RESOLUTION OF SINGULARITIES
PART(2)

1. $/^*$-exponents

Recall that we had

(1) the edge parameters $y = (y_1, \cdots, y_r)$
(2) the edge generators $g_i = y_i^{q_i} + \epsilon_i$ where $q_i = p^{e_i}$ and $ord_\xi(\epsilon_i) > q_i$ for $1 \leq i \leq r$ where $0 \leq e_1 \leq \cdots \leq e_r$.

Let us observe that for each $i, 1 \leq i \leq r$, y_i can be replaced by a unit multiple and accordingly ϵ_i by its q_i-th powered unit multiple. Also that y_i may also be replaced by $y_i - \phi_i$, usually with $\phi_i \in M_\xi^2$, and accordingly ϵ_i by $\epsilon_i + \phi_i^{q_i}$.

Definition 1.1. We are given a power $q = p^e, 0 \leq e \in \mathbb{Z}$. A $/^q$-*exponent* $\mathcal{G}$ in Z is expressed as $(\mathbf{g} \,\|\, /^q)$ with a coherent $\rho^e(\mathcal{O}_Z)$-submodule $\mathbf{g} \subset \mathcal{O}_Z$ up to the following equivalence relation among the $\mathbf{g}$. The equivalence relation is defined as follows:

$$\begin{aligned}(1.1) \qquad (\mathbf{g}(1) \,\|\, /^q) \;&=\; (\mathbf{g}(2) \,\|\, /^q) \iff \\ \mathbf{g}(1) + \rho^e(\mathcal{O}_Z) \;&=\; \mathbf{g}(2) + \rho^e(\mathcal{O}_Z)\end{aligned}$$

Definition 1.2. For a reduced irreducible subscheme $D \subset Z$ with its "generic" point ζ and for a given $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ of Def.(1.1), we define

$$ord_D(\mathcal{G}) \;=\; \min_{f \in \mathbf{g}} \Big\{ \max_{v \in R_\zeta} ord_\zeta(f - v^q) \Big\}$$

where $R_\zeta = \mathcal{O}_{Z,\zeta}$. We define the "rate of order" to be the following ratio.

$$(1.2) \qquad rord_D(\mathcal{G}) \;=\; q^{-1} ord_D(\mathcal{G})$$

Note that if $\mathcal{G}^p = (\mathbf{g}^p \,\|\, /^{pq})$ then $rord_D(\mathcal{G}^p) = rord_D(\mathcal{G})$.

Remark 1.1. Unlike the case of 'ideal exponents" we sometimes need to examine points $\eta \in Z$ with $ord_\eta(\mathcal{G}) < q$. Refer to Def.(**??**), Def.(**??**), Eq.(**??**), Th.(**??**) and Def.(1.2).

Remark 1.2. For the sake of notational simplicity we write $(f \,\|\, /^q)$ meaning $(\mathbf{g} \,\|\, /^q)$ when $\mathbf{g} = f\rho^e(R_\xi)$ with an element $f \in R_\xi$. There ξ is usually a closed point of Z and we are focusing our attention to the local nature of $(\mathbf{g} \,\|\, /^q)$ at ξ.

In the following two examples we show two new phenomena that we must keep in mind in dealing with Zariski topology of $/^q$-exponents.

Example 1.1. (*Generic-Down Pathology*)
Let $Z = Spec(\mathbb{K}[x, y])$. Let $q = p^e$ and $s = p^c$ with integers $e > c \geq 0$ and $p = char(\mathbb{K})$. Consider $D = Spec(\mathbb{K}[x, y]/(x)\mathbb{K}[x, y])$ and $(\mathbf{g} \parallel /^q) = (x^q(y - a)^s \parallel /^q)$ for every $a^{q/s} \in \mathbb{K}$. Thus we have

$ord_D(\mathbf{g} \parallel /^q) = q$ while $ord_\eta(\mathbf{g} \parallel /^q) = q + s$ for all closed points η of D.

Observe the same phenomena for $\mathbf{g} = g(x, y)^q y^s$ with any polynomial g and also for a finite sum of such.

Definition 1.3. A reduced irreducible subscheme $D \subset Z$ will be said to be "generic down type" for a $/^q$-exponent $\mathcal{G}$ if we have the following inequality

$$ord_\zeta(\mathcal{G}) \ < \ min_{\xi \in U_{cl}}\, ord_\xi(\mathcal{G}) \tag{1.3}$$

with the generic point ζ of D and an open dense subset $U \subset D$.

Example 1.2. (*Generic-up Pathology*)
Pick 5 variables (x, y, z, w, t). Let $Z = Spec(\mathbb{K}[x, y, z, w, t])$ and $\eta = (x, y, z, w) \in Spec(Z)$. Let $\phi = x^p + ty^p$ with $p = char(\mathbb{K})$ and let $\zeta = (\phi, z, w) \in Spec(Z)$ which is a prime ideal. Let $\mathbf{g} = tz^p + w^{p+1}$. Then for $\mathcal{G} = (\mathbf{g} \parallel /^p)$ have

(1) $ord_\zeta(\mathcal{G}) = p+1$ while $ord_\eta(\mathcal{G}) = p$ although η is a *specialization* of ζ. Thus *special* points can have smaller multiplicity than the *generic* point.
(2) Incidentally, if C denote the closure of the point $\sigma = (z, w)$ then

$$\begin{aligned} & ord_\sigma(\mathcal{G}) = p \ < \ ord_\zeta(\mathcal{G}) = p + 1 \\ & > \ ord_\eta(\mathcal{G}) = p \ < \ ord_\xi(\mathcal{G}) = p + 1, \ \forall \xi \in C \cap Z_{cl} \end{aligned} \tag{1.4}$$

in the ordering from generic to special.

Observe that the point η is a "singular point" of the closure of the point ζ and that the residue field κ_η is not perfect. (cf. Lem(2.3), Th.(3.2), Th.(2.1) and Th.(3.1) of later sections.)

2. Basics of Zariski $/^q$-topology

In spite of some "pathological" behavior of orders of $/^q$-exponents with respect to Zariski topology in Z we have many useful results.

Let Z_{cl} denotes the set of all *closed* points of Z and the Zariski topology of Z_{cl} is the one induced by that of Z. The specility of any closed point is its residue field is perfect.

Theorem 2.1. *Consider any $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ of Def.(1.1). For each integer $d > 0$, we define the set*

$$Sing^{(d)}_{cl}(\mathcal{G}) \;=\; \{\eta \in Z_{cl} \;\big|\; ord_\eta(\mathcal{G}) \;\geq\; d\} \tag{2.1}$$

We then assert that this set is closed in Zariski topology of Z_{cl}.

Proof. Let us write $Z(q) = Spec(\rho^e(\mathcal{O}_Z))$ where ρ denotes the q-th power map where $q = p^e$. If $d \leq\; q$, $Sing^{(d)}_{cl}(\mathcal{G}) \;=$

$$\{\; \eta \in Z_{cl} \;\big|\; Diff^{(d-1)*}_{Z/Z(q),\eta}(\mathbf{g}) \subset M_\xi \;\} \tag{2.2}$$

and if $d > q$, $Sing^{(d)}_{cl}(\mathcal{G}) \;=$

$$\{\; \eta \in Z_{cl} \;\big|\; \sum_{1 \leq j < q} \Big(Diff^{(d-j-1)}_{Z,\eta} Diff^{(j)*}_{Z/Z(q),\eta}(\mathbf{g}) \Big) \subset M_\xi \;\} \tag{2.3}$$

where $Diff^{(\mu)*}_{Z/Z(q)} = \{\; \partial \in Diff^{(\mu)}_{Z/Z(q)} \;\big|\; \partial(c) = 0, \forall c \in \mathbb{K} \;\}$. In particular if $\mu = 0$ then $Diff^{0*}_{Z/Z(q)} = \{0\}$ and $Sing^{(1)}_{cl}(\mathcal{G}) = Z_{cl}$. The Eqs. (2.2) + (2.3) follow from the lemma below. Meanwhile the claimed closeness of the theorem is an immediate consequence of the above Eqs (2.2) + (2.3) because $Diff^{(\mu)}_Z$ and $Diff^{(\mu)*}_{Z/Z(q)}$ are coherent sheaves of $\mathcal{O}_Z$-module. $\square$

Lemma 2.2. *Let us pick any point $\eta \in Sing(\mathbf{g} \,\|\, /^q) \cap Z_{cl}$ and also a regular system of parameters $x = (x_1, \cdots, x_n)$ of R_η. Let $R(q) = \rho^e(R_\eta)$ with $q = p^e$ so that R_η is freely generated as $R(q)$-module by $\{\; x^\alpha \;\big|\; \alpha \in \epsilon^n(q) \;\}$. Write $h = \sum_\alpha h_\alpha x^\alpha$ with $h_\alpha \in R(q)$ and $\alpha \in \epsilon^n(q)$. We then claim*

$$ord_\eta(\mathbf{g} \,\|\, /^q) = \min\{\; |\alpha| + ord_\eta(h_\alpha) \mid \epsilon^n(q) \ni \alpha \neq 0\} \tag{2.4}$$

Moreover for each $0 \neq \alpha \in \epsilon^n(q)$

$$\textit{if } ord_\eta(h_\alpha\, x^\alpha) \leq q \tag{2.5}$$

$$(\textit{ which can happen only if } \;\; h_\alpha \in R(q) \setminus max(\, R(q))\,) \;\; \textit{then}$$

$$ord_\eta(h_\alpha\, x^\alpha) \;=\; |\alpha| \;=\; 1 + \max\{\; m \;\big|\; Diff^{(m)*}_{Z,\eta}(h_\alpha\, x^\alpha) \subset M_\eta \;\}$$

and

$$\textit{if} \;\; ord_\eta(h_\alpha\, x^\alpha) > q \;\; \textit{then} \tag{2.6}$$

$$ord_\eta(h_\alpha\, x^\alpha) = ord_\eta(h_\alpha) + |\alpha| \;=$$

$$1 + |\alpha| + \max\Big\{\; \mu \;\big|\; \big(Diff^{(\mu)}_{Z,\eta} h_\alpha\big) \subset M_\eta \;\Big\} \;=$$

$$1 + \max\Big\{\; \mu \;\big|\; \big(Diff^{(\mu)}_{Z,\eta} \partial^{(\alpha)}(h_\alpha\, x^\alpha)\big) \subset M_\eta \;\Big\} \;=$$

$$1 + \max\Big\{\; m \;\big|\; \textstyle\sum_{1 \leq \mu < q} \Big(Diff^{(m-\mu)}_{Z,\eta} Diff^{(\mu)*}_{Z,\eta}(h_\alpha\, x^\alpha) \Big) \subset M_\eta \;\Big\}.$$

Proof. For $h = \sum_{\alpha \in \epsilon^n(q)} h_\alpha x^\alpha$ with $h_\alpha \in R(q)$ as above and for a closed point $\eta \in Z$, we have

$$\begin{gathered} ord_\eta(h) \geq b \Longleftrightarrow \\ ord_\eta(h_\alpha) \geq b - |\alpha|, \ \forall \alpha \ with\ 0 < |\alpha| < b \Longleftrightarrow \\ 1 + ord_\eta(h_\alpha) > b - |\alpha|, \ \forall \alpha \ with\ 0 < |\alpha| < b \end{gathered} \tag{2.7}$$

Let us make use of the free base

$$\partial^{(\alpha)} \quad with \quad \alpha \in \epsilon^n(q)$$

of the $\mathcal{O}$-module $Diff_{\mathcal{O}/\rho^e(\mathcal{O})}$ defined by the following equalities:

$$\partial^{(\alpha)} x^\beta \ = \begin{cases} \binom{\beta}{\alpha} x^{\beta - \alpha} & if\ \beta \in \alpha + \mathbb{Z}_0^n \\ 0 & if\ otherwise \end{cases}$$

In view of the Eq.(1.1) of Def.(1.1) and Def.(1.2), Eq.(2.7) implies

$$\begin{gathered} ord_\eta(\mathcal{G}) \geq b \Longleftrightarrow \\ ord_\eta(\partial^{(\alpha)} h) \geq b - |\alpha| \quad for\ \forall \alpha\ such\ that\ 0 < |\alpha| < b \\ \Longleftrightarrow \\ 1 + ord_\eta(\partial^{(\alpha)} h) > b - |\alpha|\ \forall \alpha\ with\ 0 < |\alpha| < b \\ \Longleftrightarrow \\ ord_\eta(Diff^{(b-|\alpha|-1)}_{R_\eta/\mathbb{K}}(\partial^{(\alpha)} h)) > 0\ \forall \alpha\ with\ 0 < |\alpha| < b \\ \Longleftrightarrow \\ Diff^{(b-|\alpha|-1)}_{R_\eta/\mathbb{K}} \partial^{(\alpha)} h \subset M_\eta\ \forall \alpha\ with\ 0 < |\alpha| < b \end{gathered} \tag{2.8}$$

For $m < q$, $Diff^{(m)*}_{R_\xi/\mathbb{K}}$ is generated by $\partial^{(\alpha)}$ for $\alpha\ with\ 0 < |\alpha| \leq m$ and hence the last condition of Eq.(2.8) implies Eq.(2.5), while the same is true for Eq.(2.6) because for $m > 0$ (say $m = b - \mu - 1$) we have

$$Diff^{(m)*}_{R_\xi/\mathbb{K}} \subset \sum_{1 \leq \mu < q} Diff^{(m-\mu)}_{R_\eta/\mathbb{K}} \partial^{(\mu)} \subset \sum_{1 \leq \mu < q} Diff^{(m-\mu)}_{R_\eta} Diff^{(\mu)*}_{R_\eta/\mathbb{K}}$$

□

Lemma 2.3. *Let us pick a pair of points η and ζ in Z such that η is a smooth point of the closure D of ζ in Z. Then we have*

$$ord_\eta(\mathcal{G}) \ \geq \ ord_\zeta(\mathcal{G}). \tag{2.9}$$

To be explicit, let us choose a regular system of parameters $u = (u, v)$ of R_η such that uR_η is the ideal of D at η. Let $\hat{R}_\eta = K[[u]]$ denote the

M_η-adic completion of R_η where K is a coefficient field containing $\mathbb{K}$. Let us write

$$\mathbf{g} = \sum_{ab} d_{ab}\, u^a v^b \quad \textit{with} \;\; d_{ab} \in K \tag{2.10}$$

Then we have

$$ord_\eta(\mathcal{G}) \;=\; \min\{\, |a| + |b| \mid d_{ab}\, u^a v^b \notin \rho^e(K[[u]])\,\} \tag{2.11}$$

and

$$ord_\varsigma(\mathcal{G}) \;=\; \min\{\, |a| \mid \exists b, d_{ab}\, u^a v^b \notin \rho^e(K[[u]])\,\} \tag{2.12}$$

Proof. Let m denote the right hand side of Eq.(2.11). The partial sum $\sum_{|a|+|b|<m} d_{ab}\, u^a v^b$ can be written as $\hat\sigma^q$ with $\hat\sigma \in K[[u]]$ by assumption and $\hat\sigma$ is $max(\hat{R}_\eta)$-adically approximated arbitrarily by $\sigma \in R_\eta$. Then we may replace h by $h - \sigma^q$ with any such σ in the expression $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$. This proves $ord_\eta(\mathcal{G}) \geq m$. To prove the reversed inequality we use differential operators as follows. Pick a separating transcendental base t of the field extension $K/\mathbb{K}$ and choose the free base $\{\, \partial^{(abc)} \,\}$ of the R_η-module $Diff_{R_\eta/\mathbb{K}}$ such that

$$\partial^{(abc)} \left(u^{a'} v^{b'} t^{c'} \right) \;= \tag{2.13}$$

$$\begin{cases} \binom{a'}{a}\binom{b'}{b}\binom{c'}{c} u^{a'-a} v^{b'-b} t^{c'-c} & \textit{if} \;\; (a'b'c') \in (abc) + \mathbb{Z}_0^n \\ \\ 0 & \textit{if otherwise} \end{cases}$$

By the definition of m, there exists (abc) such that

1. $|a| + |b| = m$ and $d_{ab}\, u^a v^b \notin \rho^e(R_\eta)$,
2. if $(ab) \equiv 0 \mod (q)$ then there exists and we choose an integer c such that $0 < |c| < q$ and $\partial^{(00c)}(d_{ab}) \neq 0$,
3. if $(ab) \not\equiv 0 \mod (q)$ then we let $c = 0$.

Let $a^* = a - q[\frac{a}{q}]$ and $b^* = b - q[\frac{b}{q}]$. Then in every one of the above 3 cases we see that $(a^*, b^*, c) \neq 0$ and $\partial^{(a^*, b^*, c)}(R(q)) = 0$ with $R(q) = \rho^e(R_\eta)$. Moreover we see that

$$\partial^{(a^* b^* c)}\big(d_{a'b'} u^{a'} v^{b'}\big) = 0, \; \forall (a', b') \; \textit{with} \; |a'| + |b'| < m.$$

On the other hand we have

$$\partial^{(a^* b^* c)}(d_{ab} u^a v^b) = d^* u^{a-a^*} v^{b-b^*} \;\; \textit{with} \; 0 \neq d^* \in K$$

and hence

$$ord_\eta(\partial^{(a^* b^* c)} \mathbf{g}) = |a - a^*| + |b - b^*| = m - (|a^*| + |b^*|). \tag{2.14}$$

It is therefore impossible to have $\tau \in R_\eta$ such that $ord_\eta(\mathbf{g} - \tau^q) > m$. In fact, we would then have

$ord_\eta\big(\partial^{(a^*b^*c)}(\mathbf{g}-\tau^q)\big) > m-(|a^*|+|b^*|)$ *while* $\partial^{(a^*b^*c)}(\mathbf{g}-\tau^q) = \partial^{(a^*b^*c)}\mathbf{g}$

which is against the above Eq.(2.14). Thus $m = ord_\eta(\mathbf{g} \,\|\, /^q)$. The proof for $ord_\zeta(\mathbf{g} \,\|\, /^q)$ is done in the same manner. Finally the inequality Eq.(2.9) follows immediately from Eq.(2.11) and Eq.(2.12). □

Theorem 2.4. *Let A be a positive integer. If $ord_\xi(\mathbf{g} \,\|\, /^q) = Aq$ for a closed point $\xi \in Z$, then $\{\ \eta \in Z \ \big|\ ord_\eta(\mathbf{g} \,\|\, /^q) = Aq\ \}$ is closed in Z within a neighborhood of $\xi \in Z$. It should be noted that the closedness in Z is much stronger than the same in Z_{cl}.*

Proof. Let

$$S_{cl}(Aq) = \{\ \eta \in Z_{cl} \ \big|\ ord_\eta(\mathbf{g} \,\|\, /^q) = Aq\ \}$$

which is closed within neighborhood of ξ in Z_{cl}. Since the question is local we replace Z by a suitable neighborhood of ξ and assume that $S_{cl}(Aq)$ is closed in Z_{cl}. Let $S(Aq)$ denote the closure of $S_{cl}(Aq)$ in Z. Pick any irreducible component C of $S(Aq)$ which contains ξ and we want to prove that $ord_\zeta(\mathbf{g} \,\|\, /^q) = Aq$ for every $\zeta \in C$. Suppose $\exists \zeta \in C$ with $ord_\zeta(\mathbf{g} \,\|\, /^q) < Aq$ from which we want to deduce a contradiction. Let D be the closure of ζ in Z. The smooth points of D make an open dense subset D_{sm} of D and we pick any $\eta \in D_{sm} \cap Z_{cl}$. Let us then choose a regular system of parameters $u = (u, v)$ of R_η such that uR_η is the ideal of D at η. Let us make use of the notation of Eq.(2.10), Eq.(2.11) having $m = Aq$ in its right hand side and also Eq.(2.13), in which the completion of R_η is denoted as $\hat{R}_\eta = K[[u]]$ with no t and no c in this case. Let l denote the right hand side of Eq.(2.12). We have $l \le m = Aq$ and want to prove $l = Aq$.

Lemma 2.5. *Pick any (ab) giving the minimum of Eq.(2.12) so that $|a| = l$. Our first claim is that we then have $a \equiv 0 \mod (q)$.*

Suppose $a \not\equiv 0 \mod (q)$. Incidentally this covers the case in which l is not divisible by q. Since $ord_\eta(\mathcal{G}) = Aq$, we have

$$ord_\eta(\partial^{(a^*0)}\mathbf{g}) \ge Aq - |a^*| \quad \textit{with } a^* = a - q[\frac{a}{q}] \ne 0 \tag{2.15}$$

because $\partial^{(a^*0)}$ kills all q-th power differences. We will prove that the existence of (ab) as above is impossible by contradiction. The equalities of Eq.(2.15) remains the same within a neighborhood of η in $D \cap Z_{cl}$. Therefore the same equality should hold for D as well because $\partial^{(a^*0)}$ kills all q-th power differences. We thus have

$$ord_\zeta(\partial^{(a^*0)}\mathbf{g}) \ \ge \ Aq - |a^*| \tag{2.16}$$

for the generic point ζ of D. On the other hand the power series expansion of $\partial^{(a^*0)}\mathbf{g}$ in $K[[u,v]]$ has a nonzero term $k_{ab}u^{a-a^*}v^b$. Therefore we have

$$\begin{aligned} (2.17) \qquad & ord_\zeta(\partial^{(a^*0)}\mathbf{g}) \ \le \\ & ord_\zeta(k_{ab}u^{a-a^*}v^b) \ = \ |a - a*| \ = \ l - |a^*| \end{aligned}$$

With Eq.(2.16) and Eq.(2.17) we deduce $Aq \le l = ord_\zeta(\mathcal{G})$ which is against the existence of (ab) as above. We have thus proven the first claim.

From now on we assume that every (ab) giving the minimum of Eq.(2.12) has $a \equiv 0 \mod q$. To complete the proof of Th.(2.4) we pick any such (ab) and want to prove $|a| \ge Aq$. Suppose $|a| < Aq$. We have $k_{ab} \ne 0$ and $b \not\equiv 0 \mod (q)$ by the assumption in Eq.(2.12). Let us define $\mu_i = b_i - q[\frac{b_i}{q}]$ for the i-th member b_i of b. We must have at least one i such that $0 < \mu_i < q$ for $b \not\equiv 0 \mod (q)$. Pick one such i and call it $i(1)$. We have $|b| \ge Aq - |a| \ge q$ because $a \equiv 0 \mod q$ and $|a| < Aq$. Therefore we have an index $i(2)$ such that either $b_{i(1)} - \mu_{i(1)} > 0$ or $b_{i(2)} > 0$ while $i(2) \ne i(1)$. For simplicity we write μ for $\mu_{i(1)}$. Let ∂ denotes the elementary differential operator of order μ in the variable $v_{i(1)}$ alone with reference to the system (u,v). We have $ord_\eta(\partial\mathbf{g}) \ge Aq - \mu$ and it holds true for all points in a neighborhood of η in $D \cap Z_{cl}$. Since ∂ kills all the q-th power differences, it then implies

$$(2.18) \qquad ord_\zeta(\partial\mathbf{g}) \ge Aq - \mu$$

for the generic point $\zeta \in D$. This Eq.(2.18) implies

$$\begin{aligned} (2.19) \qquad & ord_\zeta\Big(v_{i(1)}{}^{-\mu}u^a v^b\Big) \ \ge \\ & Aq - \mu \ = \ |a| + |b| - \mu \ > \ |a| \end{aligned}$$

where the last strict inequality is due to the existence of the index $i(2)$ having the property described above. This Eq.(2.19) is impossible unless some of the members of v could vanish on D against our selection of v. W conclude that we cannot have $l < Aq$. Thus $l = Aq$ and the Th.(2.4) is proven. □

Theorem 2.6. *Let $D \subset Z$ be an irreducible subscheme and let A be a positive integer. If $ord_\eta(\mathbf{g} \,\|\, /^q) \ge Aq$ for all $\eta \in D \cap Z_{cl}$ then $ord_\zeta(\mathbf{g} \,\|\, /^q) \ge Aq$ for the generic point $\zeta \in D$.*

Proof. This lemma is in fact a corollary of Th.(2.4). In fact, firstly we reduce the problem to be local about some smooth point ξ of D and then choose parameters $u = (u,v)$ as in the proof of Th.(2.4). We then replace replace h by $h^+ = h + u^\alpha$ with $|\alpha| = Aq$ and $\alpha \not\equiv 0 \mod (q)$ and apply Th.(2.4) to $(\mathbf{g}^+ \,\|\, /^q)$. $\square$

Lemma 2.7. *If $\xi \in Z_{cl}$ and is contained in the closure of $\zeta \in Z$ then we have*

$$ord_\xi(\mathbf{g} \,\|\, /^q) \geq ord_\zeta(\mathbf{g} \,\|\, /^q).$$

Proof. In the closure D of $\zeta \in Z$ the smooth points is open dense. Hence, thanks to Th.(2.1), we may replace ξ by a smooth point of D within Z_{cl}. Then thanks to Th.(2.1), Lem.(2.7) is nothing but Lem.(2.3). $\square$

3. $/^q$-PERMISSIBLE TRANSFORMS

Definition 3.1. The singular locus $Sing(\mathbf{g} \,\|\, /^q)$ of a $/^q$-exponent is the set $\{\eta \in Z \mid ord_\eta(\mathbf{g} \,\|\, /^q) \geq q\}$.

Theorem 3.1. *The $Sing(\mathbf{g} \,\|\, /^q)$ is closed in the Zariski topology of Z. This closedness is stronger than the closedness within Z_{cl} in the sense of Th.(2.1).*

Proof. Immediate from Lem.(2.6). $\square$

Theorem 3.2. *If D is a smooth irreducible subscheme of Z then $ord_\eta(\mathbf{g} \,\|\, /^q) \geq ord_\zeta(\mathbf{g} \,\|\, /^q)$ for every $\eta \in D$, where ζ is the generic point of D.*

Proof. Immediate from Lem.(2.3). $\square$

Definition 3.2. Let $\pi : Z' \longrightarrow Z$ be a blowup with center D. We say that π (and also D) is called *permissible* for a $/^q$-exponent $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ if D is smooth irreducible and contained in $Sing(\mathcal{G})$ in the sense of Def.(3.1). Here and as always, the permissibility is required with respect to the given NC-system Γ in the sense of Def.(**??**).

Note that $D \subset Sing(\mathcal{G})$ means that every point of D (including the generic point of D) is in $Sing(\mathcal{G})$.

Here we add one more permissibility condition as follows.

Definition 3.3. We say that π with D of Def.(3.2) is *strongly permissible* at a closed point $\xi \in D$ if furthermore $ord_\xi(\mathcal{G}) = ord_\zeta(\mathcal{G})$ with the generic point $\zeta \in D$.

This condition is strictly stronger than that of Def.(3.2) in general because of the possibility of generic-down center.

Definition 3.4. The transform $\mathcal{G}'$ of $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ by a permissible π of Def.(3.2) is defined as follows:

(1) For each closed point $\xi' \in Z'$ with $\pi(\xi') \in D$ we let I be the ideal of D at ξ and pick any $v \in I$ such that $IR_{\xi'} = vR_{\xi'}$,
(2) and then locally at ξ' we define the transform $\mathcal{G}'$ to be $= (v^{-q}\mathbf{g} \,\|\, /^q)$.
(3) We then see that above definition is independent of the choice of v due to the equivalence of Eq.(1.1) in Def.(1.1).
(4) For this reason the above definition of $\mathcal{G}'$ is globally well defined for all $\xi' \in \pi^{-1}(D)$. For points of $Z' - \pi^{-1}(D)$ the above definition is naturally extended through the isomorphism of π restricted to $Z' - \pi^{-1}(D)$.

4. $\mathfrak{S}$, $\wp$ and general $/^q$

The *permissibility* of Def.(3.2) and the notion of the transform Def.(3.4) can be extended for every LSB of Def.(**??**). Following Def.(**??**) words by words, we can define

Definition 4.1.

$$\mathfrak{S}(\mathcal{G}) \;=\; \bigcup_t \Big\{ \; LSBs \text{ over } Z[t] \text{ permissible for } \mathcal{G}[t] = (\mathbf{g}[t] \,\|\, /^q) \; \Big\}$$

We then say that "$\mathcal{G}_2$ *more singular* than $\mathcal{G}_1$" if $\mathfrak{S}(\mathcal{G}_2) \supset \mathfrak{S}(\mathcal{G})$, where $\mathcal{G}_i, i = 1, 2$. can be either ideal exponent or ^{q} exponent. We then define *equivalence* by saying

$$\mathcal{G}_1 \sim \mathcal{G}_2 \;\Leftrightarrow\; \mathfrak{S}(\mathcal{G}_1) = \mathfrak{S}(\mathcal{G}_2).$$

Then $\mathcal{G} \;\sim\; \mathcal{G}_1 \cap \mathcal{G}_2$ will mean $\mathfrak{S}(\mathcal{G}) = \mathfrak{S}(\mathcal{G}_1) \cap \mathfrak{S}(\mathcal{G}_2)$.

We now generalize the notion of $/q$-exponent itself by means of the above notion of *equivalence* $\sim$ as follows.

Definition 4.2. For a coherent $\rho^e(\mathcal{O}_Z)$-module $J \subset \mathcal{O}_Z$ and a positive power $q = p^e$ of the characteristic $p > 0$ of $\mathbb{K}$, the $/^q$-exponent $(J \,\|\, /^q)$ is defined by saying

$$\mathfrak{S}(J \,\|\, /^q) \;=\; \bigcap_{f \in J} \mathfrak{S}(f \,\|\, /^q)$$

Theorem 4.1. *For $(J \,\|\, /^q)$ as above we always have*

$$\mathfrak{S}(J \,\|\, /^q) \;\subset\; \mathfrak{S}(\mathcal{D}J, q - a)$$

for every coherent $\mathcal{O}_Z$-submodule of $Diff_Z^{*(a)}$ for every integer $a \geq 0$.

Note that $\mathcal{D}J$ is an $\mathcal{O}_Z$-module while J is only an $\rho^e(\mathcal{O}_Z)$-module in Th.(4.1).

Moreover the notions of *intersection* and *equivalence* can be extended to the mixed cases of ideal exponents and $/^q$-exponents as follows.

Definition 4.3. For a finite number of ideal exponents $E_i = (J_i, b_i)$ with $1 \le i \le c$ and $/^q$-exponents $\mathcal{G}_j = (I_j \,\|\, /^{q_j})$ with $1 \le j \le d$,

$$\begin{aligned} G \ &\sim \big(\cap_{1 \le i \le c} E_i \big) \bigcap \big(\cap_{1 \le j \le d} \mathcal{G}_j \big) \Longleftrightarrow \\ \mathfrak{S}(G) \ &= \big(\cap_{1 \le i \le c} \mathfrak{S}(E_i) \big) \bigcap \big(\cap_{1 \le j \le d} \mathfrak{S}(\mathcal{G}_j) \big) \end{aligned} \tag{4.1}$$

In particular $Sing(G) = \big(\cap_{1 \le i \le c} Sing(E_i) \big) \bigcap \big(\cap_{1 \le j \le d} Sing(\mathcal{G}_j) \big)$.

Theorem 4.2. *(Ambient $/^q$-Reduction Theorem) Given a $/^q$-exponent $\mathcal{G} = (K \,\|\, /^q)$ in Z, we let*

$$I^+ \ = \ \sum_{j=1}^{q-1} \Big(Diff_Z^{(j)*} K \Big)^{\frac{b^+}{b-j}} \quad with \ \ b^+ = (q-1)!$$

For any smooth subscheme $W \subset Z$, we let $F^+ = (I^+\mathcal{O}_W, b^+)$ which is an ideal exponent in W. We let $\mathcal{F} = (I\mathcal{O}_W \,\|\, /^q)$ which is a $/^q$-exponent in W. Then $F^+ \cap \mathcal{F}$ is an ambient reduction *of $\mathcal{G}$ from Z to W in the following sense (definition):*

> *Pick any t and any one LSB over $Z[t]$ such that all of its centers are in the strict transforms of $W[t]$. Then the LSB belongs to $\mathfrak{S}(\mathcal{G})$ if and only if it induces an LSB in $W[t]$ which belongs to $\mathfrak{S}(F^+ \bigcap \mathcal{F})$.*

Proof. Let D be a smooth irreducible subscheme of W and pick any closed point $\xi \in D$. Let $J = J(W, Z)$ be the ideal of $W \subset Z$ and let $I = I(D, Z)$ be the ideal of $D \subset Z$. Pick a regular system parameters $(u, v, u(2))$ of R_ξ such that u generates J_ξ and (u, v) does I_ξ. We then have the corresponding elementary differential operators $\partial^{(\alpha\beta\gamma)}$ which freely generate R_ξ-module $Diff_{R_\xi/\mathbb{K}}$ such that

$$\begin{gathered} \partial^{(\alpha\beta\gamma)} u^a v^b u(2)^c \ = \\ \binom{a}{\alpha}\binom{b}{\beta}\binom{c}{\gamma} u^{a-\alpha} v^{b-\beta} u(2)^{c-\gamma} \ \ for\ all \ \ (abc) \end{gathered}$$

Then we have

$$\begin{gathered} D \ is\ permissible\ for\ \mathcal{G} \ at\ \xi \\ \Longleftrightarrow \\ \exists v \in R_\xi \ such\ that\ h - v^q \in I_\xi^q \end{gathered}$$

This is equivalent to

$$\begin{gathered} \exists v \in R_\xi \ such\ that \\ (\mathbf{g} - v^q)|_W \in (I/J)_\xi^q \ and \\ \partial^{(\alpha 00)} h \in (I/J)_\xi^{q-|\alpha|} \ \ for\ all \ \ 0 < |\alpha| \le q-1 \end{gathered}$$

which is equivalent to

$$D \subset Sing(\bar{\mathbf{g}} \,\|\, /^q) \ \ and$$
$$\partial^{(0\beta\gamma)}(\partial^{(\alpha 00)}\mathbf{g}) \in (I/J)_\xi^{q-|\alpha|-|(\beta\gamma)|}$$
$$\forall\, 0 < |\alpha| \leq q-1 \quad and \quad \forall\, 0 \leq |(\beta\gamma)| \leq q-1-|\alpha|$$

which is equivalent to

$$D \subset Sing(\bar{\mathbf{g}} \,\|\, /^q) \quad and \quad \Big(Diff^{(\mu)*}_{R_\xi/\mathbb{K}}\mathbf{g}\Big)\Big|_W \subset (I/J)_\xi^{q-\mu}$$
$$for\ all \ \ 1 \leq \mu \leq q-1$$

which is equivalent to

$$D \subset Sing(\bar{\mathbf{g}} \,\|\, /^q) \bigcap Sing(F^+) = Sing(F^+ \bigcap \mathcal{F})$$

Moreover the differential operators involved here commute with the addition of any parameters t (cf Def.(**??**)), with q-th powered unit multiplication (cf Def.(1.1)) and with division by q-th powers of exceptional parameters for blowups. With a little care on added new features, we see that the rest of the proof in the $/^q$ cases is quite similar to that of ambient reduction of ideal exponents as was given in Th.(**??**). (Also refer to [**?**] and [**?**].) □

5. $/^q$-DIVISORIAL FACTORS

Theorem 5.1. *Pick a regular system of parameters $x = (z, w)$ with $z = (z_1, \cdots, z_s)$ at a closed point $\xi \in Z$ such that those components Γ_i of Γ passing through ξ are the hypersurfaces defined by the ideals $(z_i)R_\xi, 1 \leq i \leq s$. Then every $/^q$-exponent $\mathcal{G} \neq (0 \,\|\, /^q)$ is represented as $(z^{\hat{\alpha}} f \,\|\, /^q)$ with $f \in R_\xi$ where $\hat{\alpha} = (\hat{\alpha}_1, \cdots, \hat{\alpha}_r)$ with $\hat{\alpha}_i = ord_{\zeta_i}(\mathcal{G})$ for all i where ζ_i is the generic point of Γ_i.*

Proof. Pick and begin with any $g \in R_\xi$ with $\mathcal{G} = (g \,\|\, /^q)$. We will make repeated modifications on g to obtain what we want for the theorem. Let $ord_{\zeta_i}(g) = \alpha_i$ and let $f_i = g z_i^{-\alpha_i}$ where z_i is a generater of the ideal of Γ_i in R_ξ. Suppose that we found $\alpha_i < \hat{\alpha}_i$. (If there exists no such i the proof is done.) With such i we must have that $ord_{\zeta_i}(g) \equiv 0$ mod q and that $\bar{f}_i = (f_i \mod (z_i)R_\xi) \in \rho^e(\kappa_i)$ where κ_i denotes the residue field of the local ring R_{ζ_i} of Z at the generic point ζ_i of Γ_i. Let $\bar{R}_i = R_\xi/z_i R_\xi$. Since Γ_i is smooth, $\bar{R}_i$ is integrally closed in its field of fractions κ_i. Hence we have $\rho^e(\kappa_i) \cap \bar{R}_i = \rho^e(\bar{R}_i)$, which proves

$$\bar{f}_i \ \in \ \rho^e(\bar{R}_i) \quad with \quad q = p^e \tag{5.1}$$

For each $k \neq i$ with $\Gamma_k \ni \xi$, let $\alpha_k = ord_{\zeta_k}(g)$ so that $g \in (z_k^{\alpha_k})R_\xi$ which implies $f_i \in (z_k^{\alpha_k})R_\xi$ due to the fact that z is extendable to a regular system of parameters of R_ξ. Let $\bar{z}_k = (z_k \mod (z_i)R_\xi)$. By Eq.(5.1), we then have

$$
\begin{aligned}
(5.2) \qquad \bar{f}_i &\in \rho^e(\bar{R}_i) \cap (\textstyle\bigcap_{k\neq i} \bar{z}_k^{\alpha_k}\bar{R}_i) \\
&= \textstyle\bigcap_{k\neq i}\Big(\rho^e(\bar{R}_i) \cap (\bar{z}_k^{\alpha_k})\bar{R}_i\Big) \\
&= \textstyle\prod_{k\neq i}\Big(\rho^e(\bar{R}_i) \cap (\bar{z}_k^{\alpha_k})\bar{R}_i\Big)
\end{aligned}
$$

where the last equality is because the $\bar{z}_k^q$ form a portion of a regular system of parameters of $\rho^e(\bar{R}_i)$. Hence there exists

$$a_i \in \prod_{k\neq i}\Big(\rho^e(\bar{R}_i) \cap (\bar{z}_k^{\alpha_k})\bar{R}_i\Big)$$

which induces $\bar{f}_j \in \bar{R}_i$. We then replace g by $g^* = g - z_i^{\alpha_i}a_i$, so that $ord_{\zeta_i}(g^*) > \alpha_i$ and $ord_{\zeta_k}(g^*) \geq \alpha_k, \forall k \neq i$. Repeat this process as long as we can. Since $\mathcal{G} \neq (0, /^q)$, we end up with $\mathcal{G} = (z^{\hat{\alpha}} f \parallel /^q)$ having the required properties of Th.(5.1) after a finite number of repetitions as above. $\square$

Remark 5.1. The monomial $z^{\hat{\alpha}}$ of Th.(5.1) is unique up to a unit multiple in R_ξ and hence the ideal $\mathcal{P}_\xi = z^{\hat{\alpha}}R_\xi$ is locally uniquely determined by $\mathcal{G}$ at ξ. Moreover this ideal $\mathcal{P}_\xi$ is the stalk at ξ of a global coherent ideal sheaf $\mathcal{P}$ within the domain of definition of $\mathcal{G}$.

Definition 5.1. The above monomial $z^{\hat{\alpha}}$ of Th.(5.1) will be called Γ-*maximal divisor* of $\mathcal{G}$ at ξ. Write $\hat{\alpha} = q\beta + \gamma$ in such a way that $0 \leq \gamma_i < q$, $\forall i$, and call $z^{q\beta}$ the $q\Gamma$-*factor* and z^γ the $q\Gamma$-*cofactor* of $\mathcal{G}$ at ξ. The global ideal $\mathcal{P}$ will be called Γ-*maximal divisor* of $\mathcal{G}$, denoted by $\mathcal{P}(\mathcal{G})$. Moreover we have a coherent ideal $\mathcal{B}$ with stalks $\mathcal{B}_\xi = z^\beta R_\xi$ and its q-th power $\mathcal{B}^q$ will be called the $q\Gamma$-*factor* of $\mathcal{G}$. The ideal sheaf $\mathcal{B}^{-q}\mathcal{P}$ will be called the $q\Gamma$-*cofactor* of $\mathcal{G}$.

Definition 5.2. The $q\Gamma$-*cofactor* z^γ will be often written as v^γ with the subsystem $v \subset z$ consisting of exactly those z_j having $\gamma_j > 0$.

Definition 5.3. Let us write $\mathcal{G} = (\mathcal{P}\, f \parallel /^q)$ with the Γ-maximal divisor $\mathcal{P}$, which is locally $\mathcal{P}_\xi = z^{\hat{\alpha}}$ of Th.(5.1) at a closed point ξ. Such f will be called *residue* of $\mathcal{G}$ at ξ. We define

$$(5.3) \qquad resord_\xi(\mathcal{G}) \;=\; max\{\, ord_\xi(f) \,\big|\, \textit{all residues } f \textit{ of } \mathcal{G} \textit{ at } \xi\}$$

When a residue f satisfies the equality $resord_\xi(\mathcal{G}) = ord_\xi(f)$ we call f a Γ-*residual factor* or *residual factor* of $\mathcal{G}$ at ξ.

Definition 5.4. We define

$$\check{\mathcal{G}} \;=\; (\mathcal{P}^{-1}\mathbf{g} \,\|\, /^{q}) \qquad (5.4)$$
$$\textit{with } \ \Gamma\textit{-maximal } \ \mathcal{P} \ \textit{of} \ \ \mathcal{G} \;=\; (\mathbf{g} \,\|\, /^{q}),$$

where $\mathbf{g}$ is chosen to be divisible by a generator $z^{q\mathbf{b}+\gamma}$ of $\mathcal{P}$ locally at ξ. (cf. Th.(5.1) and Def.(5.1).) We will call $\check{\mathcal{G}}$ the *checked associate* of $\mathcal{G}$.

6. Standard *abc*-expression of $/^{q}$-exponent

Assume that we are given a $/^{q}$-expnent in Z, say $\mathcal{G}$. Then we need to choose specific parameters and detailed expressions of $\mathcal{G}$ in terms of its important components. There are certain common features in the pattern of their expressions. Therefore we want to set their versatile standard form which we can later refer to.

Remark 6.1. Locally at ξ we choose a system of parameters z which consists of the ones defining those components $\Gamma_j \subset Z$ of the Γ which are passing through the point $\xi \in Z$. We then extend z to a regular system of parameters $x = (z, \omega)$ of R_ξ in which the choice of ω may be free or may be contingent to the specifics of the given situation. In particular when we are dealing with a specific blowup π with center D then we may or may not require that the ideal $I(D, Z)_\xi$ be generated by a subsystem u of x. However as for the choice of z we should recall the universal permissibility of π with the NC-data Γ so that the choice of z is not affected by the choice of permissible blowup so long as we focus our investigation to local problems at the given point $\xi \in Z$. Moreover, depending upon $\mathcal{G}$ locally at ξ we choose a subsystem v of z and write $z = (v, w)$ as is done in the definition below.

Throughout this paper we will be using the following standard form of expression of any given $/^{q}$-exponent $\mathcal{G}$, which we will call *abc-expression* of $\mathcal{G}$ at the given point ξ. .

Definition 6.1. We define a *standard abc-expression* of a $/^{q}$-exponent $\mathcal{G}$ at a given closed point $\xi \in Sing(\mathcal{G})$ as follows:

$$\mathcal{G} \;=\; (\mathbf{g} \,\|\, /^{q}) \qquad (6.1)$$
$$\textit{with } \ \mathbf{g} \;=\; z^{\mathbf{a}} g \;=\; z^{q\mathbf{b}} v^{\mathbf{c}} g$$

where $z^{\mathbf{a}}$ is the maximal Γ-factor of $\mathcal{G}$, $z^{q\mathbf{b}}$ is the maximal $q\Gamma$-factor and $v^{\mathbf{c}}$ is the Γ-cofactor with $0 < \mathbf{c}_j < q$ for all j in the sense of Def.(5.1). Moreover g is a residual factor so that $ord_\xi(g)$ is equal to $resord_\xi(\mathcal{G})$. Here the parameters $x = (z, \omega)$ is chosen in accord with

Rem.(6.1), while the partition $z = (v, w)$ is determined by the equality $z^{\mathbf{a}} = z^{q\mathbf{b}} v^{\mathbf{c}}$.

As for those important numbers $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, they will be named differently in accord with the specific needs. When we are dealing with many different $/^q$-exponents simultaneously we need to choose different naming for the numbers $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

7. COTANGENT p-FLAGS

We introduce the notion of "cotangent p-flags" for an ideal $I \subset R_\xi$, especially a principal ideal $I = gR_\xi$ with $g \in M_\xi$. The notion is of *local nature* at a given closed point $\xi \in Z$ and is determined by the *initial* of the ideal I at ξ. The *initial* means the κ_ξ-module $\big(I + M_\xi^{d+1}\big)/M_\xi^{d+1}$ with $d = ord_\xi(I)$. Throughout this section the residue field κ_ξ will be assumed to be perfect.

Remark 7.1. The *cotangent p-flags* of I at ξ are written as a system of κ_ξ-submodules of $M_\xi/M_\xi^2 \subset gr_\xi(R_\xi)$ as follows:

$$\{\ L_\xi(I,a)\,,\ p^{e_a},\ 1\ \le a\ \le\ l\ \}\quad \textit{with}\quad e_a \in \mathbb{Z}_0 \tag{7.1}$$
$$\textit{which we may write}\ \ L_\xi(I,a)\ =\ L(I,a)\ =\ L(a)\ \ \textit{for short.}$$

They are characterized by the following properties:

$$(0) = L(0)\ \subsetneqq\ L(I,1)\ \subsetneqq \cdots \subsetneqq\ L(I,l)\ \subset\ L = M_\xi/M_\xi^2 \tag{7.2}$$

subject to the following conditions.

(1) We have $1 \le p^{e_1} < \cdots < p^{e_l}$ where $p^{e_l} \le d = ord_\xi(I)$

(2) If $g \in I$ and $\partial \in Diff_{R_\xi/\mathbb{K}}^{d-p^\epsilon}$ have the following property

$$ord_\xi(\partial(g))\ =\ p^\epsilon\ \ \textit{and}\ \ \bar{w}^{p^\epsilon} \in in_\xi(\partial(g))\ +\ \kappa_\xi[L(b)] \tag{7.3}$$

where $e_b < \epsilon$ and $0 \ne \bar{w} \in M_\xi/M_\xi^2$, then there exists an index a such that $e_a \le \epsilon$ and $\bar{w} \in L(a)$.

(3) for each $a, 1 \le a \le l$, the κ_ξ-module $L(a)/L(a-1)$ is generated by the images of those $\bar{w} \in L(a) \subset M_\xi/M_\xi^2$ for which there exist $\partial \in Diff_{R_\xi/\mathbb{K}}^{(d-p^{e_a})}$ and $g \in I$ satisfying Eq.(7.3) with $\epsilon = e_a$ so that $b = a - 1$.

The *cotangent p-flag* of an element $g \in M_\xi$ will mean that of the principal ideal $I = gR_\xi$. For any $f \in M_\xi$ such that $ord_\xi(g - f) > d = ord_\xi(g)$, f and g have the same *cotangent p-flags*.

Remark 7.2. Here is a list of elementary properties of the *cotangent p-flag* of an ideal I at ξ. We only consider the nontrivial case with $M_\xi \supset I \ne 0$ with $ord_\xi(I) = d > 0$.

(1) Let ϵ be the smallest non-negative integer such that

$$\Big(Diff^{(d-p^{\epsilon})}_{R_\xi/\mathbb{K}} I \ + \ M_\xi^{p^\epsilon+1}\Big)/M_\xi^{p^\epsilon+1} \ \neq 0. \tag{7.4}$$

We then have $\epsilon = e_1$ and the module Eq.(7.4) is equal to $\rho^{e_1}(L(1))$. In this case the condition Eq.(7.3) can be replaced by a "stronger" one in which we require

$$\begin{gathered} \exists\, g \in I \quad and \quad \exists\, \partial \in Diff^{d-p^\epsilon}_{R_\xi/\mathbb{K}} \\ such\ that\ \bar{w}^{p^\epsilon} = in_\xi(\partial(g)) \quad and \quad ord_\xi(\partial(g)) = p^\epsilon \end{gathered} \tag{7.5}$$

Here $\epsilon = e_1$. But in the cases of $a > 1$ this condition can be too strong to produce the whole $L(I,a)/L(I,a-1)$.

Example 7.1. Consider $I = gR_\xi$ with $g = x_{p+1}^p + \prod_{1\le i\le p} x_i$ in which $L(I,1) = \sum_{1\le i\le p} \kappa_\xi \bar{x}_i$ and $L(I,2) = L(I,1) + \kappa_\xi \bar{x}_{p+1}$. Note that $\bar{w} = \bar{x}_{p+1}$ does not satisfy Eq.(7.5). ($\bar{x}_i = in_\xi(x_i)$.)

(2) If $d = p^{e_l}$ with the maximal index l we then have

$$\begin{aligned} &rank_{\kappa_\xi}\big(L(I,l)/L(I,l-1)\big) \\ &= \ rank_{\kappa_\xi}\big(\bar{I} \ + \ \kappa_\xi[L(I,l-1)]/\kappa_\xi[L(I,l-1)]\big). \end{aligned} \tag{7.6}$$

In particular when $I = gR_\xi$ this rank is equal to 1 thanks to the perfectness of κ_ξ.

Definition 7.1. We say that g' is *cotangentially subordinate* to g if every member $L(g',b)$ of the p-flags of g' at ξ is contained in some $L(g,a)$ of the p-flags of g at ξ with $e_a \le e_b$.

For instance, pick $g \in I$ and $\partial \in Diff^{(d-\mu)}_{R_\xi/\mathbb{K}}$ such that $ord_\xi(\partial g) = \mu$. Then ∂g is *cotangentially subordinate* to g.

Theorem 7.1. *Let l be the last index of Eq.(7.1). We then have*

$$\bar{I} \ \subset \kappa_\xi[L(I,l)] \quad where \quad \bar{I} = \big(I + M_\xi^{d+1}\big)/M_\xi^{d+1} \tag{7.7}$$

Moreover $L(I,l)$ is the smallest having this inclusion property.

The theorem is proven straight forward by uthe following lemma.

Lemma 7.2. *Let $\{L_\xi(I,a),\, p^{e_a},\, 1 \le a \le l\}$ be the cotangent p-flags of Eq.(7.1). Then for each a we have*

$$\begin{gathered} ord_\xi\big(Diff^{(d-p^{e_a})}_{R_\xi/\mathbb{K}} I\big) \ = \ p^{e_a} \\ and \\ in_\xi\big(Diff^{(d-p^{e_a})}_{R_\xi/\mathbb{K}} I\big) \ + \ \kappa_\xi[L(I,a-1)] \\ = \ \kappa_\xi\, L(I,a) \ + \ \kappa_\xi[L(I,a-1)] \end{gathered} \tag{7.8}$$

Recall that $in_\xi(J) = (J + M_\xi^{\nu+1})/M_\xi^{\nu+1}$ *with* $\nu = ord_\xi(J)$ *as always.*

A proof is an excercise for the readers.

Definition 7.2. A *cotangential base of exponent* e_a of I at ξ is by definition a system $\bar{w}(a)$ of elements $\bar{w}(a)_j \in M_\xi/M_\xi^2$, which induces a free base of the κ_ξ-module $L(I,a)/L(I,a-1)$. A regular system of parameters x of R_ξ will be said to be *cotangential* of I at ξ if $in_\xi x$ contains a cotangential base of I of exponent e_a for all $a, 1 \le a \le l$.

Definition 7.3. We have $q = p^e$ and $q \le d = ord_\xi(I)$. Then define

$$\ell_{\xi,q}(I) = max\{\, a \,\big|\exists e_a < e \,\} \ \ and \ \ \ell_{\xi,p^+}(I) = max\{\, all\ a \,\} \ = \ \ell \tag{7.9}$$

with reference to the p-flags Eq.(7.1).

(1) $\ell_{\xi,q}(I)$ may be written as $\ell_q(I)$ or $\ell(I)$ for short.
(2) $\ell_{\xi,p^+}(I)$ may be written as $\ell_{p^+}(I)$ or $\ell_{\xi^+}(I)$ or $\ell_+(I)$.
(3) Keep in mind that we always have $p^{e_a} \le d$.
(4) And then define

$$\begin{gathered} L_{q-max}(I) \ = \ L(I,\ell_q) \ \ with \ \ \ell_q \ = \ \ell_{\xi,q}(I) \\ and \\ L_{[p^+]-max}(I) \ = \ L(I,\ell_{p^+}) \ \ with \ \ \ell_{p^+} \ = \ \ell_{\xi,p^+}(I) \end{gathered} \tag{7.10}$$

8. p-flags of $/^q$-exponents

We will refer to the notation of $/^q$-exponent $\mathcal{G}$ and its *abc*-expression $(z^{\mathbf{a}} g \,\|\, /^q)$ with $z^{\mathbf{a}} = z^{q\mathbf{b}} v^\gamma$ in the sense of Def.(6.1), with $q\Gamma$-cofactor v^γ and residual factor g with reference to Rem.(5.1), Def.(5.1) and Def.(5.3). As for the choice of local parameters we refer to Rem.(6.1).

$$x = (z,\omega) \ with\ z = (v,w)\ and\ v = (v_1,\cdots,v_t). \tag{8.1}$$

For the sake of notational simplicity, we sometimes write z^γ for v^γ meaning that γ is extended from $\mathbb{Z}_0^t$ to $\mathbb{Z}_0^s$ by placing zeros for those components corrresponding to w.

Remark 8.1. Given $\mathcal{G} = (z^{q\mathbf{b}} v^\gamma g \| /^q)$ we examine the following two cases of applications of the *p-flags*. (Refer to Eq.(7.1) and Eq.(7.2).)

(1) The case of the ideal $I = gR_\xi$ with a residual factor g of $\mathcal{G}$.
(2) The case of $I = v^\gamma R_\xi$ with the $q\Gamma$-cofactor v^γ of $\mathcal{G}$.

Their p-flags have different characters and must be treated differently. Their characters concern with the following uniquness question. Note that the case (2) is *up to a unit multiple* onto v^γ, while the case (1) is *up to an addition* of $\phi^q v^{\gamma *}$ to g with $\phi \in R_\xi$. Here and later as well, $\gamma*$ denotes the *q-supplement* of v^γ in the following sense.

Definition 8.1. We have $\mathbf{a} = q\mathbf{b} + \gamma \in \mathbb{Z}_0^s$ with $0 \le \gamma_i < q, \forall i$. Then the *q-supplement* of γ is the unique element $\gamma^* \in \mathbb{Z}_0^s$ such that

$$\gamma_j^* = \begin{cases} q - \gamma_j & \textit{if } \gamma_j \neq 0 \\ 0 & \textit{if otherwise} \end{cases} \tag{8.2}$$
$$\textit{where } 1 \le j \le s.$$

In other words $\alpha + \gamma^* \equiv 0 \mod (q)$ and $0 \le \gamma_j^* < q$ for all $i, 1 \le j \le s$.

Note that taking $\gamma \in \mathbb{Z}_0^t$ we have $0 < \gamma_i < q$, $\forall i$, and hence $0 < \gamma_j^* < q$, $\forall i$. Here t is the length of v while s is that of z. In the manner of Eq.(8.2) γ^* is q-supplement of γ as well that of $\mathbf{a}$ with respect to the notational convention $z^\gamma = v^\gamma$.

Recall $\ell(I)$ of Eq.(7.9) and $L_{q-max}(I) = L(I, \ell(I))$ of Def.(7.3). We will use different symbols for the residual and cofactor cases.

$$L(g,a)^{resi} \;\; \textit{for} \;\; L(g,a) \tag{8.3}$$
$$\textit{and} \;\; L_{q-max}(g)^{resi} \;\; \textit{for} \;\; L_{q-max}(g)$$

with understanding that g is a residual factor of the given $\mathcal{G}$.

$$L(v^\gamma,a)^{cofa} \;\; \textit{for} \;\; L(v^\gamma,a) \tag{8.4}$$
$$\textit{and} \;\; L_{q-max}(v^\gamma)^{cofa} \;\; \textit{for} \;\; L_{q-max}(v^\gamma)$$

with understanding that v^γ is a $q\Gamma$-cofactor of $\mathcal{G}$.

Remark 8.2. In the cofactor case the two modules $L(v^\gamma, a)^{cofa}, \forall a$, and $L_{q-max}(v^\gamma)^{cofa}$ are independent of the choice of $q\Gamma$-cofactors v^γ. Therefore we will rewrite

$$L(\mathcal{G},a)^{cofa} \;\; \textit{to be} \;\; L(v^\gamma,a)^{cofa} \tag{8.5}$$
$$\textit{and} \; L_{q-max}(\mathcal{G})^{cofa} \;\; \textit{to be} \;\; L_{q-max}(v^\gamma)^{cofa}$$

However in the residual case $L(g,a)^{resi}, \forall a$, and $L_{q-max}(g)^{resi}$ depend on the choice of g.

Remark 8.3. Consider the following condition on j for each a.

$$e(j) \;=\; max\{\; e \in \mathbb{Z}_0 \;\big|\; p^e \; divides \; \gamma_j \;\} \;\le\; e_a. \tag{8.6}$$

where $1 \le j \le t$ and $1 \le a \le l$. Since we have $0 < \gamma_j < q$ for all j, the κ_ξ-module $L(v^\gamma, a)$ is generated by those $in_\xi(v_j)$ with j satisfying Eq.(8.6). Note that if γ is replaced by the γ^* according to Def.(8.1) all the results remain unchanged because of $\gamma_j^* = q - \gamma_j, \forall j$.

Lemma 8.1. *For every index $a, 1 \le a \le l$, $L(\mathcal{G}, a)^{cofa}$ is uniquely determined by $\mathcal{G}$ and $L_{q-max}(\mathcal{G})^{cofa}$ is generated by $\{in_\xi(v_j) \,|\, 1 \le j \le t\}$ where $v = (v_1, \cdots, v_t)$.*

Proof. The parameters v_j are unique up to multiplication by units thanks to Th.(5.1) and Rem.(5.1) according to Def.(5.1), and hence $in_\xi(v^\gamma)$ is up to a constant multiple. The last assertion is deduced from the Eq.(7.2) with $d = |\gamma|$. □

About the residual factors we should refer to Eq.(5.3) of (5.3).

Lemma 8.2. *Pick any two presentations*

$$\mathcal{G} = (z^{q\mathbf{b}} v^\gamma f \,\|\, /^q) = (z^{q\mathbf{b}} v^\gamma g \,\|\, /^q)$$

Then there exists $b \in R_\xi$ *such that* $f - g = b^q v^{\gamma^*}$.

Proof. The equality of $\mathcal{G}$ implies $v^\gamma(f - g) \in \rho^q(R_\xi)$. So the lemma follows in view of Def.(8.1). □

Remark 8.4. A residual factor g of $\mathcal{G}$ is replaceable by any $f = g + b^q v^{\gamma^*}$ with $b \in R_\xi$ so long as $ord_\xi(b^q) \geq d - |\gamma^*|$ with $d = resord_\xi(\mathcal{G})$. Such a replacement can change not only $in_\xi(g)$ but also $L(g, a)^{resi}$. However we see that for all a with $p^{e_a} < |\gamma^*| + q$ the module $L(g, a)^{resi} + L_{q-max}(\mathcal{G})^{cofa}$ is independent of the choice of g by Lem.(8.2).

Lemma 8.3. *Pick any residual factor* f *of* $\mathcal{G}$ *and write* $f = \sum_\alpha f_\alpha^q x^\alpha$ *with* $f_\alpha \in R_\xi$ *in terms of the parameters* $x = (w, v, \omega)$ *of Eq.(8.1) where* $0 \leq \alpha_i < q, \forall i$. *If* $f_\alpha = 0$ *for* $x^\alpha = v^{\gamma^*}$, *then for every* $b \in R_\xi$ *such that* $ord_\xi(b^q) + |\gamma^*| \geq ord_\xi(f) = d > 0$ *we have*

$$L(f - b^q v^{\gamma^*}, a) \;\supset\; L(f, a) \quad \text{for } \forall a \text{ with } p^{e_a} < |\gamma^*| + ord_\xi(b)q \tag{8.7}$$

which implies

$$L(f, a) = \bigcap_{ord_\xi(b)q \geq d - |\gamma^*|} L(f - b^q v^{\gamma^*}, a) \quad \text{for } \forall a \text{ with } p^{e_a} < |\gamma^*| + ord_\xi(b)q$$

and

$$L(v^\gamma f, b) = \bigcap_{\substack{ord_\xi(\mathbf{g}) \geq d \\ \mathcal{G} = (\mathbf{g} \| /^q)}} L(\mathbf{g}, b) \qquad \text{for } \forall b \text{ with } p^{e_b} < d \tag{8.8}$$

where Eq.(8.7) $\Leftrightarrow$ *Eq.(8.8) is proven by multiplication by* v^γ.

Proof. Suppose that there exists $b \in R_\xi$ such that $ord_\xi(b^q) \geq d - |\gamma^*|$ and $L_{q-max}(f + b^q v^{\gamma^*}) \not\supset L_{q-max}(f)$ with the chosen f as above. Then we have an index k with $p^{e_k} \leq d$ and $\bar{u} \in L(f, k) \setminus L_{q-max}(f + b^q v^{\gamma^*})$ such that $\bar{u}^{p^{e_k}} = in_\xi(\partial f)$ with $\partial \in Diff_{Z,\xi}^{(d - p^{e_k})}$. In view of $ord_\xi(f) = d$ and $ord_\xi(\partial f) = p^{e_k}$ which is $d - (d - p^{e_k})$, we can choose ∂ of the form $\sum_{||=d-p^{e_k}} c_\beta \partial^{(\beta)}$ with elementary differential operators $\partial^{(\beta)}$ with respect to $x = (v, w, \omega)$ and $c_\beta \in \mathbb{K}$. Since f is chosen as was in Lem.(8.3) with $f_\alpha = 0$ for $\alpha = (\gamma^*, 0, 0)$, we may replace ∂ by dropiing $c_\beta \partial^{(\beta)}$ for all $\beta \in (\gamma^*, 0, 0) + q\mathbb{Z}_0$. In other words we may assume $\partial(x^\beta) = 0$ for all $\beta \in (\gamma^*, 0, 0) + q\mathbb{Z}_0$ and $|\beta| = d$ in addition to $\bar{u}^{p^{e_k}} = in_\xi(\partial f)$. With such ∂ we have $in_\xi(\partial(f + b^q v^{\gamma^*})) = in_\xi(\partial(f))$ which is $\bar{u}^{p^{e_k}}$. This means that $\bar{u} \in L_{q-max}(f + b^q v^{\gamma^*})$ against the assumption $\bar{u} \in L(f, k) \setminus L_{q-max}(f + b^q v^{\gamma^*})$. $\square$

Definition 8.2. With $d = resord_\xi(\mathcal{G}) = ord_\xi(f)$ we define

$$L_{q-max}(\mathcal{G})^{resi} = \bigcap_{\substack{b \in R_\xi \\ ord_\xi(b)q \geq d - |\gamma^*|}} L_{q-max}(f + b^q v^{\gamma^*})^{resi} \tag{8.9}$$

which is equal to a particular $L_{q-max}(f)^{resi}$ when f is chosen according to Lem.(8.3).

Definition 8.3. Furthermore we define

$$\begin{aligned} & Resi_{\xi.q}(\mathcal{G}) \Big(or\ Resi_\xi(\mathcal{G})\ or\ Resi(\mathcal{G}) \Big) \\ &= L_{q-max}(\mathcal{G})^{resi} + L_{q-max}(\mathcal{G})^{cofa} \\ &= L_{q-max}(f)^{resi} + \textstyle\sum_i \kappa_\xi\, in_\xi(v_i) \\ &= L_{q-max}(f + b^q v^{\gamma^*})^{resi} + \textstyle\sum_i \kappa_\xi\, in_\xi(v_i) \end{aligned} \tag{8.10}$$

where f is any residual factor of $\mathcal{G}$ and $b \in R_\xi$ is any such that $ord_\xi(b^q) \geq d - |\gamma^*|$. The independence on the choice of residual factors f is due to Lem.(8.2) and to Rem.(8.4). $Resi_{\xi,q}(\mathcal{G})$ will be called the *residual cotangent q-module* of $\mathcal{G}$ or *residual q-module* for short.

Definition 8.4. In view of Lem.(8.3) and by use of the notation of Def.(7.3) we also define

$$L_{[p^+]-max}(\mathcal{G})^{resi} = \bigcap_{\substack{b \in R_\xi \\ ord_\xi(b^q) \geq d - |\gamma^*|}} L_{[p^+]-max}(f + b^q v^{\gamma^*}) \tag{8.11}$$

Furthermore we define

$$\begin{aligned} & Resi_{\xi,[p^+]}(\mathcal{G}) \Big(= Resi_{[p^+]}(\mathcal{G}) \Big) \\ &= L_{[p^+]-max}(\mathcal{G})^{resi} + L_{q-max}(\mathcal{G})^{cofa} \\ &= L_{[p^+]-max}(\mathcal{G})^{resi} + \textstyle\sum_i \kappa_\xi\, in_\xi(v_i) \end{aligned} \tag{8.12}$$

$$= \bigcap_{\substack{b \in R_\xi \\ ord_\xi(b^q) \geq d - |\gamma^*|}} L_{p^+-max}(f + b^q v^{\gamma^*})^{resi} + in_\xi(v)\kappa_\xi \tag{8.13}$$

9. FRONT $\sharp$-KEY PARAMETERS

We assume a $/^q$-exponent $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ with a standard abc-presentation Eq.(6.1) of Def.(6.1). Namely

$$\mathbf{g} = z^{\mathbf{a}} g \quad with \quad z^{\mathbf{a}} = z^{q\mathbf{b}} v^{\mathbf{c}} \tag{9.1}$$

with a residual factor g such that $ord_\xi(g) = resord_\xi(\mathcal{G})$.

Definition 9.1. An element $\zeta \in M_\xi \setminus M_\xi^2$ will be called a $\sharp$-*exact parameter* of $\mathcal{G}$ if we can find

$$\partial \;\in\; Diff_{Z,\xi}^{(d-q_a)} \quad such\ that \quad \partial(\mathbf{g}) \;=\; \zeta^{q_a} \tag{9.2}$$

where $d = ord_\xi(\mathcal{G})$ and $q_a = p^{e_a}$ with some integer $0 \le e_a < e$ so that $1 \le q_a < q$. If moreover ζ induces a nonzero image in $RC_\xi(\mathcal{G})$ of Eq.(**??**) then ζ is called $\sharp$-*exact key q-parameter*, or $\sharp$-*key paramter*, of $\mathcal{G}$ at ξ.

Remark 9.1. The existence of the ∂ with the equality Eq.(9.2) of Def.(9.1) is stronger than that of of Eq.(7.5) of Rem.(7.2), which is in turn stronger than that of Eq.(7.3) of Rem.(7.1).

Theorem 9.1. *The notion of $\sharp$-key parameter of Def.(9.1) is independent of whether we choose p-flag of either $\mathbf{g}$ or $v^{\mathbf{c}} g$ or g of the standard abc-presentation Eq.(9.1) of $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$.*

Proof. Straight forward fron the definition. □

Theorem 9.2. *When $q = p$ or $e = 1$, every key q-parameter ζ is automatically* $\sharp$-exact *in the sense of Eq.(9.2) of Def.(9.1) at every closed point $\xi \in Sing(\mathcal{G})$.*

Proof. Since $\ell(I) = max\{\, a \,\big|\, \exists e_a < e \,\}$ we must have $\ell(I) = 1$ and $e_1 = 0$ so that $p^{e_1} = 1$. Thus the equality Eq.(7.5) is reduced to $in_\xi\big(\partial(g)\big) = in_\xi(\zeta)$ and $ord_\xi\big(\partial(g)\big) = 1$. Hence we can choose ζ to be $\partial(g)$ itself. □

10. $/^q$-STRATIFICATIONS

We will be assuming that $\mathbb{K}$ is algebraically closed. We are given a $/^q$-exponent $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ in Z and a Zariski-closed subset Z^* of the ambient scheme Z, We view Z^* as a closed reduced subscheme of Z. We then have a stratification of Z^* by virtue of Th.(2.1) in the following sense:

Definition 10.1. An $\mathcal{G}_{cl}$*-stratification* of Z^* is an expression of a finite disjoint union $Z^* = \cup_i Z(i)$ such that

(1) the $Z(i)$ are smooth irreducible locally closed subschemes of Z which are called *strata,* and
(2) $ord_\eta(\mathcal{G})$ is constant for all $\eta \in Z(i) \cap Z_{cl}$ for each i.

Remark 10.1. Among all possible $\mathcal{G}_{cl}$-stratifications of a given Z^*, there exists a *canonical* one which is constructed as follows:

Let us first define

$$S_d(\mathcal{G}, Z^*) \;=\; \{\eta \in Z^*_{cl} \,|\, ord_\eta(\mathcal{G}) \geq d\} \tag{10.1}$$

which is a closed subset of $Z^*_{cl} = Z^* \cap Z_{cl}$ *by Th.(2.1).*

For every integer $d \geq 1$, we let $T(d)$ denote the closure in Z^* of the subset $S_d(\mathcal{G}, Z^*)$ of Eq.(10.1). First of all let us note:

(1) For every $d \geq 1$ we have $T(d) \cap Z^*_{cl} \;=\; S_d(\mathcal{G}, Z^*)$ because the latter is closed in Z^*_{cl}.
(2) $S_1(\mathcal{G}, Z^*) \;=\; Z^*_{cl}$. In fact for every $\eta \in Z^*_{cl}$ we can find $g \in R_\eta$ such that $h - g^q \in max(R_\eta)$ because the $R_\eta/max(R_\eta)$ is perfect. Therefore we have $(\mathbf{g} \,\|\, /^q) = (\mathbf{g} - g^q \,\|\, /^q)$.
(3) Hence $T(1) = Z^*$.

We let $d_1 = min\{d > 0 \big| S_d(\mathcal{G}, Z^*) \neq Z^*_{cl}\}$ and choose $C(1)$ to be the collection of connected (and then smooth irreducible) components of $Z^* - \big(Sing(Z^*) \cup T(d_1)\big)$. This $C(1)$ will be the first set of canonical strata. Choose the next set of strata to be the collection $C(2)$ of the connected components of $T(d_1) - \big(Sing(T(d_1)) \cup T(d_2)\big)$ where d_2 is the smallest integer $> d_1$ such that $T(d_1) \neq Sing(T(d_1)) \cup T(d_2)$. Let $S(1) = \big(Sing(T(d_1)) \cup T(d_2)\big)$. Let $C(2)$ be the collection of the connected components of $S(1) - \big(Sing(S(1)) \cup T(d_3)\big)$ where d_3 is the smallest integer $> d_2$ such that $S(1) \neq Sing(S(1)) \cup T(d_3)$. Then let $S(2) = \big(Sing(S(1)) \cup T(d_3)\big)$. Repeat this process until we reach $S(l) = \emptyset$. The *canonical stratification* of Z^* with respect to $\mathcal{G}$ is then the union of those collections $C(j), j = 1, 2, \cdots$, which is altogether a finite collection.

The most basic case is $Z^* = Z$. However when $\mathcal{G}$ is given in combination with another singular object such as an ideal exponent $E = (J, b)$ in Z we often need to consider the case of $Z^* = Sing(E)$.

Remark 10.2. By virtue of Th.(**??**) we have a *canonical* refinement of any given $\mathcal{G}_{cl}$-stratification in such a way that the NC-data Γ is *normal crossing* with every one of the strata of the refinement at every point of Z. The existence of such a refinement is proven thanks to the following fact. For every smooth irreducible locally closed subset C of Z and for the subsystem $\Gamma(C)$ of Γ consisting of those not containing C, we find the smallest (an hence unique) nowhere dense closed subset S of C such that $\Gamma(C)$ is normally crossing with C at every point of $C \setminus S$. Then the final refinement can be obtained by descending induction on dimensions of strata by repeated replacement of C by $C \setminus S$ and canonical $\mathcal{G}_{cl}$-stratification of S. (Choose C to be one of the biggest dimension among the given strata having non-empty S at each of the replacements.)

We consider a $/^q$-exponent $\mathcal{G}$ in Z in the sense of Def.(1.1). Pick a closed point $\xi \in Sing(\mathcal{G})$.

On one hand we may choose a specific $\mathcal{G}_{cl}$*-stratification* of Z in the sense of Def.(10.1) and choose the stratum T containing ξ. This is a kind of top-down selection method, while it is meritably global in nature.

On the other hand we may take the set $S_{cl} = S_{cl}(\xi)$ of all those closed points of $Sing(\mathcal{G})$ at which the residual orders of $\mathcal{G}$ are equal to $resord_\xi(\mathcal{G})$. This is a kind of bottom-up selection method. We have a naturally defined *locally closed* subscheme $S = S(\xi)$ of Z such that $S_{cl}(\xi) = S(\xi) \cap Z_{cl} = S(\xi)_{cl}$. To be precise we first let C be the closure of S_{cl} in Z and let B be the closure of $(C \cap Z_{cl}) \setminus S_{cl}$ in Z. Then we obtain S as being $C \setminus B$. (Refer Th.(2.1).) We then let T be the set of smooth points of S, which is a locally closed subscheme of S.

If the point ξ is such that

$$(10.2) \qquad resord_\xi(\mathcal{G}) = \max_{\zeta \in Sing(\mathcal{G})_{cl}} resord_\zeta(\mathcal{G})$$

then $S_{cl}(\xi)$ is a Zariski closed subset of Z_{cl} and S is a closed subscheme of Z.

When we choose any straiffication of Z by means of a $/^q$-exponent given in Z it is inevitable from encountering and hence we need to deal with *generic-up-down* strata in the sense of Def.(1.3).

Let us review the example Ex.(1.2) of *generic-up-down* phenomena of the $/^q$-exponent $\mathcal{G} = (\mathbf{h}\|/^p)$ with $\mathbf{h} = tz^p + w^{p+1}$, which is given in a 5-dimensional affine space $Z = Spec(\mathbb{K}[t,x,y,z,w])$. Let ξ be the origin at which $ord_\xi(\mathcal{G}) = p+1$. In the example, our $S = S(\xi)$ turns out to be $Sing(\mathcal{G})$ which is 3-dimensional subspace. This is the closure C of the point $\sigma = (z,w)$. The order of $\mathcal{G}$ is $p+1$ at every closed point of S while it is p at the generic point σ. S contains an irreducible surface which is the closure of $\zeta = (\phi, z, w)$ with $\phi = x^p + ty^p$. Call the surface F. The singular locus of F is a line which is the closure of $\eta = (x,y,z,w)$. Call the line L, and $T = F \setminus L$ is the smooth part of

F. The order of $\mathcal{G}$ is $p+1$ at every point of T. It is also $p+1$ at every closed point of L but it is p at the generic point η.

The notable point of this example is that

$$\textit{we have}\ \ S\ \supsetneq\ F\ \supsetneq\ L\ \supsetneq\ \xi\ \ \textit{while}$$
$$S\ \ \textit{is generic-down}\ ,\ F\ \ \textit{is not but}\ \ L\ \ \textit{is}$$

in the sense of *generic-down* subscheme defined by Def.(1.3). Also note that, excluding a single exception $L = Sing(F)$, we find no other irreducible curve of generic down type contained in F. All these claims follow from Th.(2.1) and Lem.(2.3).

The example Ex.(1.2) may be slightly modified as follows:

Example 10.1. Replace $\mathbf{h}$ by $\mathbf{h}^* = \mathbf{h} + \phi^{p+1} + z^{p+1}$. Let $\mathcal{G}^* = (\mathbf{h}^* \| /^p)$. Then we get $Sing(\mathcal{G}^*)$ becomes F which is our new $S(\xi)$. Every other claim made on Ex.(1.2) holds true for the points within F. Noteworthy point is that $S(\xi)$ is not *generic-down* but it contains L which is *generic-down*.

11. Generic down theorems

The theorems in this section are used in the study of *generic-down* phenomena. It will be seen that Ex.(1.1) is a simple but typical "generic-down" case. Indeed the essential partial summand of a general "generic-down" equation is composed of such simple ones in a certain sense that we want to clarify in this section.

Remark 11.1. Let D be a reduced irreducible subscheme of Z and assume that it is a *generic-down subscheme* for a $/^q$-exponent $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ in Z in the sense of Def.(1.3), that is

$$0\ <\ l\ =\ ord_\zeta(\mathcal{G})\ <\ m\ =\ ord_\eta(\mathcal{G}) \tag{11.1}$$

with the generic point ζ of D and for almost all $\eta \in D \cap Z_{cl}$.

Remark 11.2. Pick and fix a point $\xi \in D \cap Z_{cl}$ such that D is smooth at ξ and $ord_\xi(\mathcal{G}) = m$. Then pick a regular system of parameters $x = (u, w)$ of R_ξ which is subject to the following condition.

$$(u)R_\xi\ \ \textit{is the ideal of}\ \ D \subset Z\ \ \textit{at}\ \xi. \tag{11.2}$$

In later applications, D may be given as the center of a blowup permissible for $\mathcal{G}$ as well as for the given NC data Γ. If this is the case we may require (u, w) *contains* z where z denotes a system of parameters defining those components of Γ passing through ξ. However in the following general theorems it is important that no more than

Eq.(11.2) is imposed on our choice of $x = (u, w)$ in search of invariants and globaliation in dealing with *generic down* singularities.

We will write $u = (u_1, \cdots, u_s)$ and $w = (w_1, \cdots, w_t)$. Write $x = (x_1, \cdots, x_n) = (u, w)$ with $n = s + t = dim\, Z$.

The choice of (u, w) determines the following local "etale" retraction.

$$\mathbf{r} : \xi \in D \subset Z \searrow \mathbb{A}^t = Spec(\mathbb{K}[w]) \tag{11.3}$$

in the sense of Def.(**??**). Recal that we then have the q-base algebra $B(q, \mathbf{r})$ in the sense of Eq.(**??**) and the operator algebra $\mathcal{P}(q, \mathbf{r})$ in the sense of Def.(**??**). Recall its relation with the parameters (u, w) in the manner of Eq.(**??**). We also have $\mathcal{P}_\sigma(q, \mathbf{r})$ with pulldown bound σ. Refer to Def.(**??**), Eq.(**??**) and Rem.(**??**).

We now proceed to state and prove the theorems and lemmas about "generic down" phenomena. We know that R_ξ is a $\rho^e(R)$-module freely generated by $\{u^a w^b, (ab) \in \epsilon^n(q)\}$. We then choose $\mathbf{g}$ of $\mathcal{G} = (\mathbf{g} \| /^q)$ as follows. With respect to the (u, w) we have the **-full idempotent q-operator* $\mathfrak{d}^*$ in the sense of Def.(**??**)). For the given $\mathcal{G}$ we can replace $\mathbf{g}$ by $\mathfrak{d}^*(\mathbf{g})$. In fact $\mathfrak{d}^*(\mathbf{g}) - \mathbf{g}$ belongs to $\rho^q(R_\xi)$. In effect $\mathfrak{d}^*$ annihilates all the q-the power monomial terms and keeps the other terms as they are with respect to the variables (u, w). We thus have

$$\mathfrak{d}^*(\mathbf{g}) = \sum_{(ab) \in \epsilon^n(q)} {d_{ab}}^q\, u^a w^b \tag{11.4}$$

with $d_{00} = 0$ and $d_{ab} \in R$. We then have

$$m = ord_\xi(\mathfrak{d}^*(\mathbf{g})) = \min\{\, |a| + |b| + ord_\xi(d_{ab})\, q \,\}$$
$$and \;\; l = ord_\zeta(\mathfrak{d}^*(\mathbf{g})) = \min\{\, |a| + ord_\zeta(d_{ab})\, q \,\}$$

The numbers m and l are thus defined and indpendent of the choice of (u, w) so long as the condition Eq.(11.2) is satisfied.

For the sake of notational simplicity in what follows we assume to have chosen $\mathbf{g}$ in such a way that $\mathfrak{d}^*(\mathbf{g}) = \mathbf{g}$. This is allowable by our definition of $/^q$-exponent. With such choice of $\mathbf{g}$ let us then define:

$$\Delta \;=\; \left\{ (ab) \,\middle|\, |a| + ord_\zeta(d_{ab}) q < m \right\} \tag{11.5}$$

Note that the set Δ is not empty because of the "generic down" assumption $l < m$.

Lemma 11.1. *Pick any $(ab) \in \epsilon^n(q)$ such that $a \neq 0$. We then claim*

$$ord_\zeta(d_{ab}^q u^a w^b) \;=\; |a| + ord_\zeta(d_{ab}) q \;\geq\; m.$$

Therefore we must have $a = 0$ for all $(ab) \in \Delta$ and

$$l \;=\; min\{\, ord_\zeta(d_{0b}) q \,\big|\, (0b) \in \Delta\}$$

It follows that we have $l = Aq$ with an integer $A > 0$.

Proof. Since $a \neq 0$ there exists an index k with $0 < a_k < q$. Let a/k denote the element of $\epsilon^n(k)$ whose k-th component is a_k and others are all zero. Let $\partial^{(a/k)}$ be the differential operator in the sense of Eq.(2.13). Namely $\partial^{(a/k)} u^{a'} w^{b'}$ is equal to $\binom{a'}{a/k} u^{a'-a/k} w^{b'}$. Hence

$$(11.6) \qquad ord_\chi\big(\, \partial^{a/k}(\, d^q_{ab} u^a w^b\,)\big) \;=\; ord_\chi(d^q_{ab} u^a w^b) - a_k \\ \textit{for evey point } \;\chi \in D.$$

Now for every η in a neighborhood of ξ within $D \cap Z_{cl}$, we have

$$(11.7) \qquad ord_\eta\big(\,\partial^{(a/k)}\mathbf{g}\big) \;\geq\; m - a_k$$

because $ord_\eta(\mathcal{G}) \geq m$ for every $\eta \in D \cap Z_{cl}$ and $\partial^{(a/k)}$ annihilates any q-th powered differences. Then Eq.(11.7) implies

$$(11.8) \qquad ord_\zeta\big(\,\partial^{(a/k)}\mathbf{g}\big) \;\geq\; m - a_k$$

In view of the expression Eq.(11.4) of $\mathbf{g}$ at the point ξ, Eq.(11.8) implies the following.

$$(11.9) \qquad ord_\zeta(d^q_{ab} u^{a-(a/k)} w^b) \;=\; ord_\zeta(d^q_{ab} u^{a-(a/k)}) \;\geq\; m - a_k \\ \textit{which implies } \; ord_\zeta(d^q_{ab} u^a) \;\geq\; m$$

This proves the first claim of Lem.(11.1). The rest of Lem.(11.1) follows from this. □

Lemma 11.2. *We have $m - l < q$. Hence m is not divisible by q.*

Proof. We will prove that we cannot have a positive integer A such that $l \leq Aq < (A+1)q \leq m$. This is proven as an easy consequence of Th.(2.6). In fact, if it were false then Th.(2.6) tells us that

$$S \;=\; \{\; \theta \in Z \;\big|\; ord_\theta(\mathcal{G}) \geq (A+1)q \;\}$$

is Zariski-closed in Z. But by definition $S \cap Z_{cl}$ contains

$$\{\eta \in Z_{cl} \big| ord_\eta(\mathcal{G}) \geq m\}$$

which contains $D \cap Z_{cl}$. Hence $S \supset D$ so that $(A+1)q \leq ord_\zeta(\mathcal{G}) = l = Aq$ which is absurd. □

Proof. First of all we recall that $(0b) \in \Delta$ implies

$$(11.10) \qquad l = Aq = ord_\zeta(d^q_{0b} w^b) = ord_\zeta(d^q_{0b}) < m \\ \textit{while } \; ord_\xi(d^q_{0b} w^b) \geq m$$

by Lem.(11.1), Lem.(11.2) and Lem.(11.3). In particular $b \neq (0)$. Now pick any nonzero component b_j of b and let p^{e_j} denote the highest power of p which divides b_j. Define $(b/j) \in \epsilon^n(q)$ to be the one whose j-th component is p^{e_j} and others are all zero. We then have the elementary differential operator $\partial^{(b/j)}$ such that

$$\partial^{(b/j)} u^{a'} w^{b'} = \binom{b'}{b/j} u^{a'} w^{b'-(b/j)} \tag{11.11}$$

We then follow the same sequence of reasonings as Eq.(11.6), Eq.(11.7) and Eq.(11.8). Here we take (b/j) instead of (a/k), $w^{b-(b/j)}$ instead of w^b and p^{e_j} instead of a_k. In the same manner of Eq.(11.9) we obtain

$$l = ord_\zeta(d_{0b}^q) = ord_\zeta(d_{0b}^q w^{b-(b/j)}) \geq m - p^{e_j} \tag{11.12}$$
$$\textit{so that}$$
$$l = ord_\zeta(d_{0b}^q w^b) \geq m - p^{e_j} \geq m - b_j \geq m - |b|$$

Hence $p^{e_j} \geq m - l = m - Aq$ for every $b_j \neq 0$. Moreover it follows that if $l + |b| = m$ then $|b| = b_j = p^{e_j}$. In other words if $ord_\xi(d_{0b}^q w^b) = m$ then b has one and only one nonzero component which is p^{e_j}. □

Lemma 11.3. *We have $l = Aq = ord_\zeta({d_{0b}}^q\, w^b)$ for every $(0b) \in \Delta$.*

Proof. We have $l = Aq \leq ord_\zeta({d_{0b}}^q w^b)$ for every $(0b) \in \Delta$ by the definition of the number l. On the other hand $ord_\zeta({d_{0b}}^q w^b) = ord_\zeta({d_{0b}}^q)$ which is $ord_\zeta(d_{0b})q$. This is an integral multiple of q and hence if $ord_\zeta({d_{0b}}^q w^b) > Aq$ then it must be $\geq (A+1)q > m$. This means $(0b) \notin \Delta$ against the assumption. □

Lemma 11.4. *Pick any one $(0b) \in \Delta$ and write $w^b = \prod_{1\leq j\leq t} w_j^{b_j}$. Then we have:*

(1) There always exists at least one j with $b_j > 0$.
(2) If $b_j > 0$ then there exist integers $e(b,j) \geq 0$ and $c(b,j) \geq 1$ such that $b_j = c(b,j)p^{e(b,j)}$ and $p \not| c(b,j)$.
(3) If $b_j > 0$ we have $p^{e(b,j)} \geq m - l$.
(4) If $ord_\xi(d_{0b}^q w^b) = m$ then b has one and only one nonzero component b_j which is equal to p^{e_j}.

Proof. First of all we recall that $(0b) \in \Delta$ implies

$$l = Aq = ord_\zeta(d_{0b}^q w^b) = ord_\zeta(d_{0b}^q) < m \tag{11.13}$$
$$\textit{while}\ \ ord_\xi(d_{0b}^q w^b) \geq m$$

by Lem.(11.1), Lem.(11.2) and Lem.(11.3). In particular $b \neq (0)$. Now pick any nonzero component b_j of b and let p^{e_j} denote the highest power of p which divides b_j. Define $(b/j) \in \epsilon^n(q)$ to be the one whose j-th component is p^{e_j} and others are all zero. We then have the elementary differential operator $\partial^{(b/j)}$ such that

$$\partial^{(b/j)} u^{a'} w^{b'} \;=\; \binom{b'}{b/j} u^{a'} w^{b'-(b/j)} \tag{11.14}$$

We then follow the same sequence of reasonings as Eq.(11.6), Eq.(11.7) and Eq.(11.8). Here we take (b/j) instead of (a/k), $w^{b-(b/j)}$ instead of w^b and p^{e_j} instead of a_k. In the same manner of Eq.(11.9) we obtain

$$l = ord_\zeta(d^q_{0b}) = ord_\zeta(d^q_{0b} w^{b-(b/j)}) \geq m - p^{e_j} \tag{11.15}$$

$$\text{so that}$$

$$l = ord_\zeta(d^q_{0b} w^b) \geq m - p^{e_j} \geq m - b_j \geq m - |b|$$

Hence $p^{e_j} \geq m - l = m - Aq$ for every $b_j \neq 0$. Moreover it follows that if $l + |b| = m$ then $|b| = b_j = p^{e_j}$. In other words if $ord_\xi(d^q_{0b} w^b) = m$ then b has one and only one nonzero component which is p^{e_j}. □

Definition 11.1. With the set Δ of Eq.(11.5) we define

$$g(0) \;=\; \sum_{(ab)\in\Delta} d_{ab}{}^q\, u^a w^b \;=\; \sum_{(0b)\in\Delta} d_{0b}{}^q\, w^b$$

of which the last equality if by Lem.(11.1).

Theorem 11.5. *The summand $g(0)$ of* **g** *has the following properties.*

(1) $ord_\zeta(g(0)) = l = Aq$ with a positive integer A

(2) and $Aq < m \leq ord_\xi(g(0))$.

(3) There exists a nonempty finite set B of maps from $[1,t]$ to $\mathbb{Z}_0$ which has the following properties.

$$g(0) \;=\; \sum_{\beta\in B} g(0)_\beta \quad with \quad g(0)_\beta \;=\; \phi_\beta{}^q \prod_{1\leq i\leq t} w_i{}^{q_*\beta(i)} \tag{11.16}$$

where we have:

(a) For every $\beta \in B$ there exists at least one i with $\beta(i) > 0$.

(b) $\beta(i)$ is not divisible by p for at least one pair (β, i).

(c) q_* is a unique power of p and we have $q > q_* \geq m - Aq$

(d) $\phi_\beta \in I_\xi{}^A$ and $ord_\zeta(\phi_\beta) = A$ for every $\beta \in B$.

(e) $ord_\xi(g(0)_\beta) \geq m$ for all $\beta \in B$.

(4) We always have $ord_\xi(g(0)) \geq m$.

(5) Suppose we had $ord_\xi(g(0)) = m$. (We may not have the equality. See Ex.(11.1) below.) For $\beta \in B$ with $ord_\xi(g(0)_\beta) = m$ we have one and only one index k with $\beta(k) \neq 0$ and $\beta(k) = 1$, so that $g_\beta = \phi_\beta{}^q\, w_k^{q_*}$.

Proof. We only need Lems.(11.1), (11.2), (11.3) and (11.4). The q_* is the minimum of all those $p^{e(b,j)}$ of Lem.(11.4). The last assertion follows from the fact that $ord_\xi(\phi_\beta{}^q) = Aq$ and $q_* \geq m - Aq$. □

Let us next define

$$\Delta^{\dagger} \;=\; \Big\{ (ab) \,\Big|\, m \le |a| + ord_{\zeta}(d_{ab})q < l + q \Big\} \tag{11.17}$$

Note that this set $\Delta^{\dagger}$ could be empty unlike Δ. Examples are easy to find either for $\Delta^{\dagger} = \emptyset$ or for $\Delta^{\dagger} \ne \emptyset$.

Example 11.1. Let $\mathbf{g} = u_1^{Aq} w_1 + c u_2^{Aq+1} + u_3^{(A+1)q} w_2$ where $m = Aq + 1$ and $l = Aq$ while we let either $c = 0$ or $c = 1$.

Definition 11.2. With $\Delta^{\dagger}$ as above we define

$$\begin{aligned} g(0)^{\dagger} \;&=\; \textstyle\sum_{(ab)\in\Delta^{\dagger}} {d_{ab}}^{q}\, u^a w^b \\ \textit{and}\;\; g(1) \;&=\; \mathbf{g} \;-\; g(0) \;-\; g(0)^{\dagger} \end{aligned}$$

Lemma 11.6. *We have that $g(1)$ is the sum of those terms ${d_{ab}}^{q}\, u^a w^b$ having $ord_{\zeta}(d_{ab})q + |a| \ge (A+1)q$. We also have*

(1) $ord_{\zeta}(g(0)) = Aq = l < m = \min\{ord_{\xi}(g(0)), ord_{\xi}(g(0)^{\dagger}\}$
(2) $m \le ord_{\zeta}(g(0)^{\dagger}) \le ord_{\xi}(g(0)^{\dagger})$
(3) $m < (A+1)q = l + q \le ord_{\zeta}(g(1)) \le ord_{\xi}(g(1))$.

Proof. Immediate from the definitions of Δ by Eq.(11.5) and $\Delta^{\dagger}$ by Eq.(11.17). □

Definition 11.3. Let us define the partial sum $g(1)^{+}$ of $g(1)$ to be the sum of those terms ${d_{ab}}^{q}\, u^a w^b$ belonging to $\rho^e(R_{\xi})[w]$ as well as having $ord_{\zeta}(d_{ab})q + |a| \ge (A+1)q$. Let us then define $g(1)^{-}$ by $g(1) = g(1)^{+} + g(1)^{-}$ after the notation of Def.(11.2) and Lem.(11.6). Moreover we introduce the flowing decomposition:

$$\begin{aligned} G(+) \;=\; g(0) + g(1)^{+} \;\;\textit{and}\;\; G(-) \;&=\; g(0)^{\dagger} + g(1)^{-} \\ \textit{so that}\;\; \mathbf{g} \;&=\; G(-) + G(+) \end{aligned} \tag{11.18}$$

Theorem 11.7. *We summerise the preceeding definitions.*

$$\begin{aligned} G(+) \;&=\; g(0) + g(1)^{+} \;\in\; \rho^e(R_{\xi})[w] \\ G(-) \;&=\; g(0)^{\dagger} + g(1)^{-} \;\in\; \sum_{0\ne a\in\epsilon^s(q)} u^a \rho^e(R_{\xi})[w] \\ &\textit{and} \\ \mathfrak{d}_x^{*}\mathbf{g} \;&=\; G(+) + G(-) \;\;\textit{for}\;\; \mathcal{G} \;=\; (\, \mathbf{g} \,\|\, /^{q}\,) \end{aligned} \tag{11.19}$$

*where we are considering an arbitrary residual factor g and *-full idempotent differential $\mathfrak{d}^{*}$ with respect to x of Eq.(11.2).*

12. FITTED PERMISSIBLE BLOWUPS

Recall Def.(3.2) on permissibility of a blow-up $\pi : Z' \longrightarrow Z$ with center D for a $/^q$-exponent $\mathcal{G} = (\mathbf{g} \parallel /^q)$ and its transform Def.(3.4). Also refer to Def.(3.1), Th.(3.1) and Th.(3.2).

In some cases, however, Def.(3.2) is not strong enough for the purpose of reduction of singularities. To introduce stronger notion of permissibility, we need to recall the results on Γ-*maximal divisor*, obtained by Th.(5.1) and Rem.(5.1). We also need the notion of *checked* $/^q$*-exponent* $\check{\mathcal{G}}$ associated with the given $\mathcal{G}$ in the sense of Eq.(5.4) of Def.(5.4). This is locally obtained from $\mathcal{G}$ by dividing out its Γ-maximal divisors.

Definition 12.1. Let $\pi : Z' \longrightarrow Z$ be a blowup with center D, closed irreducible and smooth in Z. It is called *fitted permissible* for $\mathcal{G}$ if the following conditions are satisfied

(1) π is permissible for $\mathcal{G}$ in the sense of Def.(3.2) and
(2) with generic point ζ of D we have

$$resord_\zeta(\mathcal{G}) \le resord_\eta(\mathcal{G}) < resord_\zeta(\mathcal{G}) + q \qquad (12.1)$$

for all $\eta \in D \cap Z_{cl}$.

As for the definition of *resord* refer to Eq.(5.3) of Def.(5.3). For better understanding of the inequality of Eq.(12.1) the reader may refer to the generic down theorems such as Th.(11.5) and Th.(11.7)

Thanks to Th.(2.1), we have a locally finite $\mathcal{G}_{cl}$-stratification of

$$Sing(\check{\mathcal{G}})_{cl} = Sing(\check{\mathcal{G}}) \cap Z_{cl}$$

in such a way that each member D of its strata is smooth and locally closed inside Z_{cl} with Zariski topology and $ord_\eta(\check{\mathcal{G}})$ is constant for closed points η of D. It then follows that the blowup with center D is fitted permissible for $\mathcal{G}$ in the sense of Def.(**??**) if it is permissible in the sense of Def.(3.2). The fitted permissibility is indeed a notion stronger than that of Def.(**??**).

Definition 12.2. A permissible blowup π for $\mathcal{G}$ (and for Γ as always) with smooth center D is said to be *schematically fitted* or *sch-fitted* for short if $ord_\eta(\check{\mathcal{G}})$ is constant for all $\eta \in D$, or equivalently $ord_\zeta(\check{\mathcal{G}}) = ord_\xi(\check{\mathcal{G}})$ the generic point ζ of D and for all points ξ of $D \cap Z_{cl}$.

Note here that since D is smooth we have $ord_\zeta(\check{\mathcal{G}}) \le ord_\sigma(\check{\mathcal{G}})$ for every point $\sigma \in D$ by Lem.(2.3). Moreover for a closed point ξ in the closure of σ we have $ord_\sigma(\check{\mathcal{G}}) \le ord_\xi(\check{\mathcal{G}})$ by Lem.(2.7). Thus if $ord_\zeta(\check{\mathcal{G}}) = ord_\xi(\check{\mathcal{G}})$ then $ord_\zeta(\check{\mathcal{G}}) = ord_\sigma(\check{\mathcal{G}})$.

Theorem 12.1. *If π with center D is residually fitted permissible but not schelly maticafitted for $\mathcal{G}$ then D must be* generic-down *type for $\mathcal{G}$.*

This theorem is nothing more than a definition by itself. What is important is its supporting background that is a criterion for "generic-down" phenomena not to happen. The reader should refer to Th.(**??**) with Eq.(**??**).

Remark 12.1. The center of a fitted permissible blowup for a $/^q$-exponent is necessarily transversal by definition to the *foliational component* (which exists only in the *generic-down* case) at every point of the center. The notion of *foliational component* are mentioned in the other sections such as the next one on $/^q$-stable singularities.

13. $/^q$-METASTABLE JUMP

Let $\pi : Z' \longrightarrow Z$ with center D be a *fitted* permissible blowup for $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ in the sense of Def.(12.2) and in particular it is permissible for $\check{\mathcal{G}}$, which denotes the *checked associate* of $\mathcal{G}$ in the sense of Def.(5.4). Let $\mathcal{G}'$ be the transform of $\mathcal{G}$ by π. Let $\xi \in Sing(\mathcal{G}) \subset Z$ be a closed point and pick a closed point $\xi' \in Sing(\mathcal{G}') \cap \pi^{-1}(\xi)$. We want to examine the effect of such a blowup to the invariant $resord_\xi(\mathcal{G})$ in the sense of Eq.(5.3) of Def.(5.3).

We normally expect $resord_{\xi'}(\mathcal{G}') \leq resord_\xi(\mathcal{G})$. (The singularity did not get worse !) But sometimes it happens that $resord_{\xi'}(\mathcal{G}') > resord_\xi(\mathcal{G})$.

Following is the most interesting case to be examined closely:

$$resord_{\xi'}(\mathcal{G}') > resord_\xi(\mathcal{G}) \textit{ while } ord_\xi(\check{\mathcal{G}}) = ord_D(\check{\mathcal{G}}), \tag{13.1}$$

where $ord_D(\check{\mathcal{G}}) = ord_\zeta(\check{\mathcal{G}})$ by definition with the generic point ζ of D.

Definition 13.1. When we have Eq.(13.1), we call ξ' a *metastable singular point* of $\mathcal{G}$ at ξ for π (in short, *metastable* point for π).

Remark 13.1. We follow the manner of Re.(6.1) in choosing parameters $z = (v, w)$ with $v = (v_1, \cdots, v_t)$ where z consists of those parameters defining the components of Γ containing ξ. We follow the manner of *abc*-expression of Def.(6.1) but here we use an expression $\mathcal{G} = (\nabla f \,\|\, /^q)$ locally at the point ξ where $\nabla = z^\alpha = z^{q\beta} z^\gamma$ is Γ-maximal divisor and $z^{q\beta}$ is its $q\Gamma$-factor in the sense of (5.1). We choose f to be a residual factor of $\mathcal{G}$ so that we have $resord_\xi(\mathcal{G}) = ord_\xi(f)$ according to Def.(5.3). We also write $z^\gamma = v^\delta$ with $0 < \delta_j < q, \forall j$.

We are given a blowup π with center D with respect to which we selecting additional parameter ω in such a way that $x = (z, \omega)$ is a regular system of paramers of R_ξ with the $z = (v, w)$ and moreover the following conditions are satisfied.

(1) The ideal I_ξ of D at ξ is $(v^\dagger, w^\dagger, \omega^\dagger)R_\xi$ where $v = (v^\dagger, v^\ddagger)$, $w = (w^\dagger, w^\ddagger)$ and $\omega = (\omega^\dagger, \omega^\ddagger)$,
(2) $v_j R_{\xi'} = I_\xi R_{\xi'}$, i.e., v_j is an exceptional parameter for π at ξ'.

In this section we take the following notational simplification in our study of *metastable singularity*:

(1) $j = 1$
(2) all the components of $v_1^{-1}\omega^\dagger$ take zero values at ξ'.
(3) every component of $v_1^{-1}(v^\dagger, w^\dagger)$ takes a value at ξ' which is either zero or 1.

These can always achieved by a simple coordinate transformation.

Proposition 13.1. (*T. Moh and H. Hauser*) *Assume that* ξ' *is a* metastable *singular point of* $\mathcal{G}'$ *for* π. *Then we have*

(1) the center D *is contained in every* Γ_j *for* $1 \le j \le t$.
(2) ξ' *is not in any of the strict transforms of* $\Gamma_j, 1 \le j \le t$, *by* π.
(3) $ord_\xi(\mathcal{G})$ *is divisible by* q, *which is equivalent to saying that* $|\gamma|+d$ *is divisible by* q *where* $d = resord_\xi(\mathcal{G})$.

The first assertion implies that $v = v^\dagger$ *and* $v^\ddagger = \emptyset$.

Proof. The first result is by the fact that if any one of the $\Gamma_j, 1 \le j \le t$ does not contain D its strict transform is equal to its total transform which contains ξ'. Moreover its exponent in the maximal Γ'-divisor of $\mathcal{G}'$ at ξ' would not be a q-power. Therefore the transform by π of the maximal residual factor of $\mathcal{G}$ at ξ will be the maximal residual factor $\mathcal{G}'$ at ξ' so that its order cannot increase. Namely ξ' should not be metastable. The reason for the second result is the same as above. As for the last result the reason is that $|\gamma| + d$ is the exponent of the exceptional divisor in the maximal Γ'-divisor of $\mathcal{G}'$ at ξ' and if it is not divisible by q then by the same reason as above ξ' should have not be metastable. □

Remark 13.2. $v_1^{-1}v_j, 1 \le j \le t-1$, takes nonzero values at ξ'. With no loss of generality we may and will assume that

$$\text{(13.2)} \quad \textit{the values } (v_1^{-1}v_j)(\xi') = 1,\ 1 \le j \le t-1, \quad \textit{and let } \theta = t-1.$$

Let us divide $w^\dagger$ into two parts $w^\dagger = (w^\dagger(1), w^\dagger(2))$ in such a way that $v_1^{-1}w^\dagger(2)$ vanish at ξ' while none of $v_1^{-1}w^\dagger(1)$ does. We again assume

$$\text{(13.3)} \qquad \textit{the values } (v_1^{-1}w_j^\dagger)(\xi') = 1,\ \forall w_j^\dagger \in w^\dagger(1)$$

so that $v_1^{-1}w^\dagger(1) - id_a$ will become a part of a regular system of parameters of $R_{\xi'}$ where $id_a = (1, 1, \cdots, 1)$ with the size a of $w^\dagger$.

Theorem 13.2. *Let* $\mathbf{d} = resord_\xi(\mathcal{G})$*. If there exists a smooth subscheme* $D \ni \xi$ *which is generic-down type for* $\mathcal{G}$ *at* ξ *in the sense of Def.(1.3) then we must have*

$$\mathbf{c} \ + \ \mathbf{d} \ \equiv \ 1 \quad \mod p$$

for the $\mathbf{c}$ *of the expression of Eq.(6.1). It follows that if any such* D *exists at all then metastable singularity cannot occur with any permissible center.*

Proof. The assertion will be proven as a consequence of Th.(**??**). In fact the generic-down assumption implies the formula Eq.(**??**) of Th.(**??**) saying

$$\mathbf{g} = v^{\mathbf{c}} z^{p\mathbf{b}} g \ \equiv \ \sum_i \ w_i \, \phi_i{}^p \quad \mod \ M_\xi^{\mathbf{d}+1} \tag{13.4}$$

where (v, w, ω) is a regular system of parameters of R_ξ containing a minimal base of the ideal $I(D, Z)_\xi$ as well as z. Moreover $p\, ord_\xi(\phi_i) + 1 = \mathbf{d}$ for every i. Let ϕ_i° denote the leading form of ϕ_i, which is homogeneous of degree $p^{-1}\mathbf{d}$. Let g° denote the initial form of g. Then we have the equality

$$v^{\mathbf{c}} z^{p\mathbf{b}}(g^\circ) \ = \ \sum_i \ w_i \, (\phi_i^\circ)^p$$

where the summands of the right have side have no common nonzero monomial terms pair-wise. Hence at least one index i and a nonzero constant c_i such that

$$v^{\mathbf{c}} z^{p\mathbf{b}}(g^\triangle) \ = \ c_j \ w_i \, (\phi_i^\circ)^p \tag{13.5}$$

where $g^\triangle$ is a partial sum of g° and homogeneous of degree $\equiv 0 \mod p$. Then the equality Eq.(13.5) implies the claimed congruence. □

Let us write $\alpha = q\beta + \gamma \in \mathbb{Z}_0^n$ with $0 \le \gamma_i < q, \forall i$. We have the *q-supplement* of γ in the sense of Def.(8.1), the same of α, which is to be the unique element $\gamma^* \in \mathbb{Z}_0^t$ such that $\alpha + \gamma^* \equiv 0 \mod (q)$ and $0 \le \gamma_j^* < q$ for all $i, 1 \le j \le t$.

Remark 13.3. Let $d = resord_\xi(\mathcal{G})$. We then begin with $Q_N(d)$-cleaning a given residual f of $\mathcal{G}$ with respect to $\{ \lambda^q v^{\gamma^*} \,\big|\, \lambda \in R_\xi \}$. (Refer to Def.(**??**) and Def.(**??**).) It follows, for instance, that $ord_\xi(f) = resord_\xi(\mathcal{G})$.

Then ξ' is *metastable* if and only if we have $\sigma \in \rho^e(R_{\xi'})$ such that

$$ord_{\xi'}\Big(v_t^{-d-|\gamma|}(v^\gamma f) - \sigma^q\Big) > d. \tag{13.6}$$

Remark 13.4. Here we choose σ in such a way that the left hand side is maximal among all choices, so that the left number of Eq.(13.6) is equal to $resord_{\xi'}(\mathcal{G}')$. In this way we can later readily investigate the question of how big the residual order $resord_{\xi'}(\mathcal{G}')$ can become at the given metastable point $\xi' \in \pi^{-1}(\xi)$.

We write

$$f \ = \ f(d) \ + \ f^\sharp \quad with \quad ord_\xi(\ f^\sharp\ ) \ > \ d \tag{13.7}$$

where $f(d)$ is a homogeneous polynomial of degree d in the variables $(v, w^\dagger, \omega^\dagger)$. Since $v_1^{-|\gamma|}v^\gamma$ is a unit in $R_{\xi'}$ we then have the *total metastable inequality*

$$ord_{\xi'}\Big(v_1^{-d}f^\sharp \ + \ \big(v_1^{-d}f(d) - (v_1^{-|\gamma|}v^\gamma)^{-1}\sigma^q\big)\Big) \ > \ d \tag{13.8}$$
$$where$$
$$v_1^{-d}f^\sharp \ \in \ (\ v_1,\ w^\ddagger,\ \omega^\ddagger\ )R_{\xi'}$$

where the last inclusion is due to

$$f^\sharp \in I_\xi^d \cap M_\xi^{d+1} = M_\xi I_\xi^d = (w^\ddagger, \omega^\ddagger)I_\xi + I_\xi^2.$$

It should be noted here that we have a regular system of parameters of $R_{\xi'}$ composed of the following two parts:

$$(\ v_1,\ v_1^{-1}w^\dagger(1) - id_a,\ v_1^{-1}w^\dagger(2),\ w^\ddagger,\ v_1^{-1}\omega^\dagger,\ \omega^\ddagger\ ) \tag{13.9}$$
$$in\ addition\ to\ v_1^{-1}v_j - 1, 1 \le j < t$$

where $id_a = (1, 1, \cdots, 1)$ with the size a of $w^\dagger(1)$.

Let $\theta = t - 1$ and we define what we call *metastable parameters* T.

$$T = (v_1^{-1}v_1 - 1, \cdots, v_1^{-1}v_\theta - 1) \quad and \quad id_\theta = (1, \cdots, 1) \tag{13.10}$$

in such a way that $id_\theta + T$ is $v_1^{-1}v$ of which $v_1^{-1}v_1 (= 1)$ is deleted.

Let us also write

$$U \ = \ (\ v_1,\ v_1^{-1}w^\dagger(1) - id_a,\ v_1^{-1}w^\dagger(2),\ w^\ddagger,\ v_1^{-1}\omega^\dagger,\ \omega^\ddagger\ ) \tag{13.11}$$
$$which\ will\ be\ written\ as\ \ (v_1, U_1, \cdots, U_\vartheta)$$

We divide U into 3 partitions as

$$U \ = \ (\ v_1,\ U(1),\ U(2)\ ) \quad where \tag{13.12}$$
$$v_1 \quad is\ the\ exceptional\ parameter\ for\ \pi\ at\ \xi'$$
$$U(1) \ = \ (\ v_1t^{-1}w^\dagger(1) - id_a,\ v_1^{-1}w^\dagger(2),\ v_1^{-1}\omega^\dagger\ )$$
$$and \ \ U(2) \ = \ (\ w^\ddagger,\ \omega^\ddagger\ )$$

We will later make use of the above partitions of our parameters. We should keep in mind that

$$(13.13)\qquad (\ v_1,\ T,\ U(1),\ U(2)\ )$$
$$\textit{is a regular system of parameters of } R_{\xi'}$$
$$\textit{so that}\ \ \hat{R}_{\xi'}\ =\ K[[\ v_1,\ T,\ U(1),\ U(2)\ ]]$$

where $\hat{R}_{\xi'}$ denotes the completion of $R_{\xi'}$ and K is its coefficient field which is a separable algebraic extension of $\mathbb{K}$.

Remark 13.5. We let

$$(13.14)\qquad \gamma^* = (\gamma^\flat, \gamma_1), \quad \textit{so that}\ |\gamma^\flat| = |\gamma^*| - \gamma.$$

Namely $V^{\gamma^\flat}$ is the product $\prod_{j=1}^{\theta} V_j^{\gamma_j^*}$ for a system V of length θ and for exponent γ^* of length $t = \theta + 1$, where γ^* is q-supplement to γ in the sense of Def.(8.1). For instance we have the *metastable unit*

$$(13.15)\qquad \left(v_1^{-|\gamma|} v^\gamma\right)^{-1} = (id_\theta + T)^{-\gamma} = (id_\theta + T)^{-q\, id}(id_\theta + T)^{\gamma^\flat}$$

where in the middle term the last component of the exponent $-\gamma$ is conventionally neglected. (Think of adding one more component $1+T_1$ to $(id_\theta + T)$ with $T_1 = 0$.)

Pick any σ of Rem.(13.4) and then let

$$(13.16)\qquad \tau = (id_\theta + T)^{-\, id_\theta}\sigma \quad \textit{with a chosen}\ \ \sigma.$$

We rewrite Eq.(13.8) with this τ as follows, and we have that ξ' is *metastable* for π if and only if we have τ such that

$$\textit{the basic metastable inequality}$$
$$(13.17)\qquad ord_{\xi'}\Big(v_1^{-d} f^\sharp\ +\ \big(v_1^{-d} f(d) - (id_\theta + T)^{\gamma^\flat}\tau^q\big)\Big)\ >\ d$$
$$\textit{where}$$
$$v_1^{-d} f^\sharp\ \in\ (\ v_1,\ U(2)\ )R_{\xi'}$$

where τ must be chosen to make the left hand side of the inequality Eq.(13.17) maximal among all choices. (Respect to Rem.(13.4).)

It should also be noted that since $f(d)$ is a homogeneous polynomial of degree d only in the variables $(v, w^\dagger, \omega^\dagger)$, $v_1^{-d} f(d)$ does not have any nonzero monomial terms divisible by any of the variables $(v_1, U(2))$ so that we have

$$v_1^{-d} f(d)\ \textit{in}\ \mathbb{K}[T, U(1)].$$

Let us write τ in two parts

$$(13.18)\qquad \tau\ =\ \tau(1)\ +\ \tau(2)\ \ \textit{with}$$
$$\tau(1)\ \in\ \mathbb{K}[U(1), T]\ \ \textit{and}\ \ \tau(2)\ \in\ (v_1, U(2))R_{\xi'}$$

Now, taking the inequality Eq.(13.17) modulo $(v_1, U(2))R_{\xi'}$, we obtain the following *key inequality*

$$\tau(1) \ \in \ \mathbb{K}[U(1), T] \ \ \textit{and}$$
$$ord_{\xi'}\Big(v_1^{-d} f(d) - \big(id_\theta + T\big)^{\gamma^\flat} \tau(1)^q\Big) > d \tag{13.19}$$

Here it is important that the combined system of Eq.(13.9) is a regular system of parameters of $R_{\xi'}$.

Theorem 13.3. *Let $F(T, U(1)) \ = \ v_1^{-d} f(d)$, which is a polynomial in $K[T, U(1)]$. If ξ' is a metastable point of the transform $\mathcal{G}'$ of $\mathcal{G}$ by π then there exists $\tau = \tau(1) + \tau(2) \in R_{\xi'}$ with respect to Rem.(13.4) and $\tau(1) \in K[T, U(1)]$ according to Eq.(13.18) in such a way that*

$$\textit{fundamental metastable equality}$$
$$F(T, U(1)) \ = \ \Big[(id_\theta + T)^{\gamma^\flat} \tau(1)^q\Big]_d \tag{13.20}$$

where $[\,]_d$ means the partial obtained by summing up all the monomial terms of degrees $\leq d$ in a polynomial (or power series) with respect to the chosen variables. Moreover we automatically have or can choose $\tau(1)$ in order to have the following properties:

(1) $F(T, U(1))$ is a polynomial of degree $\leq d$,
(2) $\tau(1) \neq 0$ because $F(T, U(1)) \neq 0$,
(3) the leading homogeneous part of $F(T, U(1))$ is a q-th power by the equality Eq.(13.20).
(4) The nonzero monomial terms of $(id_\theta + T)^{\gamma^\flat}$ are $\rho^e(R_{\xi'})$-linearly independent. In fact the components of $\gamma^\flat$ are all $\leq q - 1$ and T extends to a regular system of parameters of $R_{\xi'}$.
(5) We may choose $\tau(1) \in K[T, U(1)]$ without affecting Eq.(13.20) and Rem.(13.4). (We may even choose $deg(\tau(1)^q) \leq d$ without affecting Eq.(13.20) by itself.)
(6) We have $ord_{(T,U(1))}(\,F(T, U(1))\,) > d - |\gamma^\flat|$. This is due to the cleaning Rem.(13.3) of f by means of $v^{\gamma^\flat}$.
(7) We can choose $\tau(1)$ such that $ord_{\xi'}(\tau(1)^q) = ord_{(T,U(1))}(\tau(1)^q) = ord_{(T,U(1))}(F(T, U(1))) > d - |\gamma^\flat|$.
(8) We have $F(T, U(1)) \ \in \ K[\,T, \rho^e(U(1))\,]$.
*(9) T is Γ'-transversal at ξ' where Γ' is the NC-transform of Γ by π in the sense of Def.(**??**). In fact the subsystem $(v_1, v_1^{-1} w^\dagger(2), w^\ddagger)$ of Eq.(13.9) is the system of parameters defining those members of Γ' passing through ξ' and the combined system*

$$(\,T,\, v_1,\, v_1^{-1} w^\dagger(2),\, w^\ddagger\,)$$

extends to a regular system of parameters of $R_{\xi'}$ by Eq.(13.9).

Proof. Pick any $\tau(1)$ of Eq.(13.19) and then Eq.(13.20) is a direct consequence of Eq.(13.19). It is clear that $degF(T,U(1)) \leq d$. We have $deg(id_\theta + T)^{\gamma^\flat} = |\gamma^\flat|$ which is $|\gamma^*| - \gamma_1$ and if we let

$$\phi \ = \ v_1^d(id_\theta + T)^{\gamma^\flat}[\tau(1)^q]_{d-|\gamma^\flat|}$$

then ϕ must be either zero or a homogeneous polynomial in x of degree d. We know that $d + |\gamma^\gamma| \equiv 0 \mod q$ by Prop.(13.1). Hence $d - |\gamma^\flat| - \gamma_1 = d - |\gamma^*| \equiv 0 \mod q$ because $|\gamma + \gamma^*| \equiv 0 \mod q$ by definition. Hence we can write $d - |\gamma^\flat| = \gamma_1 + aq$ with an integer $a \geq 0$ because $0 < le\gamma_1 < q$. Thus we have $[\tau(1)^q]_{d-|\gamma^\flat|} = [\tau(1)^q]_{d-|\gamma^*|}$. Therefore we get

$$\phi \ = \ \left(v_1^{|\gamma^*|}(id_\theta + T)^{\gamma^\flat}\right)\left(v_1^{d-|\gamma^*|}[\tau(1)^q]_{d-|\gamma^*|}\right)$$

and hence ϕ is divisible by v^{γ^*}. Since f (and hence $f(d)$) is cleaned by means of v^{γ^*} by Rem.(13.3), $[\tau(1)^q]_{d-|\gamma^*|}$ must be identically zero. This implies $ord_{(T,U(1))}(\ F(T,U(1))\ ) \ > \ d - |\gamma^\flat|$ because of the equality Eq.(13.20). We conclude $ord_\xi(\tau(1)^q) \ > \ d - |\gamma^\flat|$. We can replace $\tau(1)$ so as to have $deg(\tau(1)^q) \leq d$ without affecting the equality Eq.(13.20). The equality Eq.(13.20) implies that the initial term of $F(T,U(1))$ must be q-th power in R_ξ. We can choose τ to be a polynomial in $(T,U(1))$ because $resord_{\xi'}(\mathcal{G})$ cannot be ∞. The equality Eq.(13.20) implies that $F(T,U(1)) \in K[T,\rho^e(U(1))]$. □

Corollary 13.4. *The fundamental metastable equality Eq.(13.20) is written more explicitly as follows. Let us write*

$$\tau(1)^q \ = \ \sum_{\frac{d-|\gamma^*|}{q} < l \leq \frac{d}{q}} \tau_l^{\,q}$$

where τ_l is a homogeneous polynomial of degree l in $K[T,U(1)]$. Let us use the symbol $\{\ \}_a$ to denote the homogeneous part of degree a. Then we have

$$F(T,U(1)) \ = \ \sum_{\frac{d-|\gamma^*|}{q} < l \leq \frac{d-b}{q}} \ \sum_{b \leq d-lq} \left\{(id_\theta + T)^{\gamma^\flat}\right\}_b \tau_l^{\,q}.$$

Moreover it follows that

$$F(T,U(1)) \ = \ \sum_{\frac{d-|\gamma^*|}{q} < l \leq \frac{d}{q}} \left[(id_\theta + T)^{\gamma^\flat}\right]_{d-lq} \lambda_l^q$$

with certain homogeneous polynomial λ_l of degree l in $K[T,U(1)]$, so that

$$f(d) \ = \ \sum_{lq+b=d} \Lambda_l^q \Phi_b$$

where $\Lambda_l = v_1^l \lambda_l$ *and* $\Phi_b = v_1^b \Big[(id_\theta + T)^{\gamma^\flat}\Big]_b$ *with* $0 \le b < |\gamma^\flat|$. *Note that* Φ_b *is a homogeneous polynomial of degree* b *in* $K[v]$ *and that* $\Phi_b \notin K[v^q]$.

Theorem 13.5. *Let* $F(T, U(1)) = v_1^{-d} f(d)$ *as was in Th.(13.3). Let* $\tau^{\sharp q}$ *be the sum of those terms of* $F(T, U(1))$ *which belong to* $\rho^e(R_{\xi'})$. *If* $\xi' \in Sing(\mathcal{G}')$ *is metastable of* $\mathcal{G}$ *for* π *then we can choose* $\tau^\sharp$ *instead of* $\tau(1)$ *in Eq.(13.20) as follow.*

$$F(T, U(1)) \;=\; \Big[(id_\theta + T)^{\gamma^\flat} \tau^{\sharp q}\Big]_d \tag{13.21}$$

Proof. Let ∇ denote $(id_\theta + T)^{\gamma^\flat} \tau(1)^q$ and $\nabla^\sharp$ the one after the replacement of $\tau(1)$ by $\tau^\sharp$. Let $d^* = ord_{\xi'}(F(T, U(1)) - \nabla^\sharp)$ and we want to prove $d^* > d$. Let us write

$$F(T, U(1)) - \nabla^\sharp \;=\; A - B \quad \textit{where} \tag{13.22}$$

$$A \;=\; F(T, U(1)) - \tau^{\sharp q} \quad \textit{and} \quad B \;=\; \tau^{\sharp q}\Big((id_\theta + T)^{\gamma^\flat} - 1\Big)$$

Here we see that A has no q-th powered monomial terms by definition of $\tau^{\sharp q}$ and B has no such terms because $1 \le \gamma_j^\flat < q$, $\forall j$. Hence $F(T, U(1)) - \nabla^\sharp$ has no such terms. We can also write

$$F(T, U(1)) - \nabla^\sharp \;=\; E + F \quad \textit{where} \tag{13.23}$$

$$E \;=\; \big(F(T, U(1)) - \nabla\big) \quad \textit{and} \quad F \;=\; \big(\tau(1) - \tau^\sharp\big)^q \big((id_\theta + T)^{\gamma^\flat}\big)$$

Suppose $d^* \le d$. Since $ord_{\xi'}(A) > d \ge d^*$ we should have

$$in_{\xi'}(F(T, U(1)) - \nabla^\sharp) \;=\; in_{\xi'}(F) \;=\; in_{\xi'}(\tau(1) - \tau^\sharp)^q.$$

which is q-th power. This contradicts the earlier result from Eq.(13.22). $\square$

Theorem 13.6. *Under the metastable assumption, the number* d *of Th.(13.3) cannot have* $|\gamma^*| > d \ge |\gamma^\flat|$. *With respect to*

$$\check{d} \;=\; ord_{\xi'}(v_1 1^{-d} f(d)) \;=\; ord_{\xi'}(\tau(1)^q) \tag{13.24}$$

the inequality $\check{d} > d - |\gamma^\flat|$ *of Th.(13.3) is useful (after the cleaning of Rem.(13.3)) when* $d \ge |\gamma^*|$, *while this theorem is significant when* $d < |\gamma^*| = |\gamma^\flat| + \gamma_1$.

Proof. Assume $d \ge |\gamma^\flat|$. Under the assumption that ξ' is metastable for $\mathcal{G}$, we must have $d + |\gamma| \equiv 0 \mod q$ by Prop.(13.1). Hence $d - |\gamma^\flat| - \gamma_1 = d - |\gamma^*| \equiv 0 \mod q$. Since $0 < \gamma_1 < q$ and $d - |\gamma^\flat| \ge 0$, we must first $d - |\gamma^\flat| \ge \gamma_1$ and then $d = |\gamma^*| + aq$ with a non-negative integer a. $\square$

14. $\sharp$-KEYS AGANST METASTABLES

Remark 14.1. Given a $/^q$-exponent $\mathcal{G}$ of together with a $\sharp$-*key parameters* $\zeta^\sharp$ of $\mathcal{G}$ at ξ we will define an *idempotent* differential operator $\mathfrak{d}^\sharp$ as follows.

(1) Firstly choose subsystems $\varpi \subset \omega$ and a$w^\circ \subset w$ such that $y = (\zeta^\sharp, w^\circ, v, \varpi)$ is a regular system of parameters of R_ξ. If $\zeta^\sharp$ is empty then we let $\varpi = \omega$ and $y = x$.
(2) Let us then define $\mathfrak{d}^\sharp$ to be the **-full idempotent* differential operator in $Diff_{R_\xi/\rho^e(R_\xi)[z,\varpi]}$ with respect to the parameters $\zeta^\sharp$ in the sense of Def.(**??**) and Def.(**??**).
(3) If $\zeta^\sharp = \emptyset$ then $\mathfrak{d}^\sharp = 0$.

Definition 14.1. We define

$$\mathbf{g}^\sharp = \mathfrak{d}^\sharp(\mathbf{g}) \tag{14.1}$$

with the $\sharp$-idempotent differential operator $\mathfrak{d}^\sharp$ of Rem.(14.1).

Theorem 14.1. *Assume $\zeta^\sharp \neq \emptyset$. Let $K(Z)$ be the field of fractions of R_ξ or the function field of Z. Then*

$$K(\mathfrak{d}^\sharp) \;=\; \{\phi \in K(Z) \,|\, \mathfrak{d}^\sharp(\phi) = 0\}$$

is equal to $\rho^e(K)(z,\varpi)$ which is a proper subfield of $K(Z)$. Moreover with $\mathbf{g}^\sharp$ of Eq.(14.1) we have $\mathfrak{d}^\sharp(\mathbf{g}^\sharp) = \mathbf{g}^\sharp$ and $\mathbf{g} - \mathbf{g}^\sharp \in K(\mathfrak{d}^\sharp)$.

Proof. Immediate from definitions of the symbols involved. □

Definition 14.2. The idempotent differential operator $\mathfrak{d}^\sharp$ obtained above will be called $\sharp$-*idempotent differential operator* of $\mathcal{G}$ associated with the given $\sharp$-key parameters $\zeta^\sharp$.

Remark 14.2. With $\mathfrak{d}^\sharp$ of Rem.(14.1) let us define

$$\chi \;=\; max\{\; c \in \mathbb{Z}^s \mid in_\xi(z^c) \text{ divides } in_\xi(\mathfrak{d}^\sharp\mathbf{g}) \;\} \tag{14.2}$$

where $z = (z_1, \cdots, z_s)$ is the system of equations for those members of Γ which pass through ξ. Here if $\zeta^\sharp = \emptyset$ then the max does not exist or $\chi = \infty^s$. If $\zeta^\sharp$ is not empty then $\chi \in \mathbb{Z}_0^s$. Always $z^{\mathbf{a}}$ divides z^χ but they do not coincide in general. We can write

$$\mathfrak{d}^\sharp(\mathbf{g}) = z^\chi g^\circ + \mathbf{g}^+ \quad \text{with } g^\circ \in R_\xi \tag{14.3}$$

subject to the condition that we have

$$\begin{aligned} ord_\xi(\mathbf{g}^+) \;&>\; ord_\xi(g^\circ) + |\chi| \\ &=\; ord_\xi(\mathfrak{d}^\sharp(\mathbf{g})) \end{aligned} \tag{14.4}$$

Throughout the rest of this section we will be assuming $\zeta^\sharp \neq \emptyset$.

Theorem 14.2. *All the following three*

(1) $\mathbf{g} = z^{\mathbf{a}} g$ *of Eq.(6.1)*
(2) $\mathbf{g}^\sharp = \mathfrak{d}^\sharp \mathbf{g}$ *of Eq.(14.1)*
(3) *and* $z^\chi g^\circ$ *of Eq.(14.3)*

have the same $\zeta^\sharp$ *as their* $\sharp$*-key parameters according to Def.(14.2). Moreover if* $\zeta^\sharp$ *is sharp-exact for any one of the three* $/^q$*-exponents as above then it is the same for the others.*

Proof. We can see that the κ_ξ-module Eq.(**??**) remains the same for both cases. ☐

Definition 14.3. Let us define the following $/^q$-exponent

$$\begin{aligned} \mathcal{G}(\sharp) \;&=\; (\mathbf{g}(\sharp) \,\|\, /^q) \;=\; (z^{\mathbf{a}(\sharp)} g(\sharp) \,\|\, /^q) \\ &=\; (z^{q\mathbf{b}(\sharp)} v(\sharp)^{\mathbf{c}(\sharp)} g(\sharp) \,\|\, /^q) \end{aligned} \tag{14.5}$$

in the manner of Eq.(6.1) of Def.(6.1).

Remark 14.3. The Γ-monomial z^χ of Rem.(14.2) is uniquely determined by Eq.(14.2). Now let us choose the idempotent differential operator

$$\mathfrak{d}^{(\chi)} \quad in \quad Diff_{R_\xi/\rho^e(R_\xi)[\zeta(\sharp),v(\sharp)^*]} \tag{14.6}$$

with respect to the parameters $v(\sharp)$ where $v0^*$ denotes the q-complement of $v(\sharp)$ in z. Recall that for every $\phi \in \rho^e(R_\xi)[\zeta(\sharp), v(\sharp)^*]$

$$\mathfrak{d}^{(\chi)}(v(\sharp)^\lambda \phi) \;=\; \begin{cases} v(\sharp)^\lambda \phi & \textit{if } \lambda = \chi(\sharp) \\ 0 & \textit{if otherwise} \end{cases}$$

Let us define the following notation:

Definition 14.4. $\mathcal{D}^{(A)} \;=\; \sum_{k \in \epsilon^n(q) \cap A + \mathbb{Z}_0^n} \mathfrak{d}^k$.

Remark 14.4. We have defined $\mathbf{g}(\sharp)$ in Th.(14.1). and use it for the study done later of the *protostable structure* of equations. Here we define a further partial sum $\mathbf{g}^\circ$ of $\mathbf{g}(\sharp)$ and hence of $\mathbf{g}$.

$$\mathbf{g}^\circ \;=\; \mathcal{D}(\sharp)\mathbf{g} \quad with \quad \mathcal{D}(\sharp) \;=\; \mathcal{D}^{(\chi)}(\mathfrak{d}^\sharp) \tag{14.7}$$

Note that $in_\xi(\mathbf{g}^\circ) = in_\xi(\mathbf{g})$ and that $\mathbf{g}^\circ$ is divisible by z^χ in R_ξ. Hence, from now on, we specifically choose g° of Eq.(14.3) to be $g^\circ = z^{-\chi}\mathbf{g}^\circ$ with $\mathbf{g}^\circ$ of Eq.(14.7). Thus we have

$$\mathbf{g}^\circ \;=\; z^\chi g^\circ \quad with \quad g^\circ \;=\; \mathfrak{d}^\sharp g^* \tag{14.8}$$

with $g^* = \mathfrak{d}^+ g$ in the sense of Eq.(**??**). Moreover it should be noted that $\mathcal{D}^{(\chi)}$ and $\mathfrak{d}^\sharp$ commute each other for they depend disjoint sets of variables, the former of z and the latter of $\zeta(\sharp)$. Hence Eq.(14.7) can be written as $\mathbf{g}^\circ = \mathfrak{d}^\sharp(\mathcal{D}^{(\chi)}\mathbf{g})$ and $\mathcal{D}(\sharp)$ is idempotent, too. Thus we have

$$\mathfrak{d}^\sharp\,\mathbf{g}^\circ \;=\; \mathbf{g}^\circ \;=\; \mathcal{D}(\sharp)\,\mathbf{g}^\circ \tag{14.9}$$

Lemma 14.3. *The definition of χ by Eq.(14.2) produces the same result when we replace $\mathbf{g}$ by $\mathbf{g}^\circ$ in the equation Eq.(14.2).*

We now go back to $\mathbf{g}^\sharp$ defined by Eq.(14.1) of Def.(14.1). We first simplify the notation by writing

$$\mathbf{g}(0) \;\; \textit{for} \;\; \mathbf{g}^\sharp \tag{14.10}$$

and then define what will be called $\sharp$-derivative of $\mathcal{G}$ as follows. Here we are assuming $\zeta^{\sharp 0} \neq \emptyset$.

Definition 14.5. Let $z^{\mathbf{a}(\sharp)}$ be the Γ-maximal factor of $\mathbf{g}(\sharp) = \mathbf{g}^\sharp$ of Eq.(14.13) and let $\mathbf{g}(\sharp) = z^{-\mathbf{a}(\sharp)}\mathbf{g}^{(\sharp)}$. We have

$$z^{\mathbf{a}} \;\; \textit{divides} \;\; z^{\mathbf{a}(\sharp)} \;\; \textit{which divides} \;\; z^{\chi}$$

with reference to $\mathbf{a}$ of Eq.(6.1). We define the following $/^q$-exponent:

$$\begin{gathered} \mathcal{G}(\sharp) = (\mathbf{g}(\sharp) \,\|\, /^q) \\ \textit{with} \;\; \mathbf{a}(\sharp) \;=\; q\,\mathbf{b}(\sharp) + \mathbf{c}(\sharp) \\ \textit{so that} \;\; \mathbf{g}(\sharp) = z^{\mathbf{a}(\sharp)} g(\sharp) = z^{q\,\mathbf{b}(\sharp)} v(\sharp)^{\mathbf{c}(\sharp)} g(\sharp) \end{gathered} \tag{14.11}$$

which satisfy all the conditions to be a standard *abc-expression* in the sense of Def.(6.1). Namely

(1) $z^{\mathbf{a}(\sharp)}$ is the Γ-maximal factor, $z^{q\,\mathbf{b}(\sharp)}$ is $q\Gamma$-factor and $v(\sharp)^{\mathbf{c}(\sharp)}$ is $q\Gamma$-cofactor of $\mathcal{G}(\sharp)$ with a subsystem $v(\sharp)$ of z.
(2) $q > \mathbf{c}(\sharp)_j > 0$ for all j.
(3) $g(\sharp)$ is a residual factor of $\mathcal{G}(\sharp)$.

for which we should recall Th.(5.1), Rem.(5.1), Def.(5.1) and Def.(5.3). We then have

(1) Let $ord_\xi(g(\sharp)) = d(\sharp)$ and it is equal to $resord_\xi(\mathcal{G}(\sharp))$.
(2) We have $\mathfrak{d}^\sharp(\mathbf{g}(\sharp)) = \mathbf{g}(\sharp)$ and $\mathfrak{d}^{\sharp(\sharp)}(\mathbf{g} - \mathbf{g}(\sharp)) = 0$ thanks to Eq.(**??**). Indeed $\mathbf{g} - \mathbf{g}(\sharp) \in K(\mathfrak{d}^\sharp)$ in the sense of Th.(14.1).

With the $\sharp$-idempotent operator $\mathfrak{d}^\sharp$ of Rem.(14.1), the couple $(\mathcal{G}(\sharp), \mathfrak{d}^\sharp)$ will be called $\sharp$-*derivative* of $\mathcal{G}$. Sometime $\mathcal{G}(\sharp)$ alone is called the $\sharp$-*derivative* with respect to the $\sharp$-key parameters $\zeta(\sharp)$.

Theorem 14.4. *Let $(\mathcal{G}(\sharp), \mathfrak{d}^\sharp)$ be the $\sharp$-derivative of $\mathcal{G}$ with respect to the $\sharp$-key parameters $\zeta^\sharp$ of $\mathcal{G}$ at ξ in the sense of Def.(14.6). Then the same $\zeta^\sharp$ is also a system of $\sharp$-key parameters of $\mathcal{G}(\sharp)$ at ξ and the $\mathcal{G}(\sharp)$ is the $\sharp$-derivative of $\mathcal{G}(\sharp)$ itself with respect to the $\zeta^\sharp$. Conversely any system of $\sharp$-key parameters of $\mathcal{G}(\sharp)$ is also such a system of $\mathcal{G}$ although*

$\mathcal{G}(\sharp)$ may not be the $\sharp$-derivative of $\mathcal{G}$ with respect to the new $\sharp$-key parameters.

Proof. Immediate from Th.(14.2) with its proof. $\square$

Theorem 14.5. *We always have*

$$d(\sharp) = resord_\xi(\mathcal{G}(\sharp)) \leq resord_\xi(\mathcal{G}) = \mathbf{d} \tag{14.12}$$

for the $\sharp$-derivative $\mathcal{G}(\sharp)$ of $\mathcal{G}$ at ξ. The difference of the two is $|a(\sharp)-\mathbf{a}|$ in the sense of Eq.(14.14).

In the examples below we will follow the standard *abc*-expression in the sense of Eq.(6.1) with specified symbols.

Example 14.1. (Case: $\zeta^{\sharp(0)} \neq \emptyset$) Let us consider the case of $q = p = 2$ and $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ with $h = (\zeta_1 v_1 + \omega_1^2)v_1$ so that $z^{\mathbf{a}} = v^{\mathbf{c}} = v_1$ and $g = \zeta_1 v_1 + \omega_1^2$. In this case $\zeta^{\sharp(0)} = (\zeta_1)$ and $\mathcal{G}(0) = (\mathbf{g}(0) \,\|\, /^q)$ with $\mathbf{g}(0) = \zeta_1 v_1^2$, $z(0)^{a(0)} = z(0)^{q\,b(0)} = v_1^2$ and $v(0)^{c(0)} = 1$. Note that $in_\xi(\mathbf{g}) \neq in_\xi(\mathbf{g}(0))$.

Example 14.2. (Case:$\zeta^{\sharp(0)} = \emptyset$) Let us consider the case of $q = p = 2$ and $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ with $h = (z_1 v_1 + v_1^4 + \omega_1^3)v_1$ so that $z^{\mathbf{a}} = v^{\mathbf{c}} = v_1$ and $g = z_1 v_1 + v_1^4 + \omega_1^3$. Also $g^* = g$. In this case $\zeta^{\sharp(0)} = \emptyset$ and $\mathcal{G}(0) = \mathcal{G}$, while

$$\mathcal{G}(1) = (z^{a(1)} g(1) \,\|\, /^q) = (z^{q\,b(1)} v(1)^{c(1)} g(1) \,\|\, /^q)$$

where $z^{a(1)} = v_1^2$, $z^{q\,b(1)} = v_1^2$, $v(1)^{c(1)} = 1$ and $g(1) = z_1 + v_1^4$. Moreover

$$\mathcal{G}(2) = (z^{a(2)} g(2) \,\|\, /^q) = (z^{q\,b(1)} v(2)^{c(2)} g(2) \,\|\, /^q)$$

where $z^{a(2)} = z_1 v_1^2$, $z^{q\,b(2)} = v_1^2$, $v(2)^{c(2)} = z_1$ and $g(2) = 1$. Note that $in_\xi(\mathbf{g}) = in_\xi(\mathbf{g}(1)) = in_\xi(\mathbf{g}(2))$ while $h \neq \mathbf{g}(1) \neq \mathbf{g}(2)$. We have $\mathcal{G}(\sharp^+)$ is equal to all $\mathcal{G}(k), k \geq 2$, but it is different from both $\mathcal{G}$ and $\mathcal{G}(1)$.

Theorem 14.6. *Let us consider the case of $q = p$ for $\mathcal{G} = (\mathbf{g} \,\|\, /^p)$ with $\mathbf{g} = z^{\mathbf{a}} g$ where $z^{\mathbf{a}} = z^{q\mathbf{b}} v^{\mathbf{c}}$ according to Eq.(6.1). Assume that $ord_\xi(\mathcal{G}) = d > 1$. Let ζ be a $\sharp$-key paraameter of $\mathcal{G}$. Then there exists $\partial_0 \in Diff_{Z,\xi}^{(d-k)}$ and $\partial_1 \in Diff_{Z,\xi}^{(k)}$ with $0 < k \leq d$ such that*

(1) *$k \not\equiv 0 \mod p$ and $\partial_0(\mathbf{g}) = \zeta^k$*
(2) *$ord_\xi(\partial_1(\mathbf{g})) = d - k$ and $\partial_1\partial_0 = \partial_0\partial_1$*
(3) *$\partial_1\partial_0(\mathbf{g}) = \zeta^k \partial_1(\mathbf{g})$, say $= \mathbf{g}^\sharp(0)$*
(4) *$\partial_1\partial_0(\mathbf{g} - \mathbf{g}^\sharp) = 0$.*

Moreover it follows that $\mathfrak{S}(\mathcal{G}) \subset \mathfrak{S}(\mathcal{G}^\sharp)$ where $\mathcal{G}^\sharp = (\mathbf{g}^\sharp \,\|\, /p)$.

Lemma 14.7. *The definition of χ by Eq.(14.2) produces the same result when we replace $\mathbf{g}$ by $\mathbf{g}^\circ$ in the equation Eq.(14.2).*

Proof. Firstly $\zeta^\sharp$ is independent of z by the definition Def.(**??**) after Rem.(**??**). Hence the idempotent differential operator $\mathfrak{d}^\sharp$ of Def.(14.2) after Rem.(14.1) is identically zero in $\rho^e(R_\xi)[z]$. Hence $\zeta(\sharp)$ is the same before or after the application of $\mathfrak{d}^\sharp$ and so is $\mathfrak{d}^\sharp$ itself. Moreover the divisibility condition in the definition of χ by Eq.(14.2) implies that the initial form $in_\xi(\mathfrak{d}^\sharp h)$ is unchanged by the application of the differential operator $\mathcal{D}^{(\chi)}$ to $\mathfrak{d}^\sharp(\mathbf{g})$. $\square$

We now go back to $\mathbf{g}^\sharp$ defined by Eq.(14.1) of Def.(14.1). We first simplify the notation by writing

$$\mathbf{g}0 \quad \textit{for} \quad \mathbf{g}^\sharp \tag{14.13}$$

and then define what will be called $\sharp$-derivative of $\mathcal{G}$ as follows. Here we are assuming $\zeta^{\sharp 0} \neq \emptyset$.

Definition 14.6. Let $z^{\mathbf{a}(\sharp)}$ be the Γ-maximal factor of $\mathbf{g}(\sharp) = \mathbf{g}^\sharp$ of Eq.(14.13) and let $\mathbf{g}(\sharp) = z^{-\mathbf{a}(\sharp)}\mathbf{g}^{(\sharp)}$. We have

$$z^{\mathbf{a}} \ \ \textit{divides} \ \ z^{\mathbf{a}(\sharp)} \ \ \textit{which divides} \ \ z^{\chi}$$

with reference to $\mathbf{a}$ of Eq.(**??**). We define the following $/^q$-exponent:

$$\begin{aligned} \mathcal{G}(\sharp) &= (\mathbf{g}(\sharp) \parallel /^q) \\ \textit{with} \ \ \mathbf{a}(\sharp) \ &= \ q\,\mathbf{b}(\sharp) + \mathbf{c}(\sharp) \\ \textit{so that} \ \ \mathbf{g}(\sharp) &= z^{\mathbf{a}(\sharp)} g(\sharp) = z^{q\,\mathbf{b}(\sharp)} v(\sharp)^{\mathbf{c}(\sharp)} g(\sharp) \end{aligned} \tag{14.14}$$

which satisfy all the conditions to be a standard *abc-expression* in the sense of Def.(6.1). Namely

(1) $z^{\mathbf{a}(\sharp)}$ is the Γ-maximal factor, $z^{q\,\mathbf{b}(\sharp)}$ is $q\Gamma$-factor and $v(\sharp)^{\mathbf{c}(\sharp)}$ is $q\Gamma$-cofactor of $\mathcal{G}(\sharp)$ with a subsystem $v(\sharp)$ of z.
(2) $q > \mathbf{c}(\sharp)_j > 0$ for all j.
(3) $g(\sharp)$ is a residual factor of $\mathcal{G}(\sharp)$.

for which we should recall Th.(5.1), Rem.(5.1), Def.(5.1) and Def.(5.3). We then have

(1) Let $ord_\xi(g(\sharp)) = d(\sharp)$ and it is equal to $resord_\xi(\mathcal{G}(\sharp))$.
(2) We have $\mathfrak{d}^\sharp(\mathbf{g}(\sharp)) = \mathbf{g}(\sharp)$ and $\mathfrak{d}^{\sharp(\sharp)}(\mathbf{g} - \mathbf{g}(\sharp)) = 0$ thanks to Eq.(14.1). Indeed $\mathbf{g} - \mathbf{g}(\sharp) \in K(\mathfrak{d}^\sharp)$ in the sense of Th.(14.1).

With the $\sharp$-idempotent operator $\mathfrak{d}^\sharp$ of Rem.(14.1), the couple $(\mathcal{G}(\sharp), \mathfrak{d}^\sharp)$ will be called $\sharp$-*derivative* of $\mathcal{G}$. Sometime $\mathcal{G}(\sharp)$ alone is called the $\sharp$-*derivative* with respect to the $\sharp$-key parameters $\zeta(\sharp)$.

Theorem 14.8. *Let $(\mathcal{G}(\sharp), \mathfrak{d}^\sharp)$ be the $\sharp$-derivative of $\mathcal{G}$ with respect to the $\sharp$-key parameters $\zeta^\sharp$ of $\mathcal{G}$ at ξ in the sense of Def.(14.6). Then the same $\zeta^\sharp$ is also a system of $\sharp$-key parameters of $\mathcal{G}(\sharp)$ at ξ and the $\mathcal{G}(\sharp)$ is the $\sharp$-derivative of $\mathcal{G}(\sharp)$ itself with respect to the $\zeta^\sharp$. Conversely any* *system of $\sharp$-key parameters of $\mathcal{G}(\sharp)$ is also such a system of $\mathcal{G}$ although $\mathcal{G}(\sharp)$ may not be the $\sharp$-derivative of $\mathcal{G}$ with respect to the new $\sharp$-key parameters.*

Proof. Immediate from Th.(14.2) with its proof. $\square$

Theorem 14.9. *We always have*

$$d(\sharp) \;=\; resord_\xi(\mathcal{G}(\sharp)) \;\le\; resord_\xi(\mathcal{G}) \;=\; \mathbf{d} \tag{14.15}$$

for the $\sharp$-derivative $\mathcal{G}(\sharp)$ of $\mathcal{G}$ at ξ. The difference of the two is $|a(\sharp)-\mathbf{a}|$ in the sense of Eq.(14.14).

In the examples below we will follow the standard *abc*-expression in the sense of Eq.(6.1) with specified symbols.

Example 14.3. (Case: $\zeta^{\sharp 0} \neq \emptyset$) Let us consider the case of $q = p = 2$ and $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ with $h = (\zeta_1 v_1 + \omega_1^2)v_1$ so that $z^{\mathbf{a}} = v^{\mathbf{c}} = v_1$ and $g = \zeta_1 v_1 + \omega_1^2$. In this case $\zeta^{\sharp 0} = (\zeta_1)$ and $\mathcal{G}0 = (\mathbf{g}0 \,\|\, /^q)$ with $\mathbf{g}0 = \zeta_1 v_1^2$, $z0^{a0} = z0^{q\,b0} = v_1^2$ and $v0^{c0} = 1$. Note that $in_\xi(\mathbf{g}) \neq in_\xi(\mathbf{g}0)$.

Example 14.4. (Case:$\zeta^{\sharp 0} = \emptyset$) Let us consider the case of $q = p = 2$ and $\mathcal{G} = (\mathbf{g} \,\|\, /^q)$ with $h = (z_1 v_1 + v_1^4 + \omega_1^3)v_1$ so that $z^{\mathbf{a}} = v^{\mathbf{c}} = v_1$ and $g = z_1 v_1 + v_1^4 + \omega_1^3$. Also $g^* = g$. In this case $\zeta^{\sharp 0} = \emptyset$ and $\mathcal{G}0 = \mathcal{G}$, while

$$\mathcal{G}(1) = (z^{a(1)} g(1) \,\|\, /^q) = (z^{q\,b(1)} v(1)^{c(1)} g(1) \,\|\, /^q)$$

where $z^{a(1)} = v_1^2$, $z^{q\,b(1)} = v_1^2$, $v(1)^{c(1)} = 1$ and $g(1) = z_1 + v_1^4$. Moreover

$$\mathcal{G}(2) = (z^{a(2)} g(2) \,\|\, /^q) = (z^{q\,b(1)} v(2)^{c(2)} g(2) \,\|\, /^q)$$

where $z^{a(2)} = z_1 v_1^2$, $z^{q\,b(2)} = v_1^2$, $v(2)^{c(2)} = z_1$ and $g(2) = 1$. Note that $in_\xi(\mathbf{g}) = in_\xi(\mathbf{g}(1)) = in_\xi(\mathbf{g}(2))$ while $h \neq \mathbf{g}(1) \neq \mathbf{g}(2)$. We have $\mathcal{G}(\sharp^+)$ is equal to all $\mathcal{G}(k), k \geq 2$, but it is different from both $\mathcal{G}$ and $\mathcal{G}(1)$.

15. Moh's theory for $q = p$

In this section we are primarily interested in the case of $q = p$ and examine the special feature of metastable phenomena in the special case with $e = 1$ of $q = p^e$. However we first start with the assumption and notation for the case of general $q = p^e, e > 0$, before we specialize our interest to the case of $q = p$. We are given a closed point $\xi \in Sing(\mathcal{G})$.

$$\mathcal{G} = (\,\mathbf{g} \,\|\, /^q) \quad with \quad \mathbf{g} = z^{q\beta} v^{\gamma}\, h$$

where $z^{q\beta}$ is the q-factor, v^γ is the q-cofactor and h is a residual factor of $\mathcal{G}$. Also $z = (v, w)$ is the system of parameters defining those components of Γ passing through ξ, $0 < \gamma_i < q$ for every i and $d = ord_\xi(\mathbf{g}) = resord_\xi(\mathcal{G})$. Moreover h is cleaned by v^{γ^*} according to Rem.(13.3) with the supplement γ^* of γ in the sense of Def.(8.1). We write $h = h(d) + h^\sharp$ with $ord_\xi(\mathbf{g}^\sharp) > d$ according to Eq.(13.7).

We also have the transform $\mathcal{G}'$ of a given $/^q$-exponent by a permissible blowup $\pi : Z' \longrightarrow Z$ with center D.

Attention: In this section, we are not assuming $d > 0$ a priori. However if $d = 0$ then we must have $\gamma \neq 0$.

From now on we pick a closed point ξ' in $Sing(\mathcal{G}') \cap \pi^{-1}(\xi)$ and assume that ξ' is a metastable point of $\mathcal{G}'$ for the blowup π.

We will follow the notation of the earlier sections in regards to our selection of parameters according to Rem.(6.1). We have $x = (z, \omega)$, $z = (v, w)$, $w = (w^\dagger, w^\ddagger)$ and $\omega = (\omega^\dagger, \omega^\ddagger)$ so that $(v, w^\dagger, \omega^\dagger)$ generates the ideal of D at ξ. Our exceptional parameter at ξ' is chosen to be v_1 according to Eq.(13.2) of Rem.(13.2). We let $v = (v_1, \cdots, v_t)$ and let $c_{j-1} \in \mathbb{K}$ denote the value of $v_1^{-1} v_j$ at ξ' for every $j > 1$. We define variables $T_{j-1} = v_1^{-1} v_j - c_j$ and $T = (T_1, \cdots, T_\theta)$ with $\theta = t - 1$ in the manner of Eq.(13.10). Write $c = (c_1, \cdots, c_\theta)$. We choose σ by Eq.(13.4), and τ by Eq.(13.16). We write $\tau = \tau(1) + \tau(2)$ by Eq.(13.18). We choose variables $U = (v_1, U(1), U(2))$ and ϑ of Eq.(13.11) and Eq.(13.12). Thus $(v_1, T, U(1), U(2))$ of Eq.(13.13) is the regular system of parameters Eq.(13.9) of $R_{\xi'}$.

We then have the basic metastable inequality Eq.(13.17)

$$ord_{\xi'}\Big(v_1^{-d} h^\sharp \ + \ \big(v_1^{-d} h(d) - (c+T)^{\gamma^\flat} \tau^q\big)\Big) \ > \ d \tag{15.1}$$

with $\gamma^\flat$ is obtained from γ by deleting its first component. Let $\tau(*)$ denote the initial homogeneous part of $\tau(1)$ of Eq.(13.17) so that $\tau(*)^q$ is a homogeneous polynomial of degree $d - |\gamma^\flat| + k$ with an integer k such that $0 < k \leq |\gamma^\flat|$ by Th.(13.3).

For notational simplicity, let us write

$$\begin{aligned} A_{d+1} \ &= \ v_1^{-d} h^\sharp \ - \ (c+T)^{\gamma^\flat} \tau(2)^q \ \in \ (\, v_1, \, U(2)\,) R_{\xi'} \\ B_{d+1} \ &= \ \Big(\tau(1)^q (c+T)^{\gamma^\flat} - [\, \tau(1)^q (c+T)^{\gamma^\flat}\,]_{d+1}\Big) \end{aligned} \tag{15.2}$$

and

$$C_{d+1}(1) \ = \ \tau(*)^q \big\{\, (c+T)^{\gamma^\flat} \,\big\}_{|\gamma^\flat| - k + 1} \tag{15.3}$$

where $\{\ \}_a = [\ ]_a - [\ ]_{a-1}$ after the notation of Cor.(13.4), that is

$$\begin{aligned} C_{d+1}(1) &= \tau(*)^q\Big(\,[\,(c+T)^{\gamma^\flat}\,]_{|\gamma^\flat|-k+1} - [\,(c+T)^{\gamma^\flat}\,]_{|\gamma^\flat|-k}\Big) \\ & \textit{which is in } \ K[T, U(1)^q]. \\ C_{d+1}(2) &= [\,(\tau(1)^q - \tau(*)^q)(c+T)^{\gamma^\flat}\,]_{d+1} \\ &\in \rho^e(R_{\xi'})[\,(c+T)^{\gamma^\flat}\,]_{|\gamma^\flat|-k} \end{aligned}$$

We then rewrite the above Eq.(15.1) as

$$ord_{\xi'}\Big(\,v_1^{-d}h - (c+T)^{\gamma^\flat}\tau^q\,\Big) \ > \ d \tag{15.4}$$

$$\begin{aligned} & \textit{where} \\ & v_1^{-d}f - (c+T)^{\gamma^\flat}\tau^q \\ = & \ \big(\,A_{d+1} - B_{d+1} - C_{d+1}(2)\,\big) + C_{d+1}(1) \end{aligned}$$

in which

(1) $ord_{\xi'}(B_{d+1}) > d+1$
(2) $C_{d+1}(1)$ have no nonzero common monomial terms with any one of A_{d+1}, B_{d+1} and $C_{d+1}(2)$.
(3) $C_{d+1}(1)$ is homogeneous of degree $d+1$ in $K[T, U(1)]$ unless it is zero,
(4) $C_{d+1}(1) \in K[T, U(1)^q]$ and it is a partial sum of the power series expansion of

$$\big(A_{d+1} - B_{d+1} - C_{d+1}(2)\big) + C_{d+1}(1) \ \in \ K[[v_1, T, U(1), U(2)]]. \tag{15.5}$$

Definition 15.1. The polynomial $C_{d+1}(1) \in K[T, U(1)^q]$ of Eq.(15.3) will be called *resord-core* of the transform $\mathcal{G}'$ of $\mathcal{G}$ by π at the metastable point ξ' with respect to $x' = (v_1, T, U(1), U(2))$ which is a regular system of parameter of $R_{\xi'}$. It is *homogeneous polynomial of degree* $d+1$ *and a partial sum* in the power series expansion of Eq.(15.5).

We have the following special case of the theorem of T.T.Moh, [**?**], and we reprove it in the manner which we prefer for the purpose of the subsequent $/^q$-reduction theorems. In the following theorem we use a special agreement that if the residual is in $\rho^e(R_\xi)$ we conventionally say $resord(\mathcal{G}) = 0$.

Theorem 15.1. (T.T. Moh) *Let us consider the case of $q = p$. Then we have $resord_{\xi'}(\mathcal{G}') \le resord_\xi(\mathcal{G}) + 1$. The essence of this assertion is that if ξ' is a metastable point of the the transform $\mathcal{G}'$ of $\mathcal{G}$ by π then the* resord-core $C_{d+1}(1)$ *of Def.(15.1) is nonzero.*

Proof. In the case of $e = 1$ and $q = p$ the polynomial $(c+T)^{\gamma^\flat}$ has a special property that it has nonzero coefficients exactly to the following monomials:

$$\{\, T^\delta \,\big|\, 0 \le \delta_j \le \gamma^\flat_j < p \ \forall j,\ 1 \le j \le \theta \,\} \tag{15.6}$$

This is by the binomial theorem. Recall Eq.(15.3) which says

$$C_{d+1}(1) \;=\; \tau(*)^q \big\{\, (c+T)^{\gamma^\flat} \,\big\}_{|\gamma^\flat|-k+1}$$

where $d+1-deg(\tau(*)^q) = |\gamma^\flat| - k + 1$. We claim that

$$|\gamma^\flat| \ge d+1-deg(\tau(*)^q) \ge 1 \quad \textit{and} \quad C_{d+1}(1) \ne 0. \tag{15.7}$$

Thanks to Th.(13.6) we have either $d < |\gamma^\flat|$ or $d \ge |\gamma^*|$. We thus have to examine these two cases. Note that in any case we have $d \ge deg(\tau(*)^q) \ge 0$. First consider the case of $d < |\gamma^\flat|$ and then

$$|\gamma^\flat| \ge d+1 \ge d+1-deg(\tau(*)^q) \ge 1$$

As for $C_{d+1}(1) \ne 0$, the inequality $|\gamma^\flat| - k + 1 = d+1-deg(\tau(*)^q) \ge 1$ while $|\gamma^\flat| - k + 1 \le |\gamma^\flat|$ for $k \ge 1$. Therefore the factor of $C_{d+1}(1)$:

$$\big\{\, (c+T)^{\gamma^\flat} \,\big\}_{|\gamma^\flat|-k+1} \tag{15.8}$$

must be a nonzero nonconstant homogeneous. Hence $C_{d+1}(1) \ne 0$. Next consider the case of $d \ge |\gamma^*| = |\gamma^\flat| + \gamma_1 > |\gamma^\flat|$. We then have $deg(\tau(*)^q) > d - |\gamma^\flat|$ by Th.(13.3) so that $|\gamma^\flat| \ge d+1-deg(\tau(*)^q)$. Thus $|\gamma^\flat| \ge |\gamma^\flat| - k + 1$. Moreover $d+1-deg(\tau(*)^q) = (d - deg(\tau(*)^q)) + 1 \ge 1$. Thus Eq.(15.7) is proven. In both cases $C_{d+1}(1)$ has a factor which is nonzero nonconstant homogeneous. We have seen by Th.(13.3) that $C_{d+1}(1)$ is a nonzero partial sum of the power series expansion of $\big(A_{d+1} - B_{d+1} - C_{d+1}(2)\big) + C_{d+1}(1)$ in $K[[v_1, T, U(1), U(2)]]$. Hence $ord_{\xi'}\big(A_{d+1} - B_{d+1} - C_{d+1}(2) + C_{d+1}(1)\big) \le deg(C_{d+1}(1)) = d+1$ which proves the theorem of Moh. □

Remark 15.1. We want to pay special attention to the polynomial $C_{d+1}(1)$ called the *resord-core* defined by Def.(15.3). It is *nonzero homogeneous of degree* $d+1$ in $\mathbb{K}[x']$ with $x' = (v_1, T, U(1), U(2))$ according to Def.(15.1) . It is a partial sum of the expansion of

$$v_1^{-d} h - (c+T)^{\gamma^\flat} \tau^q \;=\; A_{d+1} - B_{d+1} - C_{d+1}(2) + C_{d+1}(1)$$

defined by Eq.(15.5). Recall $C_{d+1}(1)$ of Eq.(15.3) and define

$$\begin{aligned} T_{d+1} \;&=\; (v_1^{-|\gamma|} v^\gamma) C_{d+1}(1) \\ &=\; (v_1^{-|\gamma|} v^\gamma)\Big(\tau(*)^q \big\{\, (c+T)^{\gamma^\flat} \,\big\}_{|\gamma^\flat|-k+1}\Big) \end{aligned} \tag{15.9}$$

where it is important to note that

(1) $v_1^{-|\gamma|}v^\gamma$ is a unit in $R_{\xi'}$ because ξ' is metastable,
(2) $\tau(*)$ is nonzero homogeneous in $\mathbb{K}[v_1, T, U(1)]$
(3) $deg(\tau(*)^q) + (|\gamma^\flat| - k + 1) = d + 1$ which is $resord_{\xi'}(\mathcal{G}')$.
(4) $0 < |\gamma^\flat| - k + 1 \leq |\gamma^\flat|$

T_{d+1} will be called the *order-bounding polynomial* of the transform $\mathcal{G}'$ for π at the metastable point ξ'. We sometimes write $T_{\xi'}(\mathcal{G}')$ for the T_{d+1}.

Remark 15.2. We start from the situation immediately after a metastable singular point ξ' appeared according to the notation of the theorem of Moh, so that we have

$$resord_{\xi'}(\mathcal{G}') \ = \ d+1 \quad where \quad d \ = \ resord_{\xi}(\mathcal{G}). \tag{15.10}$$

We refer to the regular system of parameters $x' = (v_1, T, U(1), U(2))$ which are chosen according to Eq.(13.2), Eq.(13.3),Eq.(13.9), Eq.(13.10), Eq.(13.15), Eq.(13.11), Eq.(13.12), etc.

Remark 15.3. When a metastable point ξ' is created according to Th.(15.1) we have the cases of the inequalities of Eq.(15.7), in each of which we can choose a system of key q-parameters $T(*)$ for the transform $\mathcal{G}'$ at ξ'. The $T(*)$ is extracted from T_{d+1} of Eq.(15.9) as follows: Having always $|\gamma^\flat| \geq (d+1) - deg\, \tau(*)^p \geq 1$, we define and examine $T(*)$ in the following two cases separately.

$$\textit{The first case of} \ \ T(*) \ = \ Y(1) \ \ \textit{as follows:} \tag{15.11}$$

This is the case of $(d+1) - deg\,\tau(*)^p \ = |\gamma^\flat| - k + 1 = 1$. In this case we have $d = deg\,\tau(*)^p \ \equiv 0 \mod p$. The residual factor f of $\mathcal{G}$ have the same initial term as $v_1^d\tau(*)^p$ which is a p-th power. We then have

$$in_{\xi'}(\,T_{d+1}\,) \ = \ in_{\xi'}\Big(\tau(*)^p\{(c+T)^{\gamma^\flat}\}_1\Big)b$$

where b is the nonzero value taken by the unit $v_1^{-|\gamma|}v^\gamma$ at ξ'. Hence we can choose a system $T(*)$ of key q-parameters to be the singleton:

$$T(*) \ = \ Y(1) \ = \ \{\,\gamma_2^{-1}\sum_j \gamma_{j+1}T_j\,\} \tag{15.12}$$

This parameter is indeed a generator of

$$L_{p-max}(T_{d+1}) \ = \ L_{p-max}(C_{d+1(1)}) \ \Big(\subset \ L_{p-max}(\mathbf{g}')\Big)$$

where h' is a residual factor of $\mathcal{G}'$ at ξ'.

$$\textit{The second case of} \ \ T(*): \tag{15.13}$$

This is the rest of the cases in which

$$|\gamma^\flat| \geq (d+1) - deg\,\tau(*)^p \;= |\gamma^\flat| - k + 1 > 1$$

Let $\lambda = (d+1) - deg\,\tau(*)^p$ and look for $T(*) = Y(\lambda)$ depending upon the number λ. For this purpose we need:

Lemma 15.2. *We have $\mathbb{K} \ni c_j \neq 0$ for $\forall j$ and we let*

$$P_\lambda \;=\; \{\; (c+T)^{\gamma^\flat} \;\}_\lambda$$

which is the homogeneous part of degree λ of $(c+T)^{\gamma^\flat}$. We consider the case such that $|\gamma^\flat| \geq \lambda > 1$. Then there exists no proper $\mathbb{K}$-submodule L of $\sum_j \mathbb{K}T_j$ such that $P_\lambda \in \mathbb{K}[L]$.

Proof. Suppose we had $L \subsetneqq \sum_j \mathbb{K}T_j$ such that $P_\lambda \in \mathbb{K}[L]$ for any λ. We will derive a contradiction to this supposition. To do this we first consider the special case in which $\gamma_j^\flat = 1$, $\forall j$, and then we have the proof of the general case reduced to the special case. Thus for the special case under the above supposition we must have a nonzero derivation of the form

$$\partial \;=\; \sum_j \; a_j \frac{\partial}{\partial x_j} \quad \textit{with} \quad a_j \in \mathbb{K}$$

belonging to $Der_{\mathbb{K}[T]/\mathbb{K}}$ such that

$$\partial(P_\lambda) \;=\; 0 \quad \textit{identically} \tag{15.14}$$

For each index j we have

$$\begin{aligned} \left(\tfrac{\partial}{\partial x_j}\right) P_\lambda \;&=\; \left\{\; \left(\tfrac{\partial}{\partial x_j}\right)(c+T)^{\gamma^\flat} \;\right\}_{\lambda-1} \\ &=\; \left\{ \textstyle\prod_{i\neq j}(c_i+T_i)^{\gamma_i} \right\}_{\lambda-1} \end{aligned} \tag{15.15}$$

which is equal to

$$\sum_{\substack{\Lambda \,\subset\, [1,\theta] \,\setminus\, \{j\} \\ |\Lambda| \,=\, \lambda-1}} \qquad c(\Lambda) \prod_{i\in\Lambda} T_i \tag{15.16}$$

where $[1,\theta]$ is the set of indices j of T_j and we denote

$$c(\Lambda) \;=\; \prod_{i\in[1,\theta]\setminus\Lambda} \quad c_i$$

Here it should be noted that since we assumed $\gamma_j^\flat \;=\; 1, \forall j$, we have $\lambda \leq |\gamma^\flat| = \theta$ so that such Λ as above exist and every $|\Lambda| = \lambda - 1 \leq \theta - 1$.

We now proceed to prove that these quantities of Eq.(15.18) are $\mathbb{K}$-linearly independent and this independence will contradict Eq.(15.14). Let us consider the differential operator

$$\delta_{(\Lambda,k)} \;=\; \prod_{i\in\Lambda\setminus\{k\}} \frac{\partial}{\partial x_i} \tag{15.17}$$

for every pair (Λ, k) where $k \in \Lambda$ and $\Lambda \subset [1,\theta]\setminus\{j\}$ with $|\Lambda| = \lambda - 1$.

The result of Eq.(15.15 is equal to

$$\begin{aligned} &\gamma_j!\Big\{\Big(\tfrac{\partial}{\partial x_j}\Big)^{\lambda-1-\gamma_j}\textstyle\prod_{i\neq j}(c_i+T_i)^{\gamma_i}\Big\}_1 \\ &+\gamma_j!(c_j+T_j)\Big\{\Big(\tfrac{\partial}{\partial x_j}\Big)^{\lambda-2-\gamma_j}\textstyle\prod_{i\neq j}(c_i+T_i)^{\gamma_i}\Big\}_0 \end{aligned} \tag{15.18}$$

where it should be noted that $|\gamma^\flat| \geq \lambda \geq 1$ by the property (4) of Eq.(15.9).

We then must have a nonzero differential operator of the form

$$\partial \;=\; \sum_j a_j\Big(\frac{\partial}{\partial x_j}\Big)^{\lambda-1} \quad \textit{with} \;\; a_j\in\mathbb{K}$$

belonging to $Diff_{\mathbb{K}[T]/\mathbb{K}}$ such that

$$\partial(P_\lambda) \;=\; 0 \;\; \textit{identically} \tag{15.19}$$

For each index j we have

$$\begin{aligned} \Big(\tfrac{\partial}{\partial x_j}\Big)^{\lambda-1}P_\lambda \;&=\; \Big\{\;\Big(\tfrac{\partial}{\partial x_j}\Big)^{\lambda-1}(c+T)^{\gamma^\flat}\;\Big\}_1 \\ =\; &\textstyle\sum_{k=0}^{\lambda-1}\binom{\gamma_j}{k}(c_j+T_j)^{\gamma_j-k}\Big\{\Big(\tfrac{\partial}{\partial x_j}\Big)^{\lambda-1-k}\prod_{i\neq j}(c_i+T_i)^{\gamma_i}\Big\}_{1-\gamma_j+k} \\ =\; &\textstyle\sum_{k=\gamma_j-1}^{\gamma_j}\binom{\gamma_j}{k}(c_j+T_j)^{\gamma_j-k}\Big\{\Big(\tfrac{\partial}{\partial x_j}\Big)^{\lambda-1-k}\prod_{i\neq j}(c_i+T_i)^{\gamma_i}\Big\}_{1-\gamma_j+k} \end{aligned} \tag{15.20}$$

where $(c_j+T_j)^{\gamma_j-k} = 0$ when $\gamma_j - k < 0$ and $\Big(\tfrac{\partial}{\partial x_j}\Big)^{\lambda-1-k}$ is the zero operator when $\lambda - 1 - k$

We next want to prove that the polynomials of Eq.(15.21) for varying indices j are $\mathbb{K}$-linearly independent. If this is proven then the supposition Eq.(15.19) is impossible and the lemma is established. Let us now proceed to prove the linear independence.

$$\begin{aligned} &\gamma_j^{-1}c^{\gamma^\flat}c_j^{-\lambda}(-1)^{\lambda-1}T_j^{\lambda-1} \\ &+\;\textstyle\sum_{\substack{m+n=\lambda-1\\ m>0}}\{\;\gamma_j^{-1}(c+T)^{\gamma^\flat}\;\}_m\;c_j^{-1-n}(-1)^nT_j^n \end{aligned} \tag{15.21}$$

Therefore the derivative $\partial(P_\lambda)$ is clearly a nonzero polynomial because $a_j \neq 0$ for at least one j and $\lambda - 1 \geq 1$. This is contradictory. $\square$

Thus in the second case Eq.(15.13) we can choose $T(*) = Y(\lambda) = \sum_j \mathbb{K}T_j$ to be a system of key q-parameters for $\mathcal{G}'$ at ξ'.

Remark 15.4. Now for the sake of notational simplicity we drop *prime* from the symbols and write ξ for ξ', x for x' and so on. Let us then note that

$$\begin{aligned} \mathcal{G} \ &= \ (z^{q\beta}h \,\|\, /^p) \quad with \quad v^\gamma = 1 \ (v = \emptyset) \\ & with \ \ resord_\xi(\mathcal{G}) = ord_\xi(\mathbf{g}) = d+1 \end{aligned} \tag{15.22}$$

Moreover we can choose h such that

$$\begin{aligned} in_\xi(\mathbf{g}) \ &\in \ \mathbb{K}[L_{d+1}] \\ with \ \ L_{d+1} \ = \ L_{p-max}(\mathbf{g}) \ &\supset \ T_{d+1} \ \neq \ 0 \end{aligned} \tag{15.23}$$

in the sense of Def.(7.3) where T_{d+1} is defined by Eq.(15.9).

Remark 15.5. We are thus in the situation in which Th.(**??**) and Cor.(**??**) are applicable to the $\mathcal{G}$ of Eq.(15.22) where $resord_\xi(\mathcal{G}) = d+1$ in this case instead of d of Cor.(**??**). The role of the key q-parameters ζ of Cor.(**??**) is played here by a nonempty system extracted from T_{d+1} of Eq.(15.9). For instance, $\zeta = T(*)$ of Rem.(15.3). Therefore we are assured:

Theorem 15.3. *In the situation of Rem.(15.4) any finite sequence of fitted permissible blowups of $\mathcal{G}$ does not create any more metastable points for the transforms of $\mathcal{G}$ within the inverse images of ξ', until after the residual order drops from $d+1$ to $\leq d$.*

If the residual order drops $< d$ at any point of the transform, then we consider that our mission is accomplished by induction on d in virtue of Moh's theorem.

Theorem 15.4. *Asuume $q = p$. There are two cases of the metastable locus in Z' within a neiborhood of $\pi^{-1}(\xi)$ with respect to a fitted permissible blowup $\pi : Z' \to Z$,*

1. *When $in_\xi(g)$ is a p-power for some residual factor g of $\mathcal{G}$, the locus of metastable pointsis equal to the intersection of the exceptional divisor with the strict transform of the hyperplane $\sum_{1\leq i\leq t} \mathbf{c}_i v_i$*
2. *In the other cases,the locus of metastable pointsis equal to the intersection of all the strict transforms of $v_i = 0$ with the exceptional divisor.*

16. $/^p$-metastable produce

Theorem 16.1. *Consider a blowup $\pi : Z' \to Z$ with center D which is fitted permissible for $/^q$-exponent $\mathcal{G}$ in Z. If D is generic-down type for $\mathcal{G}$ at a closed point $\xi \in Sing(\mathcal{G})$ then there exists no closed point $\xi' \in \pi^{-1} \cap Sing(\mathcal{G}')$ with the transform $\mathcal{G}'$ of $\mathcal{G}$ by π.*

Theorem 16.2. *Consider a blowup $\pi : Z' \to Z$ with center D which is fitted permissible for $/^q$-exponent $\mathcal{G}$ in Z. Then the set of metastable points of $\mathcal{G}'$ ia closed in $Sing(\mathcal{G}')$.*

17. Weierstrass $/^q$-division

We are given a $/^q$-exponent $\mathcal{G} = (z^{\mathbf{a}}g \,\|\, /^q)$ with $z^{\mathbf{a}} = z^{q\mathbf{b}}v^{\mathbf{c}}$ and a regular system of parameters $x = (z, \omega), z = (v, w)$ at a closed point $\xi \in Sing(\mathcal{G})$ in the manner of the abc-expression of Eq.(6.1) of Def.(6.1). We set the following standard assumption.

$$(17.1) \qquad q = p^e > 1 \quad and \quad \mathbf{c} < \mathbf{d} = ord_\xi(g) = resord_\xi(\mathcal{G})$$

with $\mathbf{c}$ of the cofactor $v^{\mathbf{c}}$. The purpose of this section is to define and study the technique of WT(= Weierstrass Tchirnhausen) under the assumption of Eq.(17.1) combined with additional conditions on $\mathbf{d}$ mod p and v.

Remark 17.1. Consider a system of $\sharp$-key parameters $\zeta = (\zeta_1, \cdots, \zeta_s)$ in the sense of Eq.(9.2). It is said lndependent when the $in_\xi(\zeta_j)$ are κ_ξ-linearly independent modulo $in_\xi(v)\kappa_\xi$. We let

$$(17.2) \qquad A(\zeta, v) \;=\; \rho(R_\xi)[\zeta, v] + M_\xi^{\mathbf{d}+1} \quad and \quad A(\zeta) \;=\; A(\zeta, \emptyset)$$

and define "allowable" changes of ζ to mean that theie changes are induced by a $\mathbb{K}[v]$-automorphism of R_ξ.

Remark 17.2. Let ζ be any $\sharp$-key parameter of $\mathcal{G}$. Namely we have an operator $\partial \in Diff_{Z,\xi}^{(\mathbf{d}-r)}$ such that $\partial(g) \;=\; \zeta^r$ for some positive power r of p. If θ is an "allowable" parametric change of x in the sense of Rem.(17.1) then it naturally extend to tansformation of differential opeators. Let $\theta(\partial)$ be the transform of ∂. We then have $\theta(\partial)g = (\theta(\zeta))^r$ and hence $\theta(\zeta)$ is also a $\sharp$-key parameter of $\mathcal{G}$. In fact $\theta(x)$ is another regular system of parameters of R_ξ while v is unchanged,

From now on we assume that $\mathbb{K}$ is algebraically closed and hence infinite so that Th.(18.1) and its corollaries are applicable arbeit certain additional conditions.

Theorem 17.1. *We pick $N \gg 1$, for instance $p^N \geq \mathbf{d}$. Assume Eq.(17.1) and follow Rem.(17.1). Moreover we assume that $q = p$, $d \not\equiv$*

$0 \mod p$ and $v = \emptyset$. Assume that a maximal independent system of $\sharp$-key parameters has length $s \geq 1$ in the sense of Rem.(17.1). Refer to $g \in A(\zeta)$ of Rem.(17.2). Now consider a blowup $\pi : Z' \to Z$ with center D fitted permissible for $\mathcal{G}$. Pick a closed point $\xi' \in \pi^{-1}(\xi) \cap Sing(\mathcal{G}'$ with the transform $\mathcal{G}'$ of $\mathcal{G}$ by π. Then by a suitable allowable parametric change from x to (ζ, y) we can express residual g into the following forms:

(1) *If $s = 1$ and $\mathbf{d} \equiv 1 \mod p$ then there exists $\phi \in \rho(R_\xi)$ such that $ord_\xi(\phi) = \mathbf{d} - 1$ and $g = \zeta\phi$.*
(2) *Assume $s > 1$ and let $\zeta(2) = \zeta \setminus \zeta_1$. There then exists $\mathfrak{z} \in (\zeta(2), y)$ which is exceptional at ξ' and such that*

$$g \;=\; U^* Q(\zeta, y) \quad \textit{and} \quad g \;\equiv\; Q(\zeta, y) \mod M_\xi^{d+1} \tag{17.3}$$

$$\textit{where}$$

$$Q(\zeta, y) \;=\; \zeta_1^d \;+\; \textstyle\sum_{2 \leq j \leq d} Q_j(\zeta(2), y)\zeta_1^{d-j}$$

$$\textit{in which}$$

$$Q_j(\zeta(2), y) \in \mathbb{K}[\zeta(2), y^p] = \rho^N(R_\xi)[\zeta(2)] \cap \mathbb{K}[\zeta(2), y],$$

$$\textit{which is homogeneous of degree } j,\ \forall j, \quad \textit{and}$$

$$\mathfrak{z}^{-1}\zeta_k \in M_{\xi'} \quad \textit{if } 1 < k \textit{ and } z_k \neq \mathfrak{z}$$

Here U^* is a unit in R_ξ and $\mathfrak{z} \in \zeta(2) = \zeta \setminus \zeta_1$.

(3) *$\mathfrak{z}^{-1}\zeta_1$ can be either $\in M_{\xi'}$ or $\in R_\xi \setminus M_{\xi'}$.*
(4) *When $\mathfrak{z}^{-1}\zeta_1 \notin M_{\xi'}$ and $\mathfrak{z} \not\equiv \zeta_1 \mod M_\xi^2$, we have $ord_{\xi'}(\mathfrak{z}^{-d}g) \leq 1$. (Make use of the Tchirnausen expression in which we replace ζ_1 by $\zeta_1 - c\mathfrak{z}$ with a suitabble constant c.)*
(5) *If neither $\zeta(2)$ nor y contain any exceptional parameter at ξ' then we can choose $\zeta_1 = \mathfrak{z}$ and $\mathfrak{z}^{-d}g$ becomes a unit of $R_{\xi'}$.*

Proof. The proof is straight forward from Th.(17.1). As for the last claim on ζ_k all we need to do is to replace it by $\zeta_k - b_k\mathfrak{z}$ with a suitable $b_k \in \mathbb{K}$. $\square$

18. Weierstrass Tchirnhausen in $p > 0$

We will have the same $/^p$-exponent $\mathcal{G} = (z^{p\mathbf{b}} v^{\mathbf{c}} g \| /^p)$ with residual factor g. We will be mostly interested in the case of trivial cofactor $v^{\mathbf{c}} = 1$ with $v = \emptyset$. Recall that in this case every key parameter is $\sharp$-exact by Th.(9.2).

We pick any regular system of parameters of R_ξ, say $x = (x_1.\cdots, x_n)$. Let us pick and fix any integer $N \geq \mathbf{d} = ord_\xi(g)$ and let $R(N)$ denote $\rho^N(R_\xi)$. Note that R_ξ is a finite $R(N)$-module generated freely by $\{x^\alpha \big| \alpha \in \epsilon^n(p^N)\}$. We have $\mathbf{d} = ord_\xi(g) = resord_\xi(\mathcal{G})$.

Theorem 18.1. *Assume that $g - x_1^d \in (x_1^{d+1}, x(2))R_\xi$ with an integer $d > 1$ and $x(2) = (x_2, \cdots, x_n)$. Choose $p^N \geq d$. Then we claim that R_ξ is equal to*

$$\sum_{\substack{0 \leq a+bd < p^N,\, 0 \leq a < d \\ \beta \in \epsilon^{n-1}(p^N)}} x_1^a g^b\, x(2)^\beta R(N)$$

This $R(N)$-module, denoted by $\mathcal{Q}_N(g,d)$, is also an $R(N)[x(2)]$-module. We write $\mathcal{Q}_N(g)$ for $\mathcal{Q}_N(g,d)$ when $d = ord_\xi(d)$.

Proof. Let $I = (x_1^{d+1}, x(2))R_\xi$ and pick any one of the generators of R_ξ as $R(N)$-module, say $h = x_1^\alpha x(2)^\beta$ with $0 \leq \alpha < p^N$. We write $\alpha = ad + b$ where $a \geq 0$ and $d > b \geq 0$. Let S denote the localization of $\mathbb{K}[x_1]$ at $x_1 = 0$, that is

$$(\mathbb{K}[x_1] \setminus (x_1)\mathbb{K}[x_1])^{-1}\mathbb{K}[x_1]$$

We then define $\nu(f) \in S$ for every $f \in R_\xi$ by $f - \nu(f) \in x(2)R_\xi = I$. Note that $\nu(f)$ is well defined and that $\nu(g^a)$ is a unit multiple of x_1^{ad} in S. Now we replace x_1^{ad} by g^a in the given $h = x_1^\alpha x(2)^\beta$ in the following manner. Namely we observe

$$\begin{aligned} h - x_1^b x(2)^\beta g^a \ &= \ (x_1^b x(2)^\beta)(x_1^{ad} - g^a) \\ &= \ (x_1^b x(2)^\beta\, F_a(x_1^{ad}, g^a)(x_1^d - g) \\ \textit{where } x_1^d - g \ &\in \ (x_1^{d+1}, x(2))R_\xi \ = \ I \end{aligned} \tag{18.1}$$

where $F_a(X,Y) = (X-Y)^{-1}(X^a - Y^a)$. We apply Eq.(18.1) to every one of the generaters of the finite $R(N)$-module R_ξ. Since $x_1^b x(2)^\beta g^a \in \mathcal{Q}_N(g,d)$, we obtain $R_\xi \subset \mathcal{Q}_N(g,d) + I$. Each element of I is written as a R_ξ-linear combination of x_1^{d+1} and $x(2)$ and hence we obtain $R_\xi \subset \mathcal{Q}_N(g,d) + I^m$ for every integer $m > 0$. For $m \gg 1$ we have $max(R(N)) \subset I^m$ and hence we get

$$R_\xi \subset \mathcal{Q}_N(g,d) + max(R(N))R_\xi$$

which implies $R_\xi = \mathcal{Q}_N(g,d)$ by Nakayama's lemma. □

Corollary 18.2. *Let g and $x \supset x(2)$ be the same as in Th.(18.1). There exists an expression*

$$
\begin{aligned}
(18.2) \qquad g \ &= \ UP(x) \ \textit{with a unit } U \ \textit{and} \\
P(x) \ &= \ x_1^d \ + \ \textstyle\sum_{1 \leq j \leq d} P_j(x(2)) x_1^{d-j} \\
& \textit{where } \ P_j(x(2)) \in \rho^N(R_\xi)[x(2)], \ \forall j
\end{aligned}
$$

Proof. In terms of the generators of $\mathcal{Q}_N(g,d)$ we express x_1^d in $\mathcal{Q}_N(g,d)$ and divide the result into two parts, the first one is the sum of those terms which contain g and the second is $-\sum_{1 \leq j \leq d} P_j(x(2)) x_1^{d-j}$. The first one is a unit multiple of g because $g \in x_1^d + (x_1^{d+1}, x(2))R_\xi$. □

From now on we write $\mathbf{d} = ord_\xi(g) = resord_\xi(\mathcal{G})$.

Theorem 18.3. *Under the same assumption as in Th.(18.1) we moreover assume $\mathbf{d} \not\equiv 0 \mod p$. Then choose $x_1 = w_1 - d^{-1}P_1$ and the equality Eq.(18.2) turns into*

$$
\begin{aligned}
(18.3) \qquad g \ &= \ UQ(w_1, x(2)) \ \textit{with a unit } U \ \textit{and} \\
Q(w_1, x(2)) \ &= \ w_1^{\mathbf{d}} \ + \ \textstyle\sum_{2 \leq j \leq \mathbf{d}} Q_j(x(2)) w_1^{\mathbf{d}-j} \\
& \textit{where } \ Q_j(x(2)) \in \rho^N(R_\xi)[x(2)], \ \forall j
\end{aligned}
$$

Moreover consider a blowup $\pi : Z' \to Z$ with smooth irreducible center D with $ord_D(g) = ord_\xi(g)$. We then have $ord_{\xi'}(\mathfrak{z}^{-\mathbf{d}} g) < \mathbf{d}$ for every point $\xi' \in \pi^{-1}(\xi)$ where w_1 is an exceptional parameter for π at ξ'.

Proof. Easy from Cor.(18.2). □

Theorem 18.4. *Under the same assumption as in Th.(18.3) we choose any maximal system of $\sharp$-key parameters $\zeta = (\zeta_1, \cdots, \zeta_s)$ of $\mathcal{G}$ with $s \geq 1$. Assume that $v = \emptyset$ and $\zeta_i = x_i, \forall i \leq s$, (ignoring the usual assumption $z \subset x$), We can then have an expression*

$$
\begin{aligned}
(18.4) \qquad g \ &= \ \textstyle\sum_{1 \leq i \leq s} V(i) P(i) \ + \ V(s) R \ \ \textit{where} \\
& V(i) \ \ \textit{are units in } R_\xi \\
& w_i - \zeta_i \in \big(x(j), \textit{ all } j > i, w_k, \textit{ all } k < i \big) R_\xi \\
(18.5) \qquad P(i) \ &= \ w_1^{\mathbf{d}} \ + \ \textstyle\sum_{2 \leq j \leq \mathbf{d}} P(i)_j w_1^{\mathbf{d}-j} \\
& \textit{with } \ P(i)_j \in \rho^N(R_\xi) \ \ \textit{for all } \ (i,j)
\end{aligned}
$$

and

$$R = R(w, x(s+1)) \ \in \sum_{0 \leq \alpha_j < \mathbf{d}, \forall j} w^\alpha \rho^N(R_\xi)[x(s+1)]$$

having $ord_\xi(R) > \mathbf{d}$.

Proof. By suitable change of parameters $x_j, \forall j > 1$, in to $x_j + c_j x_1$ with suitable constants c_j, we may assume that the monomial term $x_1^{\mathbf{d}}$ appears in the residual g. Then we apply Th.(18.3 and obtain an expression of the form $g \;=\; UQ(w_1, x(2))$ with a unit U and $Q(w_1, x(2)) = w_1^{\mathbf{d}} + \sum_{2\le j\le \mathbf{d}} Q_j(x(2)) w_1^{\mathbf{d}-j}$ where $Q_j(x(2)) \in \rho^N(R_\xi)[x(2)]$. Then pick the $*$-ful idempotent operator $\partial \in Diff_{R(N)(x(2))}/R(N)$ with respect $x(2)$. Then let $P_j = (id - \partial)Q_j, \forall j \ge 2$, and $P = w_1^{\mathbf{d}} + \sum_{2\le j\le \mathbf{d}} Pj w_1^{\mathbf{d}-j}$. We next want to apply the same process to the remainder $g - P$ and repeat. Let us change names as $g = g(1)$, $U = U(1)$, $P_j = P(1)_j$, $P = P(1)$ and $g(2) = g(i) - P(1)$. T Thus inductively we obtain a sequence of $g(i)$, $U(i)$, $P(i)_j$ and $P(i)$ for every $i \le s$. We let $V(i) = \prod_{k\le i} U(k)$ and

$$g \;=\; V(s)R \;+\; \sum_{1\le i\le s} V(i)P(i) \quad \mathit{R{=}g(s){-}P(s)}$$

$\square$

Remark 18.1. Consider the elementary differential operater

$$\partial_{w_1}^{(\mathbf{d}-1)} \in Diff^{(\mathbf{d}-1)}_{R_\xi/\rho^N(R_\xi)[x(2)]} \quad \textit{with} \;\; \partial_{w_1}^{(\mathbf{d}-1)}(w_1^m) = \binom{m}{\mathbf{d}-1} w_1^{m-\mathbf{d}+1}$$

We then have

$$d^{-1}\Big(\partial_{w_1}^{(\mathbf{d}-1)} \circ U^{-1}\Big)(g) \;=\; w_1$$

which shows that w_1 is a $\sharp$-key parameter of $\mathcal{G} = (z^{p\mathbf{b}} g \| \,/^p)$.

Theorem 18.5. *Under the same assumption as in Th.(18.1) we assume* $p \le \mathbf{d} \equiv 0 \mod p$. *We can then choose a pair of key parametrs, say* (x_1, x_2), *which extends to a regular system of parameters of* R_ξ, *say* x, *such that there exists a nonzero monomial term of* g *having both* x_1 *and* x_2 *as its factors. Moreover there then exist key parameters* (w_1, w_2) *having the following properties.*

(1) $(w_1, x(2))$ *is a regular system of parameters of* R_ξ *where* $x(i) = (x_i, \cdots, x_n)$, $i = 1, 2$.
(2) $(w_1, w_2, x(3))$ *is a regular system of parameters of* R_ξ *and*
(3) g *can be put in the form*

$$g \;=\; U\Big(w_1 Q(w_2) + H(x)\Big) \tag{18.6}$$

$$\textit{with a unit } U \in R_\xi \textit{ and}$$

$$Q(w_2) \;=\; w_2^{\mathbf{d}-1} \;+\; \textstyle\sum_{2\le j\le \mathbf{d}-1} Q_j w_2^{\mathbf{d}-1-j}$$

$$\textit{with} \;\; Q_j \in \rho^N(R_\xi), \;\forall j, \;\textit{and}$$

$$H(x) \in \sum_{1\le j\le \mathbf{d}-1} \rho^N(R_\xi)[w_1, x(3)]w_2^{\mathbf{d}-1-j}$$

Proof. Since x_1 is a key parameter we can write

$$g = A(x)x_1 + F(x(2)) \text{ with } ord_\xi(A(x)) = \mathbf{d} - 1$$

with $F(x(2)) \in \rho^N(R_\xi)[x(2)]$. Since $\mathbf{d} - 1 \not\equiv 0 \mod p$ we must have at least one key parameter of $(A(x)\| /^p)$ whose initial is linearly independent of $in_\xi(x_1)$. Let us call this parameter x_2. Apply Cor.(18.2) and Th.(18.3) to $A(x)$ with respect to $x_2 \subset x$ so as to produce Weierstass Tchirnhausen polynomial in w_2, say $Q^*(w_2, x_1, x(3))$, together with a unit factor U following the manner of Th.(18.3). We then write $A(x) = UQ^*(w_2, x_1, x(3))$ with the unit $U \in R_\xi$. Then obtain $Q(w_2)$ of te teorem by dividing $Q^*(y_2, x_1, x(3))$ into a sum $Q(w_2) + h(x)$ with $Q(w_2) \in \rho^N(R_\xi)[w_2]$ and $h(x) \in \rho^N(R_\xi)[x(2)]$. Then apply Th.(18.1) to $Q(w_2) - w_2^{\mathbf{d}-1} \in (x_1, x(3))R_\xi$ and we obtain $\alpha(x) \in R_\xi$ such that

$$h(x) + U^{-1}F(x) \; - \; \alpha(x)Q(w_2) \; = \; H$$

is contained in

$$\sum_{2\le j\le \mathbf{d}-1} \rho^N(R_\xi)[w_1, x(3)]w_2^{\mathbf{d}-1-j}$$

Note that H of the theorem is thus defined. The theorem is proven by letting $w_1 = x_1 + \alpha(x)$. □

Remark 18.2. For g of Eq.(18.6) we have

$$\partial^{(1)}_{x_1} g \; = $$
$$V\Big(Q\big(w_2, x(3)\big)\Big) + U\Big(\partial^{(1)}_{x_1}\big(x_1^2 H(x)\big)\Big) + (\partial^{(1)}_{x_1} U)x_1^2 H(x)$$

where $V = U + x_1\partial^{(1)}_{x_1}U$ which is a unit in R_ξ and

$$\partial^{(1)}_{x_1} \in Diff^{(1)}_{R_\xi/\rho^N(R_\xi)[w_2,x(3)]}$$

Therefore we have

$$W^{-1}\Big((d-1)^{-1}\partial^{(\mathbf{d}-2)}_{w_2} \circ \partial^{(1)}_{x_1}\Big)g \; = \; y_2 \tag{18.7}$$

where $W = V + \partial^{(\mathbf{d}-2)}_{w_2}V$ which is a unit in R_ξ and

$$\partial^{(\mathbf{d}-2)}_{w_2} \in Diff^{(\mathbf{d}-2)}_{R_\xi/\rho^N(R_\xi)[x_1,x(3)]}$$

Eq.(18.7) shows that w_2 is a $\sharp$-key parameter of $\mathcal{G}$.