an irreducible hypersurface $\mathcal{E}$ in $\mathfrak{Z}'$ which consists of the scheme hypersurfaces $\mathcal{E}(i, a) \subset \varpi'(i, a)$ defined by the ideals $(a)A[a^{-1}I]$. This $\mathcal{E}$ is called the "exceptional divisor" of the blowup π. Note that we obtain an NC-data in $\mathfrak{Z}'$ as follows.

$$\Gamma' \ = \ (\Gamma'_1, \cdots, \Gamma'_t, \mathcal{E}) \tag{6.2}$$

where Γ'_k is the site hypersurface consisting of scheme hypersurfaces defined by the ideals $(a^{-1}I_k)A[a^{-1}I]$ with the ideal $I_k \subset A$ of $\Gamma_k(i)$. the Γ'_k is called the "strict transform" of Γ_k by π.

Theorem 6.1. *We are given* $\mathfrak{Z}$ *furnished with* (E,Γ) *as above. Let* $\tilde{\mathfrak{Z}}$ *be an* E*-saturation of* $\mathfrak{Z}$ *(ignoring* Γ*) and we have the exponent* $\tilde{E}\supset E$ *in* $\tilde{\mathfrak{Z}}$*, uniquely determined by* E*. Let us assume that there exists a finite sequence of successively permissible blowups*

$$\pi(k):\mathfrak{Z}^{(k)}\to\mathfrak{Z}^{(k-1)}, 1\le k\le r, \text{ with } \mathfrak{Z}^{(0)}=\mathfrak{Z} \tag{6.3}$$

such that

(1) *the centers of the* $\pi(k)$*'s are all mapped down into the given* $\Gamma=(\Gamma_1,\cdots,\Gamma_t)$*,*
(2) *with the strict transform* $\Gamma_i(r)$ *of* Γ_i *and with the transform* $E(r)$ *of* E *by Eq.(6.3) we have* $\Gamma_i(r)\cap Sing(E(r))=\emptyset$

The sequence Eq.(6.3) can be naturally and uniquely extended to

$$\tilde{\pi}(k):\tilde{\mathfrak{Z}}^{(k)}\to\tilde{\mathfrak{Z}}^{(k-1)}, 1\le k\le r, \text{ with } \tilde{\mathfrak{Z}}^{(0)}=\tilde{\mathfrak{Z}} \tag{6.4}$$

Define the $\tilde{\Gamma}(r)$ *in* $\tilde{\mathfrak{Z}}^{(k)}$ *to be the natural extension of* $\Gamma^{(r)}$ *which is the tansform of* Γ *by Eq.(6.3). (*$\Gamma^{(r)}$ *is an NC-data in* $\mathfrak{Z}^{(r)}$ *by the permissibility of Eq.(6.3).) We then assert that the* $\tilde{\Gamma}(r)$ *is an NC-data within a neighborhood of* $Sing(\tilde{E}(r))\subset\mathfrak{Z}^{(r)}$*.*

Lemma 6.2. *Pick any permissible center* D *for* E *in* $\mathfrak{Z}$*.* $\tilde{\mathfrak{Z}}$ *be an* E*-saturation of* $\mathfrak{Z}$ *(ignoring* Γ*). We have a unique extension* $\tilde{E}\supset E$ *in* $\tilde{\mathfrak{Z}}$ *and a unique subscheme* $\tilde{D}\supset D$ *(and* $Sing(\tilde{E})\supset Sing(E)$*) in* $\tilde{\mathfrak{Z}}$*. Then* $\tilde{D}$ *is a permissible center for* $\tilde{E}$ *(but not necessarily for* $\tilde{\Gamma}$*). We have the natural extension of the blowup* $\pi:\mathfrak{Z}'\to\mathfrak{Z}$ *to* $\tilde{\pi}:\tilde{\mathfrak{Z}}'\to\tilde{\mathfrak{Z}}$ *together with the exceptional divisor* $\mathcal{E}$ *to* $\tilde{\mathcal{E}}$*. We claim that the exceptional divisor* $\tilde{\mathcal{E}}(i,a)$ *has normal crossing with* $\tilde{\varpi}(i,a)$ *for every point* $Q=(i,a)\in\tilde{\mathcal{I}}$*.*

The proofs of Lem.(6.2) and Th.(6.1) are straight forward from the definitions.

Remark 6.1. The induction hypothesis Eq.(6.3) of Th.(6.1) must be (and in fact will be) verified in the following sense.

> For every (E°,Γ°) in any A-site $\mathfrak{Z}^\circ=\mathfrak{Z}^{(0)}$ with $dim(\mathfrak{Z}^\circ)<n$ there exists a finite sequence of permissible blowups such that $Sing(\mathfrak{Z}^{(r)})$ is empty.

Incidentally this hypothesis is proven directly and easily for $n\le 2$ and has been known to be true for $n=3$.

Here we propose our "first inductive strategy" as follows.

Remark 6.2. "Induction on the number $d(E,G)$":
If $d(E,G)\le 1$ the resolution of singulrities is easy. If $d=d(E,G)>d(\tilde{E})=\tilde{d}$ then we pick a connected component C of $Sing(E)$ and the

concentration $E[C]$ of E along C. To be precise the exponent $E[C]$ in $\mathfrak{Z}$ is derived from E by changingg into unit all those ideals of E which are away from C, so that $E[C]|_U = E_U$ for a neighborhood $U \supset C$ and $Sing(E[C] = C$. We then perform the following inductive process.

(1) we have $d(\hat{C}[d]) = d$ with the unique extension $\hat{C}[d] \supset C[d]$ of $C[d]$ into the $(C[d], \Gamma)$-saturation $\tilde{\mathfrak{Z}}$ of to $\mathfrak{Z}$.
(2) We then apply resolution of singularities successively to $C[d\triangleright] \cap \Gamma_k$ and to its transform, and so on, in order to achieve the end result that the strict transforms of the $\Gamma_k, k = 1, 2, \cdots, t$ become away from the singular locus of the trandform $E^\dagger$ of E.
(3) We do this for every connected component of $Sing(E)$.
(4) Th end result is $d(E^\dagger, \Gamma^\dagger) < d$.

We thus can reduce the resolution problem to the case $d = \tilde{d}$.

(To continue) followed by:

7. Local analysis of global program

8. Reduction from $p^e, e > 1$, down to p

9. $/^p$-decomposition, $\sharp$-flags vs $\flat$-flags

10. $\sharp$-flags structure relation, local vs global

11. sum-division, stable part and metastable part

12. The end