RESOLUTION OF SINGULARITIES
PART (3)

1. STABLE AND PRESTABLE $/^p$-EXPONENTS

We consider a $/^p$-exponent $\mathcal{G}$ at a closed point $\xi \in Z$. We restrict our attention to the case of $q = p$, that is $e = 1$.

Definition 1.1. We say that $\mathcal{G}$ is "stable" at a closed point $\xi \in Sing(\mathcal{G})$ if it has a *standard abc-expression* in the sense of Def.(**??**), having the following properties:

$$\begin{aligned} \mathcal{G} \;&=\; (\mathbf{g} \,\|\, /^p) \quad \textit{with} \;\; \mathbf{g} = z^{\mathbf{a}} g = z^{p\mathbf{b}} v^{\mathbf{c}} g \\ \textit{where} \;\; \mathbf{d} \;&=\; resord_\xi(\mathcal{G}) \;=\; ord_\xi(g) \;\le\; 1 \end{aligned} \tag{1.1}$$

and moreover if $\mathbf{d} = 0$ then $v^{\mathbf{c}}$ should not be a unit, while if $\mathbf{d} = 1$ then g must be a key parameter of $\mathcal{G}$ in the sense that g is transversal to Γ, i.e, (v, g) is extendable to a regular system of parameters of R_ξ.

Recall that g is residual and $z \supset v$ is a system of parameters defining those components of the NC-data Γ which go through ξ.

Theorem 1.1. *Let us assume that $\mathcal{G}$ is stable at ξ. Then pick any fitted permissible blowup $\pi : Z' \to Z$ with center D for $\mathcal{G}$. Then the transform $\mathcal{G}'$ of $\mathcal{G}$ by π is stable at every point $\xi' \in \pi^{-1}(\xi) \cap Sing(\mathcal{G}')$.*

Proof. Since π is fitted we have every member of z identically zero on D. If $ord_\xi(g) = 0$ meaning that g is a unit we then $\mathbf{c} \neq 0$ implying $v^{\mathbf{c}}$ is not a unit. Except for the metastable case at ξ' the transform $\mathcal{G}'$ of $\mathcal{G}$ by π has its residual belonging to $\rho^q(R_{\xi'}$ and $(\mathcal{G}')$ is stable at ξ'. For the metastable case at a closed point ξ' of $Sing(\mathcal{G}') \cap \pi^{-1}(\xi)$ we have $resord_{\xi'}(\mathcal{G}') = 1$ by Th.(**??**). If $g = 0$ then $\mathcal{G}'$ is the same. If $g \in \rho^e(R_\xi), \neq 0$, then the residual of $\mathcal{G}'$ has order 0 or 1. Finally consider the case of $ord_\xi(g) = 1$ so that g is $\sharp$-key parameter of $\mathcal{G}$. By the definition of "fitted permissible" and by the generic-down theorem, we have either $g \in I(D, Z)$ or g is transversal to D. In each of the two cases consider the subcases of whether g is exceptional at ξ' or otherwise. We may discard those components of Γ which do not contain the center D without affecting the validity of the theorem (which makes the assumption on π stronger to be *fitted*). □

Definition 1.2. Let us consider a *standard abc-expression* of a $/^p$-exponent $\mathcal{G} = (\mathbf{g} \,\|\, /^p)$ with $\mathbf{g} = z^{p\mathbf{b}}v^{\mathbf{c}}g$ and $resord_\xi(\mathcal{G}) = ord_\xi(g)$, say $= \mathbf{d}$. Let N be an integer $> \mathbf{d}$. We say that $\mathcal{G}$ is "$/^q$-prestable" of depth p^N at ξ if it has either one of the following holds:

1. We have $g \in \rho(R_\xi)$ and $\mathbf{c} \neq (0)$.
2. We have $\mathbf{d} \not\equiv 0 \mod p$ and the residual factor of $\mathcal{G}$ is written as $g = UP$ where U is a unit in R_ξ while P is in $\rho^N(R_\xi)[\zeta]$ with a $\sharp$-key parameter ζ of $\mathcal{G}$.
3. We have $\mathbf{d} \equiv 0 \mod p$ and $g = UP$ where U is a unit while $P \in \rho^N(R_\xi)[\zeta_1, \zeta_2]$ where both (ζ_1, ζ_2) is a pair of independent $\sharp$-key parameters of $\mathcal{G}$.

2. $/^p$-PRESTABLE CASES

Throughout this section we are primarily interested in the fitted permissible transforms of a $/^p$-exponent, having $q = p$ and $e = 1$:

$$\mathcal{G} = (\mathbf{g} \| /^p) \textit{ with } \mathbf{g} = z^{\mathbf{a}}g = z^{p\mathbf{b}}v^{\mathbf{c}}g \tag{2.1}$$

locally expressed at a given closed point $\xi \in Sing(\mathcal{G}) \subset Z$ in the manner of Eq.(**??**) of Def.(**??**). We then examine the transforms $\mathcal{G}'$ of $\mathcal{G}$ by π locally defined at any chosen closed point $\xi' \in \pi^{-1}(\xi) \cap Sing(\mathcal{G}')$. We write

$$(\mathcal{G}'(i), \eta'(i)), 1 \leq i \leq \nu' \; \} \tag{2.2}$$

locally at ξ' in the manner of Eq.(**??**) of Def.(**??**), where

$$\begin{gathered} \mathcal{G}' = (\mathbf{g}' \| /^p) \textit{ with } \mathbf{g}' = z'^{\mathbf{a}'}g' = z'^{q\mathbf{b}'}v'^{\mathbf{c}'}g' \\ \textit{and } \mathcal{G}'(i) = (\mathbf{g}'(i) \| /^p) \\ \textit{with } \mathbf{g}'(i) = z'^{\mathbf{a}'(i)}g'(i) = z'^{q\mathbf{b}'(i)}v'(i)^{\mathbf{c}'(i)}g'(i). \end{gathered} \tag{2.3}$$

Remark 2.1. We have the following cases:

1. Case I: We have $resord_{\xi'}(\mathcal{G}') > resord_\xi(\mathcal{G})$.
2. Case II: We have $resord_{\xi'}(\mathcal{G}') = resord_\xi(\mathcal{G})$.
3. Case III: We have $resord_{\xi'}(\mathcal{G}') = resord_\xi(\mathcal{G}) - 1$.
4. Case IV: We have $resord_{\xi'}(\mathcal{G}') < resord_\xi(\mathcal{G}) - 1$.

If Case I happens then by Moh's Th.(**??**) we must have $resord_{\xi'}(\mathcal{G}') = resord_\xi(\mathcal{G}) + 1$ and v' of Eq.(2.3) must be empty by Prop.(**??**). Then we apply the Prop.(2.2) and Prop.(2.1) which are proven below. It should also be kept in mind that if Case IV happens then our inductive strategy is considered successful by Moh's theorem. The undesired phenomena are any indefinite sequences of alternately repeated Case I coupled with Case III possibly having some additions of Case II inserted between such couples.

Remark 2.2. Recall the exceptional parameter $\mathfrak{z}$ which is selected in the manner of Rem.(**??**). It should be kept in mind that

> $\mathfrak{z}$ appears in $in_\xi(v^{\mathbf{c}}g)$ because of the fitted permissibility of π. This is significant in Weierstrass Tchirnhausen expression when $v = \emptyset$. (Recall Th.(**??**.)

Viewing $\mathfrak{z}$ as an element of R_ξ as well as that of R'_ξ we consider the following cases:

1. Case A: $\mathfrak{z} \in v$. Equivalently $\mathfrak{z} \in \eta(\nu + 1)$.
2. Case B: $\mathfrak{z} \notin v$. Equivalently $\mathfrak{z} \in \eta(i)$ with some $i \leq \nu$.
3. Case A': $\mathfrak{z} \in v'$. Equivalently $ord_\xi(\mathcal{G}) \not\equiv 0 \mod p$.
4. Case B': $\mathfrak{z} \notin v'$. Equivalently $ord_\xi(\mathcal{G}) \equiv 0 \mod p$.

Proposition 2.1. *If there exists $i < \nu + 1$ such that $ord_\xi(\mathcal{G}(i)) = ord_\xi(\mathcal{G})$ then $\eta(i)$ must contain at least one $\sharp$-key parameter ζ of $\mathcal{G}$ at ξ. It follows that the Case I cannot happen for any fitted permissible transform of $\mathcal{G}$ at any closed point $\xi' \in \pi^{-1}(\xi) \cap Sing(\mathcal{G}')$. Moreover the transform $\zeta' = \mathfrak{z}^{-1}\zeta$ of the given parameter ζ is a $\sharp$-key parameter of $\mathcal{G}'$ at ξ' unless we have Case III or Case IV.*

Proposition 2.2. *Assume that v is empty. Then there exists $i < \nu+1$ having the property of Prop.(2.1) so that any fitted permissible blowup for $\mathcal{G}$ must produce Case II unless either Case III or Case IV happens.*

Proposition 2.3. *Consider a fitted permissible blowup $\pi : Z' \to Z$ for $\mathcal{G}$ with center D and a closed point $\xi' \in \pi{-}1(\xi) \cap Sing\mathcal{G}'$ with the transform $\mathcal{G}'$ of $\mathcal{G}$. Assume that the chosen exceptional parameter $\mathfrak{z}$ for $\mathfrak{F}/\mathcal{G}$ is $\sharp$-key parameter for $\mathcal{G}$. Then either Case IV or Case III in which the residual factor f' of the transform $\mathcal{G}'$ contains a key parameter transveral to the exceptional divisor of π in Z'.*

Remark 2.3. Note that the assumption of Prop.(2.2) is satisfied at any metastable point whence v is empty. Then Prop.(2.1) becomes applicable. Therefore thanks to Moh's Th.(**??**) and Prop.(2.1), we see that after Case I have occured our next inductive objective will be accomplished if the residual order can be made to drop two or more (either once Case IV or twice Case III before the next Case I).

After Case I we may have Case II repeated and then possibly Case III followed by Case II repeated again. After such successions we may have Case I again. Such a cycle of Cases I-II-III-II-I may be repeated.

Therefore our task is to show such cycles cannot repeat indefinitely in order to make a successsful step forward according to our inductive strategy. Thus our immediate interest is to clarify what are possible (or rather impossible) courses of Cases after a Case I had occurd.

Remark 2.4. Let us introduce the following number and keep it as an important reference for comparison with corresponding numbers of subsequent transforms.

$$\mathbf{d} = \mathbf{d}(\mathcal{G}) = \Big(resord_{\xi}(\mathcal{G}) \quad \textit{at the starting point} \Big). \tag{2.4}$$

We will be applying a fitted permissible blowup successively one after another. However for the sake of simplicity we may choose a notational change back at the each of later steps during such a sequence of successive transformations. In any event such a notational reset should be understood only for the purpose of clarifying essential effects taking place at a particular step. However one thing we must keep in mind is that the number called **d** is the one chosen and fixed at the very stating point of Eq.(**??**) and the meaning the symbol will not be changed later.