RESOLUTION OF SINGULARITIES (PART 1)

Part 1. Basics

Conventional Notation:
Our *ambient scheme*, often denoted by Z, is a *smooth irreducible* scheme of finite type over a base field $\mathbb{K}$. Our main interest lies in the case of a perfect $\mathbb{K}$ having characteristic $p > 0$. Ocasionally for technical reasons we consider specific cases of *imperfect* $\mathbb{K}$ as well. We have $dim(Z) = n \geq 1$, A point $\xi \in Z$ is usually a closed point, $\mathcal{O}_Z$ denotes the structure sheaf of Z and write $R = R_\xi = \mathcal{O}_{Z,\xi}$, $M = M_\xi = max(R)$, $\kappa = \kappa_\xi = R/M$ for short so long as the meaning is clear by context.

1. Differential operators in characteristic $p > O$

Let us first consider the case in which κ is separable algebraic over $\mathbb{K}$ and pick any regular system of parameters $x = (x_1, \cdots, x_n)$ of R. Then there exist a free base $\{\partial^{(a)} = \partial_x^{(a)}, a \in \mathbb{Z}_0^n\}$ of the R-module of differential operators $Diff_{Z,\xi} = Diff_{R/\mathbb{K}}$, uniquely determined by the following property.

$$\partial^{(\alpha)} x^\beta = \begin{cases} \binom{\beta}{\alpha} x^{\beta-\alpha} & \text{if } \beta \in \alpha + \mathbb{Z}_0^n \\ 0 & \text{if otherwise} \end{cases} \tag{1.1}$$

called “elementary” differential operators with respect to x.

Using a system of indeterminates $t = (t_1, \cdots, t_n)$ we have

$$\partial^{(\alpha)}(f)(x) = \text{ the coefficient of } t^\alpha \text{ in } f(x+t). \tag{1.2}$$

We pick $q = p^e, e \geq 0$, and let ρ denote the Frobenius p-th power so that $\rho^e(f) = f^q$.

Remark 1.1. We have $Diff_{R/\rho^e(R)} \subset Diff_{R/\mathbb{K}}$. Let us denote

$$\epsilon^n(q) \;=\; \{a \in \mathbb{Z}_0^n \,|\, 0 \leq a_j \leq q-1, \forall j\} \tag{1.3}$$

Then $\{\partial^{(a)}, a \in \epsilon^n(q)\}$ is a free base of R-module $Diff_{R/\rho^e(R)}$. It is dual to the free base $\{x^b \mid b \in \epsilon^n(q)\}$ of R as $\rho^e(R)$-module.

Let $Diff^{(m)}_{R_\xi/\mathbb{K}}$ (also $Diff^{(m)}_{Z,\xi}$) denote the R-submodule of $Diff_{Z,\xi}$ which consists of those differentil operators of orders $\leq m$.

(1.4) *Define* $Diff^{(m)*}_{R_\xi/\mathbb{K}} = \{ \partial \in Diff^{(m)}_{R_\xi/\mathbb{K}} \mid \partial(\mathbb{K}) = 0 \}$

We let $Diff^{*}_{R_\xi/\mathbb{K}} = Diff^{*}_{Z,\xi} = \bigcup_{all\ m\geq 0} Diff^{(m)*}_{R_\xi/\mathbb{K}}$.

2. IDEMPOTENT DIFFERENTIAL OPERATORS

Review on logarithmic differential calculus. Refer to the work of H.Kawanoue, [**?**]. We then extend it to those of "idempotent" and "primitive" operators which we introduce in this section and the next.

Consider a field extension $L = K(x)$ with $x = (x_1, \cdots, x_s)$ which is a *q-independent base* of L/K in the follwoing sense.

(1) For each $i, x_i^q = a_i \in K$, and
(2) every relation among x over K is generated by $X_i^q - a_i, 1 \leq i \leq n$, in the polynomial algebra $K[X]$.

Definition 2.1. Define the $\mathbb{Z}/p\mathbb{Z}$-module $Dlog_x(L/K)$ which is freely generated by $\{x^a\partial^{(a)} \in Diff_{L/K} \,|\, a \in \epsilon^s(q)\}$. They will be called *q-logarithmic differential operators* of L/K with respect to x.

Definition 2.2. We then define "idempotent differential operators" as folows.

$$\mathfrak{d}^{(a)} = \sum_{k\in\epsilon^s(q)\cap(a+\mathbb{Z}_0^s)} C_{ak}\ x^k\partial^{(k)} \tag{2.1}$$

where C_{ak} are chosen as follows: $C_{aa} = 1$ and for $b \neq a$

$$C_{ab} = \begin{cases} -\sum_{k\in a+\mathbb{Z}_0^s, b\in k+\mathbb{Z}_0^s, b\neq k} C_{ak}\binom{b}{k} & \textit{if } b \in (a+\mathbb{Z}_0^s), \neq a, \\ 0 & \textit{if otherwise.} \end{cases}$$

We then have that

$$\mathfrak{d}^{(a)}x^b = \begin{cases} x^b & \textit{if } b = a \\ 0 & \textit{if otherwise} \end{cases} \tag{2.2}$$

(1) $\mathfrak{d}^{(a)}$ is idempotent for every $a \in \epsilon^s(q)$, i.e., $\mathfrak{d}^{(a)}\mathfrak{d}^{(a)} = \mathfrak{d}^{(a)}$
(2) they are mutually independent, i.e., $\mathfrak{d}^{(a)}\mathfrak{d}^{(b)} = 0$ for all $a \neq b$
(3) and $\sum_{a\in\epsilon^s(q)} \mathfrak{d}^{(a)} = \mathbf{1}$, the identity operator in $Diff_{L/K}$.

Definition 2.3. We then define $\mathfrak{d}^* = \sum_{1\leq j\leq m} \mathfrak{d}_j$ and call it a **-full ID* for L/K.

Theorem 2.1. *With L/K Let $\mathfrak{d}^*(i), i = 1, 2,$ be a pair of the *-full ID operators. We then have $\mathfrak{d}^*(2)\mathfrak{d}^*(1) = \mathfrak{d}^*(2)$ and also $\mathfrak{d}^*(1)(h) - \mathfrak{d}^*(2)(h) \in K$ for every $h \in L$.*

It should be noted that those $\mathfrak{d}^*(i)$ may be the ones defined with respect to resular system of parameters at birationally correponding points of different birational models respectively. For instance think of one model and another obtained by a sequence of blowups.

Proof.

$$\mathfrak{d}^*(2)\mathfrak{d}^*(1) = \mathfrak{d}^*(2)(\mathbf{1} - \mathfrak{d}_{\mathbf{0}}(\mathbf{1})) = \mathfrak{d}^*(\mathbf{2})(\mathbf{1}) = \mathfrak{d}^*(\mathbf{2})$$

$$\mathfrak{d}^*(2)\Big(\mathfrak{d}^*(1)(h) - \mathfrak{d}^*(2)(h)\Big) = \mathfrak{d}^*(2)\mathfrak{d}^*(1)(h) - \mathfrak{d}^*(2)\mathfrak{d}^*(2)(h)$$

$$= \mathfrak{d}^*(2)(h) - \mathfrak{d}^*(2)(h) = 0. \;\; \textit{Hence} \;\; \mathfrak{d}^*(1)(h) - \mathfrak{d}^*(2)(h) \in K.$$

$$\mathfrak{d}^*(1)(h) \in (f^q)R \Longrightarrow \mathfrak{d}^*(1)(h) = f^q g \;\; \textit{with} \;\; g \in R \Longrightarrow$$

$$\mathfrak{d}^*(2)(h) = \mathfrak{d}^*(2)\mathfrak{d}^*(1)(h) = \mathfrak{d}^*(2)(f^q g) = f^q\mathfrak{d}^*(2)(g) \in (f^q)R.$$

□

3. PRIMITIVE AND NILPOTENT DIFFERENTIAL OPERATORS

Consider $L = K(u)$ with q-independent base u for L/K.

Definition 3.1. For each $a \in \epsilon^s(q)$ with the length s of u we define $\delta_u^{(a)} = u^{-a}\mathfrak{d}^{(a)}$. Here the division is done inside $Diff_{L/K}$.

Theorem 3.1. *With $L = K[u]$ we have the following equality.*

$$(\mathbb{Z}/p\mathbb{Z})[\; u, \{\partial^{(a)}\}\;] \;=\; (\mathbb{Z}/p\mathbb{Z})[\; u, \{\delta^{(a)}\}\;] \tag{3.1}$$

where the set $\{\ \}$ is for all indices $a \in \epsilon^s(q)$.

! !! !! !

Proof. The first algebra of Eq.(3.1) contains all the logarithmic $u^a\partial^a$ and hence all the idempotent $\mathfrak{d}^{(a)}$ by Def.(2.2). Since each $\mathfrak{d}^{(a)}$ is a $\mathbb{Z}/p\mathbb{Z}$-linear combination of those $u^b\partial^{(b)}$ with $b \in a + \mathbb{Z}$. Therefore $\mathfrak{d}^{(a)}$ is divisible by u^a in the first algebra of Eq.(3.1). We thus have the first algebra contain the second in Eq.(3.1). Conversely for each $a \in \epsilon^s(q)$ we have $\partial^{(a)} - \delta^{(a)}$ is a $(\mathbb{Z}/p\mathbb{Z})[u]$-linear combination of those $\partial^{(b)}$ with $b \in (\alpha + \mathbb{Z}_0^s) \setminus \{\alpha\}$ whence $|b| > |a|$. We then replace $\partial^{(b)}$ by $\delta^{(b)}$ wherever possible and repeat the replacements in an expressin of any given element of of the first algebra of Eq.(3.1). Such replacements cannot be repeated indeffinitely by the finiteness of $\epsilon^s(q)$. This proves the equality Eq.(3.1). □

Theorem 3.2. *Consider the following special case.*

(1) *L and K are the fields of regular local rings R and $S \subset R$,*
(2) *(u, w) is a regular system of parameters of R while (u^q, w) is that of S and $R = S[u]$ with q-independent u.*

We then claim that $P = \{\delta_u^{(a)}, \forall a \in \epsilon^s(q)\}$ has the following properties.

(1) *P is a free base of the R-module $Diff_{R/S}$ as well as that of L-module $Diff_{L/K}$.*
(2) *P is dual to the free base $\{\, u^a, a \in \epsilon^s(q)\,\}$ of L/K, i.e, for every $a \in \epsilon^s(q)$ and for $a' \in \epsilon^s(q)$ we have*

$$\delta_u^{(a)} u^{a'} = \begin{cases} 1 & \text{if } a = a' \\ 0 & \text{if otherwise} \end{cases} \tag{3.2}$$

! !! !! !

Proof. The first algebra of Eq.(3.1) contains all the logarithmic $u^a\partial^a$ and hence all the idempotent $\mathfrak{d}^{(a)}$ by Lem.(**??**). Since each $\mathfrak{d}^{(a)}$ is a $\mathbb{Z}/p\mathbb{Z}$-linear combination of those $u^b\partial^{(b)}$ with $b \in a + \mathbb{Z}$. Therefore $\mathfrak{d}^{(a)}$ is divisible by u^a in the first algebra of Eq.(3.1). We thus have the first algebra contain the second in Eq.(3.1). Conversely for each $a \in \epsilon^s(q)$ we have $\partial^{(a)} - \delta^{(a)}$ is a $(\mathbb{Z}/p\mathbb{Z})[u]$-linear combination of those $\partial^{(b)}$ with $b \in (\alpha + \mathbb{Z}_0^s) \setminus \{\alpha\}$ whence $|b| > |a|$. We then replace $\partial^{(b)}$ by $\delta^{(b)}$ wherever possible and repeat the replacements in an expressin of any given element of of the first algebra of Eq.(3.1). Such replacements cannot be repeated indeffinitely by the finiteness of $\epsilon^s(q)$. This proves the equality Eq.(3.1). □

Theorem 3.3. (1) *For $0 \in \epsilon^s(q)$ we have*

$$\delta_u^{(0)} \;=\; \mathfrak{d}_u^{(0)} \;=\; identity \;-\; \sum_{0 \neq a \in \epsilon^s(q)} u^a \delta^{(a)} \tag{3.3}$$

which is idempotent and $\in Hom_{\rho^e(R_\xi)[w]}(R_\xi, \rho^e(R_\xi)[w])$.

(2) *$\delta_u^{(0)}\delta_u^{(a)} = \delta_u^{(a)}$ for every $a \in \epsilon^s(q)$, and*

$$\delta_u^{(a)}\delta_u^{(0)} = \begin{cases} \delta_u^{(0)} & \text{if } a = 0 \\ 0 & \text{if } a \neq 0 \end{cases}$$

(3) *If $a \neq 0$ and $b \neq 0$ then $\delta_u^{(a)}\delta_u^{(b)} = 0$. For $a \neq 0$, $\delta_u^{(a)}$ is* square nilpotent.
(4) *$\sum_{a \in \epsilon^s(q)} \rho^e(R)[w]\delta^{(a)} = Hom_{\rho^e(R)[w]}\big(\ R,\ \rho^e(R)[w]\ \big)$*
(5) *$\partial = \sum_a \theta_a^q \delta^{(a)}$ is square-nilpotent if and only if $\partial \in Diff_Z^*$. This means $\theta_0 = 0$.*

Proof. The divisibility (1) is by Th.(3.1). The properties of (2) of the $\delta_u^{(a)}$ are straight forward from the corresponding ones of the $\mathfrak{d}_u^{(a)}$ by Eq.(2.2) and Eq.(2.2). The last assertion of the characterization of $\delta_u^{(a)}$ is by the assumption $\delta_u^{(a)} \in Diff_{L/K}$ and by Eq.(3.2). $\square$

Definition 3.2. Let us define:

$$\mathcal{P}^q(u/w) = \sum_{a \in \epsilon^s(q)} \rho^e(R)\delta^{(a)} = Hom_{\rho^e(R)[w]}\big(\ R,\ \rho^e(R)[w]\ \big)$$

and

$$\mathcal{P}^{*q}(u/w) = \sum_{0 \neq a \in \epsilon^s(q)} \rho^e(R)\delta^{(a)} = \mathcal{P}^q(u/w) \cap Diff_Z^*$$

Note that they depend only on w but not on u at all.

Remark 3.1. Consider the case of $w = \emptyset$ in Def.(3.2) whence we write $\mathcal{P}^{*q}$ for $\mathcal{P}^{*q}(u/w)$. We then have

$$\mathcal{P}^{*q} \ = \ \{\partial \in Hom_{\rho^e(R)}\big(R, \rho^e(R)\big) \mid \partial^2 = 0$$

However there could be many differential operators in $Diff^*_{R/\rho^e(R)}$ which are square-nilpotent but not in $\mathcal{P}^{*q}$. See below.

Example 3.1. Let $u = (u_1, u_2, u_3)$ and $q = 2$. Then $\partial = u_1\delta_u^{(0,1,0))} + \delta_u^{(0,0,1))}$ is square nilpotent but not in $\mathcal{P}^2$.

4. Ideal Exponents

Readers may refer to [?], [?] and [?].

An *idealistic exponent* is a pair (J, b) where

(1) J is an ideal given in the *ambient scheme* Z
(2) and b is a positive integer.
   (a) We sometimes need ambient extensions from Z to
   $$Z[t] \;=\; Z \times_{\mathbb{K}} Spec(\mathbb{K}[t])$$
   with a finite number of additional variables t.

   (b) Define the *order* and *singular locus* by

$$ord_\xi(J, b) \;=\; b^{-1} ord_\xi(J) \quad and \tag{4.1}$$
$$Sing(J, b) \;=\; \{\ \xi \in Z \mid ord_\xi(J, b) \geq 1\ \}.$$

Definition 4.1. A blow-up $\pi : Z' \to Z$ with center D is said *permissible* for $E = (J, b)$ if D is regular (smooth if $\mathbb{K}$ is perfect) irreducible and $\subset Sing(E)$.

Definition 4.2. The *transform* of $E = (J, b)$ by π is $E' = (J', b)$ with $J' = (I(D, Z)\mathcal{O}_{Z'})^{-b} J\mathcal{O}_{Z'}$ where $I(D, Z)$ denotes the ideal sheaf defining $D \subset Z$.

In other words the b-times exceptional divisor is *removed* from the total transform. Note that $I(D, Z)\mathcal{O}_{Z'}$ is invertible as $\mathcal{O}_{Z'}$-module.

5. INFINITELY NEAR SINGULARITIES

In this section we consider an arbitrary base field $\mathbb{K}$.

Definition 5.1. An LSB over Z is defined to mean a diagram of the following form.

$$
\begin{array}{ccccc}
 & \pi_{r-1} & & & \pi_{r-2} \\
Z_r & \rightarrow & U_{r-1} \subset Z_{r-1} & & \rightarrow \\
 & & \bigcup & & \\
 & & D_{r-1} & & \\
 & & & \cdots &
\end{array}
$$

$$
\begin{array}{ccccc}
\pi_1 & & \pi_0 & \\
\rightarrow & U_1 \subset Z_1 & \rightarrow & U_0 \subset Z_0 = Z \\
 & \bigcup & & \bigcup \\
 & D_1 & & D_0
\end{array}
$$

where $U_i \subset Z_i$ is open, D_i is a "regular" irreducible closed in U_i and the $\pi_i : Z_{i+1} \rightarrow U_i$ is the blow-up with center D_i.

Any blowup with empty center is the identity morphism and such trivial ones are also included in the definition of LSB for technical reasons (in particular for locaization of blowups).

Definition 5.2. We define the t-indexed disjoint union:

$$
\mathfrak{S}(E) \; = \; \bigcup_t \{ \textit{ the LSBs over } Z[t] \textit{ permissible for } E[t] = (J[t], b) \; \} \tag{5.1}
$$

which is the totality of the *infinitely near singular points* of E in Z, with arbitrary finite systems t of indeterminates. Say "E_2 is *more singular* than E_1" if $\mathfrak{S}(E_2) \supset \mathfrak{S}(E)$, and define the *equivalence relation* by

$$
E_1 \sim E_2 \Longleftrightarrow \mathfrak{S}(E_1) = \mathfrak{S}(E_2) \tag{5.2}
$$

Remark 5.1. It is essencial to make use of t in the definition and consequent theorems. For instance consider an analytically irreducible plane curve singularity with $\mathbb{K} = \mathbb{C}$ for simplicity, with t we can deduce the first characteristic exponent, while without t we get nothing more than the integer part of the rational number. More generally the use of t is crucially important for the validity of Th.(6.3) below.

Remark 5.2. In general the use of any t is consequential in effect of that of any larger system. Moreover we may use any smooth $\mathbb{K}$-schemes instead of those $Spec(\mathbb{K}[t])$ and the resulting theorems remain equivalent (thanks to the Ambient Reduction of Th.(6.2) below).

Definition 5.3. When the base field $\mathbb{K}$ is arbitrary, we may take its algebraic closure $\tilde{\mathbb{K}}$ and the base field extension by $\mathbb{K} \to \tilde{\mathbb{K}}$.

$$\tilde{Z} = Z \times_{\mathbb{K}} \tilde{\mathbb{K}} \quad \textit{and} \quad \tilde{E} = E \times_{\mathbb{K}} \tilde{\mathbb{K}} \tag{5.3}$$

We will let σ denote the projection $\tilde{Z} \to Z$ so that $\tilde{E} = \sigma^{-1}(E)$, the pullback of $E = (J, b)$ by σ that is $(J\mathcal{O}_{\tilde{Z}}, b)$ on $\tilde{Z}$. Then we have $\mathfrak{S}(\tilde{E})$ which will be also written as $\tilde{\mathfrak{S}}(E)$.

6. Three basic technical theorems

Recall what we called the *Three Key Theorems* which were proven in [?] and [?].

Theorem 6.1. *(Differentiation theorem)*
For every $\mathcal{O}_Z$-submodule $\mathcal{D}$ of $Diff_Z^{(i)}$, we have

$$\mathfrak{S}(Diff_Z^{(i)} J, b-i) \ \supset \ \mathfrak{S}(J,b)$$

Theorem 6.2. *(Ambient Reduction Theorem)*
Given an ideal exponent $E = (J, b)$ in Z, we let

$$J^\sharp \ = \ \sum_{j=0}^{b-1} \Big(Diff_Z^{(j)} J \Big)^{\frac{b^\sharp}{b-j}} \quad with \quad b^\sharp = b!.$$

For any smooth subscheme $W \subset Z$, we let $F = (J^\sharp \mathcal{O}_W, b^\sharp)$. Then F is an ambient reduction *of E from Z to W in the following sense (definition):*

> *Pick any t and any LSB over $Z[t]$, such that all of its centers are in the strict transforms of $W[t]$. Then we have $LSB \in \mathfrak{S}(E)$ if and only if the LSB induces to W the one belonging to $\mathfrak{S}(F)$.*

Theorem 6.3. *(Numerical Exponent Theorem)*
Let $E_i = (J_i, b_i), i = 1, 2,$ be two ideal exponents in Z. If $\mathfrak{S}(E_1) = \mathfrak{S}(E_2)$ then $ord_\xi(J_1)/b_1 = ord_\xi(J_2)/b_2$ for every $\xi \in Z$ where any one of the two is ≥ 1.

7. The Characteristic Algebra

We are primarily interested in the case of a "perfect" base field $\mathbb{K}$. An important point of the "perfect" case is the the geometric definition coincides with the algebraic one for the characterisitic algebra. They do not in general. The geometric and the algebraic have different charaters with respect to base field extenstions.

If $\mathbb{K}$ is imperfect we then take the algebraic closure $\tilde{\mathbb{K}}$ of $\mathbb{K}$ and the base field extention from $\mathbb{K}$ to $\tilde{\mathbb{K}}$. We then have $\tilde{Z} = Z \times_{\mathbb{K}} \tilde{\mathbb{K}}$, projection morphism $\sigma : \tilde{Z} \to Z$, $\tilde{E} = E \times_{\mathbb{K}} \tilde{\mathbb{K}}$ and we let $\tilde{\mathfrak{S}}(E) = \mathfrak{S}(\tilde{E})$ compared with $\mathfrak{S}(E)$. We examine the "inseparable descent" with respect to σ.

Definition 7.1. The *"geometric " characteristic algebra* of $E = (J, b)$ is defined to be the following graded $\mathcal{O}_{\mathcal{Z}}$-algebra.

$$\wp_{geo}(E) \;=\; \sum_{a=0}^{\infty} J_{max}(a) T^a \tag{7.1}$$

where T is a dummy variable to indicate homogeneous degrees and

$$J_{max}(a) \;=\; \bigcup \{I \mid \mathfrak{S}(I, a) \supset \mathfrak{S}(J, b)\} \tag{7.2}$$

Definition 7.2. Given an ideal exponent $E = (J, b)$ we define its *saturation* to mean $\hat{E} = (J_{max}(b), b)$ with the same b where $J_{max}(b)$ of Eq.(7.2) which is the homogeneous part of degree b of $\wp(E)$.

Definition 7.3. The *"algebraic " characteristic algebra* $\wp_{alg}(E)$ of $E = (J, b)$ is defined to be the integral closure of the following subalgebra.

$$\mathcal{O}_Z[J^\sharp T^{b^\sharp}] = \sum_{\alpha=0}^{\infty} (J^\sharp)^\alpha \, T^{b^\sharp \alpha} \subset \sum_{\beta=0}^{\infty} \mathcal{O}_Z \, T^\beta = \mathcal{O}_Z[T] \tag{7.3}$$

where $b^\sharp = b!$ and

$$J^\sharp = \sum_{0 \le \mu \le b-1} \left(Diff^{(\mu)}_{Z/\mathbb{K}} J \right)^{b^\sharp/(b-\mu)}$$

Thus $\wp(E)$ is clearly finitely presented as a graded $\mathcal{O}_Z$-algebra with globally coherent homogeneous parts.

Theorem 7.1. *We always have $\wp_{alg}(E) \supset \wp_{geo}(E)$ If the base field $\mathbb{K}$ is perfect then we have $\wp_{geo}(E) = \wp_{alg}(E)$. When we have the equality we will denote it simply by $\wp(E)$.*

This last assertion of the theorem asserts that the *algebraic* condition Eq.(7.3) of is equivalent to the *geometric* one Eq.(7.1). This has been proven in my earlier paper [**?**]. For the detail of the proof of the *algebraic* characterization Eq.(7.3) of $\wp(E)$, the reader should refer to the proofs of Lemmas 2.1 - 2.2 and the equality ($\flat$) of page 918 of the paper [?]. They are given in the proof of the *Main Theorem* of [?] asserting the finite presentation of $\wp(E)$.

Theorem 7.2. *The graded $\mathcal{O}_Z$-algebra $\wp(E) = \sum_a J_{max}(a)$ for $E = (J, b)$ is the smallest $\mathcal{O}_Z$-subalgebra of $\mathcal{O}_Z[T]$ such that*

(1) $J \ \subset \ J_{max}(b)$
(2) $Diff_Z^{(\mu)} J_{max}(a) \ \subset \ J_{max}(a-\mu)$ for all $0 \leq \mu < a$ and
(3) $\wp(E)$ is integrally closed in $\mathcal{O}_Z[T]$.

For a proof of the second property above, we may use Differentiation theorem Th.(6.1) applied to $J_{max}(a)$ of Eq.(7.2) and Th.(7.1), together with the following lemma.

Lemma 7.3. *For every $a = \sum_{i=0}^{b-1}(b-i)\alpha_i$ with $\alpha \in \mathbb{Z}_0^b$ and for every $\mu < a$,*

$$Diff_Z^{(\mu)}\Big(\prod_{i=0}^{b-1}(Diff_Z^{(i)}J\ )^{\alpha_i}\Big) \ \subset \sum_{\substack{\beta\in\mathbb{Z}_0^b \\ \sum_{i=0}^{b-1}\beta_i(b-i)=a-\mu}} \Big(\prod_{i=0}^{b-1}(Diff_Z^{(i)}J\ )^{\beta_i}\Big)$$

For its proof once again we refer to Remarks (2.1)-(2.2) of [?].

Remark 7.1. For comparison we first recall the case of characteristic zero, for instance $\mathbb{K} = \mathbb{C}$. Consider a plane curve defined by

$$f(x, y) = \sum_{ij} c_{ij}x^iy^j \quad with \quad c_{ij} \in \mathbb{K}$$

such that its multiplicity is $m = ord_{(0,0)}(f)$ and its *first characteristic exponent* is $n/m = \delta = min\{\, i/(m-j) \mid j < m, c_{ij} \neq 0 \,\}$.

Now for $E = (f\mathbb{K}[x, y], m)$, we can prove that $\wp(E) = \sum_{l=0}^{\infty} J_{max}(l)T^a$ is determined by δ within a neighborhood of $\xi \in Z$ as follows:

$$J_{max}(l) \ = \ \{\ x^iy^j \mid \frac{i}{\delta} + j \geq l, i \geq 0, j \geq 0\ \}\mathbb{K}[x, y], \quad \forall l \geq 0.$$

As is seen below, the above assertion fails to be true in general when $char(\mathbb{K}) = p > 0$.

Next, let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. Consider a plane curve defined by $f = y^q - x^n$ with $q = p^e, e > 0$, and $n > q, (n, p) = 1$. Then we have a " 3 "-dimensional Newton polygon, so to speak, in the sense that

$$J_{max}(l) \ = $$
$$\{ \ x^i y^j f^k \mid i \geq 0, j \geq 0, k \geq 0, \ ineq(l) \ \}\mathbb{K}[x,y], \ \forall l \geq 0$$
$$where \ \ ineq(l) \ \ means$$
$$i\tfrac{q-1}{n-1} + j\tfrac{n(q-1)}{q(n-1)} + kq \geq l.$$

8. Comments on the imperfect base field

Consider the case of Z "smooth" over $\mathbb{K}$ which is "imperfect".

We then use the algebraic closure $\tilde{\mathbb{K}}$ of $\mathbb{K}$ after Def.(5.3) and Eq.(5.3) with $\sigma : \tilde{Z} \to Z$, $\tilde{E}$, $\tilde{\mathfrak{S}}(E)$, $\tilde{\wp}(E) = \wp(\tilde{E})$,"geometric" and "algebraic". "Geometrically" $\wp_{geo}(\tilde{E})$ is ' more effective than $\wp_{geo}(E)$.

Theorem 8.1. *In general, including the cases of imperfect* $\mathbb{K}$*,* $\wp_{alg}(E)$ *is equal to the "inseparable descent" of* $\wp_{alg}(\tilde{E})$ *from* $\tilde{\mathbb{K}}$ *to* $\mathbb{K}$ *in the sense of Def.(8.1) below, while* $\wp_{geo}(E)$ *contains the inseparable descent of* $\wp_{geo}(\tilde{E})$ *but not equal in general.*

Definition 8.1. We define the "naive" *inseparable descent* of a $\mathcal{O}_{\tilde{Z}}$-module $\tilde{A}$ by σ from $\tilde{\mathbb{K}}$ to $\mathbb{K}$. This "descent" is defined as follows:

(1) Choose and fix a free base of $\tilde{\mathbb{K}}$ as $\mathbb{K}$-module including 1:

$$\Big\{c_i, i \in \{1, C\}\Big\} \quad \textit{where} \;\; C \subset \tilde{\mathbb{K}} \setminus \mathbb{K} \tag{8.1}$$

(2) Every element $f \in \tilde{A}$ is uniquely written as $f_1 + \sum_{i \in C} b_i f_i$ with $b_i \in \mathcal{O}_Z$ where σ_* denotes the direct image of $\tilde{A}$ by σ.
(3) Then the "descent" of $\tilde{A}$ with respect to the chosen Eq.(8.1) to be the collection of f_1 for all $f \in \tilde{A}$.

In general the "naive" descent depends upon the choice of Eq.(8.1). When it is independent of, we call it the *inseparable descent* of $\tilde{A}$.

Theorem 8.2. *The "descent" defined by Def.(8.1) for* $\wp_{alg}(\tilde{E})$ *is independent of the choice of Eq.(8.1) and it is equal to* $\wp_{alg}(E)$*, which is the finitely presented graded* $\mathcal{O}_Z$*-algebra having the "algebrac" characterization Eq.(7.3) of Th.(7.1).*

The proof is by $Diff_{\tilde{Z}} = Diff_Z \otimes_{\mathcal{O}_Z} \mathcal{O}_{\tilde{Z}}$ and by the descent of integral closure.

Now back to the perfect $\mathbb{K}$ and examine the changes of $\wp$ with respect to localizaions at non-closed points, such as generic points of singular locus of E.

Pick a system of parameters $t = (t_1, \cdots, t_d)$ with $t_j \in \mathcal{O}_{Z,\xi}, \forall j$, such that

(1) the $t_j, 1 \le j \le d$, are algebraically independent over $\mathbb{K}$ and t is extendable to a system of "separating transcendental base of the function field $\mathbb{K}(Z)$.
(2) Let ∇ denote the multiplicative group of nonzero elements of $\mathbb{K}[t]$, and apply the localization ∇^{-1} to Z, E and $\wp(E)$.

Example 8.1. Let $D_i, 1 \leq i \leq s$, be the reduced irreducible components of $Sing(E)$ having $dim(D_j) = dim(Sing(E))$. Then pick $t_j \in \cap_{1 \leq j \leq s} \mathcal{O}_{Z,\zeta_j}$ for everry j in such a way that t induces a separating transcendental base of the function field of D_i over $\mathbb{K}$ for every i. Then t has the properties (1) and (2) as above.

Theorem 8.3. *Consider $\nabla^{-1}E$ as an ideal exponent in $\nabla^{-1}Z$ which is a smooth scheme over the new base field $\mathbb{K}(t)$. We then claim that*

(1) *$\wp_{geo}(\nabla^{-1}E)$ is equal to $\nabla^{-1}\wp_{geo}(E)$, while*
(2) *$\wp_{alg}(\nabla^{-1}E)$ is equal to the "inseparable descent" of $\wp_{alg}(\widetilde{\nabla^{-1}E})$ where the $\widetilde{\ }$ denote the base field extension from $\mathbb{K}(t)$ to its algebraic closure $\widetilde{\mathbb{K}(t)}$.*

The first claim is due to the fact that

$$\nabla^{-1}(\mathfrak{S}(E)) \;=\; \mathfrak{S}(\nabla^{-1}E) \tag{8.2}$$

where $\mathfrak{S}$ denotes the totality of infinitely near singularities in the sense of Def.(5.2). The second claim is a special case of Th.(8.2).

Theorem 8.4. *Let us consider the case of an arbitrary base field $\mathbb{K}$. Even then $\wp_{geo}(E)$ is finitely presented as a graded $\mathcal{O}$-algebra, while it could be bigger than $\wp_{alg}(E)$ which is obviously finitely presented.*

Since E is written by a finite number of polynomials in a finite number of variables, we may assume that the base field is finitely generated field over a prime field. In other words we may assume that $\mathbb{K}$ is finitely generated over a perfect field K, say $\mathbb{K} = K(u_1, \cdots, u_t)$. We may even assume $Z[u]$ is smooth by replacing $Spec(K[u])$ by a sufficiently small but dense affine open subset. Then Thm.(8.4) follows by virtue of Th.(7.1) and Th.(8.3) with $\nabla = K[u] \setminus \{0\}$.

9. Edge Decompositions

For a regular system of parameters $x = (x_1, \cdots, x_n)$ of R_ξ, let $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)$ with $\bar{x}_i = in_\xi(x_i), 1 \le i \le n$. We have

$$gr_\xi(R_\xi) = \bigoplus_{d \ge 0} M_\xi^d / M_\xi^{d+1} = \kappa_\xi[\bar{x}_1, \cdots, \bar{x}_n]$$

This section along with the earlier ones on *infinitely near singularities* and *characteristic algebra* are essencially same with what have been presented at the conference June 2006 in Trieste, Italy, [**?**].

Theorem 9.1. *(Edge Generators Theorem)*
We can find

(1) a regular system of parameters $x = (y, z)$ of R_ξ where $y = (y_1, \cdots, y_r)$ with $0 < r \le n$,
(2) a sequence of powers of p: $q_i = p^{e_i}, 0 \le e_1 \le \cdots \le e_r$,
(3) $g_i = y_i^{q_i} + \epsilon_i \in J_{max}(q_i)_\xi$ with $ord_\xi(\epsilon_i) > q_i$

such that for every $a \ge 0$

$$J_{max}(a)_\xi \quad \subset \quad M^{a+1} \; + \sum_{\substack{\beta \in \mathbb{Z}_0^r \\ a = \sum_{j=1}^r q_j \beta_j}} \Big(\prod_{j=1}^r g_j^{\beta_j} \Big) R_\xi. \tag{9.1}$$

Proof. Observe that $\wp(E) \subset bl_\xi(R_\xi)$ by Th.(7.2) and Lem.(**??**). Then we select a minimal system of generators of the forms $\bar{g}_i = \bar{y}_i^{q_i}, 1 \le i \le r$, for the natural homomorphic image of $\wp(E)$:

$$\bar{\wp}(E) \;=\; Im\Big(\wp(E) \to g_\xi(R_\xi) = bl_\xi(R_\xi)/M_\xi \, bl_\xi(R_\xi) \Big) \tag{9.2}$$

where $\bar{y}_i$'s are chosen according to Lem.(**??**) and $bl_\xi(R_\xi)$ is the blowup algebra of Eq.(**??**). □

Remark 9.1. If there happens to have $q_j = 1$ for some j then we may replace y_j by g_j, aiming a "possible" ambient reduction to $y_j = 0$.

Theorem 9.2. *(Edge Decomposition Theorem)*
We obtain the following equivalence which holds within a sufficiently small neighborhood U of $\xi \in Z$:

$$E \;\sim\; \Big(\textstyle\bigcap_{i=1}^r \; E_i \Big) \;\bigcap\; F \tag{9.3}$$

which means

$$\mathfrak{S}(E) \;=\; \Big(\textstyle\bigcap_{i=1}^r \; \mathfrak{S}(E_i) \Big) \;\bigcap\; \mathfrak{S}(F)$$

where $E_i = (g_i \mathcal{O}_U, q_i), 1 \le i \le r$, and $F = (I, c)$ with $ord_\xi(I) > c$.

Proof. Here only question is how to choose F. One way is as follows. Let d be any integer bigger than q_r and b of $E = (J, b)$. Then we let

$$F \;=\; \bigcap_{1\le a\le d} \big(J_{max}(a) \cap M_\xi^{a+1}, a \big) \tag{9.4}$$

We then can easily prove

$$\begin{aligned} \mathfrak{S}(E) \;&=\; \textstyle\bigcap_{1\le a\le d} \big(J_{max}(a), a \big) \\ &\subset \; \Big(\textstyle\bigcap_{i=1}^{r} \; \mathfrak{S}(E_i) \Big) \bigcap \; \mathfrak{S}(F) \;\subset\; \mathfrak{S}(E) \end{aligned}$$

where the last inclusion is by Eq.(9.1) for $a = b$. □

Definition 9.1. Given an ideal exponent E and a closed point $\xi \in Sing(E)$, a *set of edge data* of E at ξ will mean a combination of the following objects and their expressions:

1. The *edge parameters* $y = (y_1, \cdots, y_r)$,
2. the *edge generators* $g = (g_1, \cdots, g_r)$ with $g_i = y_i^{q_i} + \epsilon_i$ and
3. the *edge decomposition*

$$E \;\sim\; \Big(\bigcap_{i=1}^{r} \; E_i \Big) \bigcap \; F \quad \textit{where} \quad E_i = (g_i\mathcal{O}_Z,\, q_i)$$

Definition 9.2. The primary inductive strategy:

Our approach to the inductive proof will be based upon the following system of numbers.

$$\mathrm{Inv}_\xi(E) \;=\; (n, n-r, q_1, \cdots, q_r). \tag{9.5}$$

with respect to the *lexicographical ordering*. The system will be called *the edge invariants* of E at ξ. The first number n is $dim_\xi Z$ and the other numbers $\{r, q_i = p^{e_i}, 1 \le i \le r,\}$ are the ones defined by Th(9.1).

Remark 9.2. If $n = 1$ then the problem is trivial. If $n - r = 0$, it is easy. If $n - r = 1$ then it is a question similar to resolution of curve singularities. What is more, if $q_1 = 1$ that is $e_1 = 0$ then at least "locally" at ξ we can apply the ambient reduction theorem Th.(6.2) from Z to the hypersurface $g_1 = y_1 = 0$. This provision "locally" will be cleared later by a "global" procedure of selecting and modifying those y_i. The inductitive proof will thus start working.

10. Edge data for plane curves

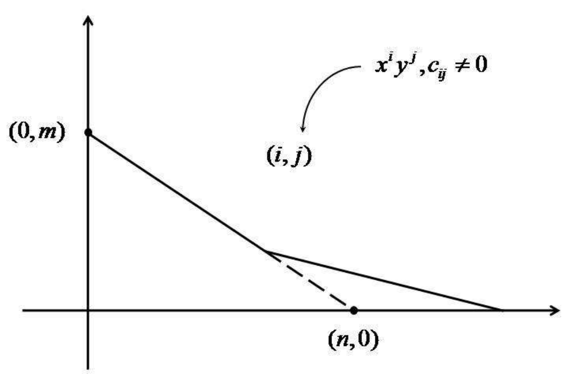
Remark 10.1. Assume that $\mathbb{K} = \mathbb{C}$, the complex number field. Consider a plane curve defined by

$$f(x, y) = \sum_{ij} c_{ij} x^i y^j \quad with \quad c_{ij} \in \mathbb{K}$$

Assume that its multiplicity is $m = ord_{(0,0)}(f)$ and $c_{0,m} = 1$. Moreover assume that its first characteristic exponent is

$$\delta = \min\{ \ i/(m - j) \mid j < m, c_{ij} \neq 0 \ \} > 1$$

which is n/m in the following figure.

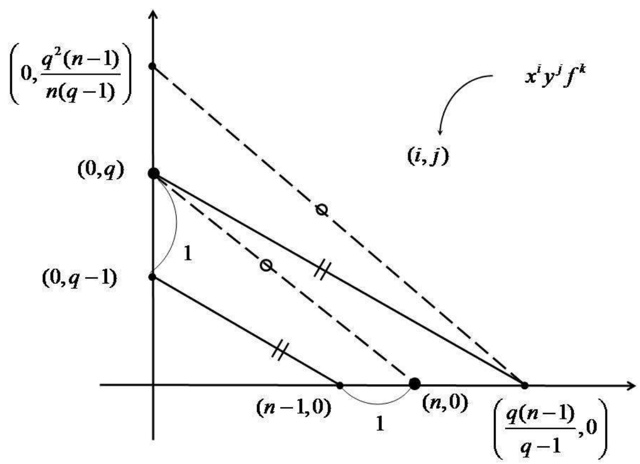


Here for $E = (f\mathbb{K}[x, y], m)$, it can be proven that $\wp(E) = \sum_{l=0}^{\infty} J_{max}(l) T^a$ is determined by δ within a neighborhood of $\xi \in Z$ as follows:

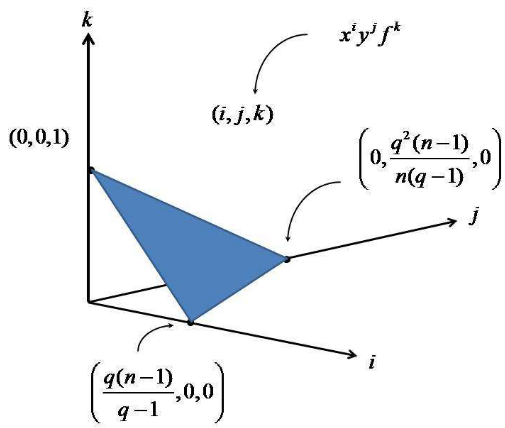
$$J_{max}(l) \ = \ \{ \ x^i y^j \mid \frac{i}{\delta} + j \geq l, i \geq 0, j \geq 0 \ \} \mathbb{K}[x, y], \quad \forall l \geq 0.$$

However the same type of statement fails in general when the characteristic $p > 0$. Let $\mathbb{K}$ be an algebraically closed field with $char(\mathbb{K}) = p > 0$.

Consider a plane curve defined by $f = y^q - x^n$ where $q = p^e, e > 0$ and $n > q, (n,p) = 1$.



where parallel lines produce two end points $(0, \frac{q^2(n-1)}{n(q-1)})$ and $(\frac{q(n-1)}{(q-1)}, 0)$.



The "shaded triangle" shows the lower bound for the generators of the homogeneous part $J_{max}(q)$ of $\wp(E)$. It figures a "3"-dimensional "Newton polygon". To be precise we have

$$J_{max}(l) \; = \; \{\; x^i y^j f^k \mid i \geq 0, j \geq 0, k \geq 0, \; ineq(l) \;\} \mathbb{K}[x,y], \; \forall l \geq 0$$
$$where \;\; ineq(l) \;\; means \;\; i\tfrac{q-1}{n-1} + j\tfrac{n(q-1)}{q(n-1)} + kq \geq l.$$

(Drawings are done by Mr. Ayamasa Nagai.)

11. Transforms of edge data

We want to examine transforms of the edge parameters y and the edge generaters g by means of permissible blowups for the given E. .

Theorem 11.1. *Pick a blowp $\pi : Z' \longrightarrow Z$ with center D permissible for E. Then the edge invariants never increases. To be precise pick any closed point $\xi' \in \pi^{-1}(\xi) \cap Sing(E')$ where E' denotes the transform of E by π. Then we have $Inv_{\xi'}(E') \leq Inv_{\xi}(E)$ in the lexicographical ordering.*

Theorem 11.2. *Let $\pi : Z' \to Z$ be a permssible blowup for E. Let $I = I(Z, D)_\xi$, Pick a closed point $\xi \in D$ and a closed point $\xi' \in \pi^{-1}(\xi) \cap \left(\cap_{1 \leq i \leq r} Sing(G'_i)\right)$ where G'_i is the transform of G_i for each i. Pick any system z such that (y, z) is a regular system of parameters of R_ξ. Then we can find an exceptional parameter $\mathfrak{z}$ at ξ' such that*

(1) $\mathfrak{z} \in \mathbb{K}[z]$ and $\mathfrak{z}^{-1} y_i \in R_{\xi'}$ for all $i, 1 \leq i \leq r$,
(2) If $Inv_{\xi'}(E') = Inv_\xi(E)$ then there exists $c_i \in \mathbb{K}$ with $\mathfrak{z}^{-1} y_i - c_i \in M_{\xi'}$ for all i.

.

The following lemmas are needed for the proofs of those theorems.

Lemma 11.3. *The permissibility implies that the ideal I of the center contains $y_i - \phi_i$, say $= \mathfrak{y}_i$, with $\phi_i \in M_\xi^2$ for every i.*

Proof. We must have $g_i = y_i^{q_i} - \epsilon_i \in I^{q_i}$ with $I = I(D, Z)_\xi$. Hence ϵ_i mod I is a q_i-th power in $\bar{R} = R_\xi / I$. It is also in $max(\bar{R})^{q_i+1}$ because $\epsilon_i \in M_\xi^{q_i+1}$. Therefore $\epsilon_i \mod I$ must be in $max(\bar{R})^{2q_i}$ from which there follows the existence of the claimed ϕ_i. $\square$

Lemma 11.4. $(\mathfrak{y}_1, \cdots, \mathfrak{y}_r, \mathfrak{z})$ *is extendable to a base of I as well as to a regular system of parameters of R_ξ and $\mathfrak{z}^{-1}\mathfrak{y}_i \in R_{\xi'}$ for all i.*

Proof. Write $g_i = \mathfrak{y}_i^{q_i} + \epsilon_i^*$. Then we must have ϵ_i^* in $I(D, Z)_\xi^{q_i} \cap M_\xi^{q_i+1}$ which is equal to $M_\xi \, I(D, Z)_\xi^{q_i}$. Now pick any parameter $\mathfrak{z} \in I(D, Z)_\xi$ which generates the ideal of $\pi^{-1}(D) \subset Z'$ at ξ'. Then $\mathfrak{z}^{-q_i}\epsilon_i^* \in M_{\xi'}$ and hence $\mathfrak{z}^{-q_i}\mathfrak{y}_i^{q_i} \in M_{\xi'}$ by the assumption $\xi' \in Sing(E'_i)$ where E'_i denotes the transform of E_i by π. We conclude $\mathfrak{z}^{-1}\mathfrak{y}_i \in M_{\xi'}$ for all i. This proves Lem.(11.4) as well as the following Lem.(11.5). $\square$

Lemma 11.5. $y' = (\mathfrak{z}^{-1}\mathfrak{y}_1, \cdots, \mathfrak{z}^{-1}\mathfrak{y}_r, \mathfrak{z})$ *is extendable to a regular system of parameters of $R_{\xi'}$.*

Lemma 11.6. *If $Inv_{\xi'}(E') \geq Inv_\xi(E)$ then $Inv_{\xi'}(E') = Inv_\xi(E)$. Moreover the transform E'_i of E_i by π is equal to $(g'_i \mathcal{O}_{Z'}, q_i)$ with $g'_i =$*

$\mathfrak{z}^{-q_i} g_i$ for all i and we obtain an edge decomposition of the transform E' of E by π at the point ξ'

$$\mathfrak{S}(E') = \Big(\bigcap_{i=1}^{r} \mathfrak{S}(E'_i) \Big) \bigcap \mathfrak{S}(F') \tag{11.1}$$

where F' is the transform of F by π.

Proof. The assumption $\xi' \in Sing(E')$ implies $\xi' \in Sing(E'_i)$ for all i. Therefore The assumption on the edge invariants implies that $g'_i, 1 \leq i \leq r$, must generates the *edges* of $\wp(E')$ at ξ' for we naturally have Eq.(11.1). □

Lemma 11.7. *So long as $\xi' \in Sing(E')$ the exceptional parameter $\mathfrak{z}$ of Lem.(11.3) can be chosen from the polynomial ring $\mathbb{K}[z]$.*

Proof. The morphism from a neighborhood of $\xi \in Z$ to $Spec(\mathbb{K}[\mathfrak{y}, z])$ is etale at ξ. Hence $R_\xi/(\mathfrak{y})$ is etale over $\mathbb{K}[\mathfrak{y}, z]/(\mathfrak{y})$ which is naturally identified with $\mathbb{K}[z]$. Therefore the ideal $I(D, Z)_\xi/(\mathfrak{y})$ in $R_\xi/(\mathfrak{y})$ is generated by elements of $\mathbb{K}[z]$. We conclude $I(D, Z)_\xi$ is generated by $\mathfrak{y}$ and a system of elements of $\mathbb{K}[z]$. One of these elements should be chosen to be $\mathfrak{z}$ by Lem.(11.5). □

We now proceed to describe the next stage of our strategy for resolution of singularities:

We consider finite sequences of blowups starting with $Z0 = Z$:

$$\pi(i) : \xi(i+1) \in Z(i+1) \longrightarrow \xi(i) \in Z(i) \, \forall i \geq 0 \tag{11.2}$$

which are permissible with center $D(i)$ for $E(i)$, where $E0 = E$ and $E(i+1)$ is the transform of $E(i)$ by $\pi(i)$. We then choose a sequence of closed points $\xi(i)$ accompanied Eq.(11.2) such that

$$\xi(i+1) \in Sing(E(i+1)) \cap \pi(i)^{-1}(\xi(i)) \quad \textit{with} \quad \xi 0 = \xi \tag{11.3}$$

We restrict our attention to only those sequences satisfying the condition that we have the same edge invariants:

$$\mathrm{Inv}_{\xi(i+1)}(E(i+1)) = \mathrm{Inv}_{\xi(i)}(E(i)) = \mathrm{Inv}_\xi(E) \, \forall i. \tag{11.4}$$

Definition 11.1. We consider finite sequences Eq.(11.2) accompanied by Eq.(11.3) satisfying the condition Eq.(11.4) for a given E and a closed point $\xi \in Sing(E)$. We call any one of those sequences an *Inv-sequence* (edge invariants test sequence) for the given E and $\xi \in Sing(E)$.

Remark 11.1. (1) Examine what "improvement" we can make on the singularities of a given data only by means of Inv-sequences. Search for new invariants (or characters of singularity) which will show such "improvement" of singularity.

(2) After certain "improvement" of singularities by a suitable choice of a preparatory Inv-sequence, we proceed to the next stage to find a way of lowering edge invariants.

12. Normal crossing data Γ

From now on we assume that we are given a *normal crossing data*

$$\Gamma \;=\; (\Gamma_1, \cdots, \Gamma_s)$$

in Z, called the *NC-data* for short.

Definition 12.1. A blow-up $\pi : Z' \to Z$ with center D is called *permissible* for Γ if D is smooth irreducible and have *normal crossing* with Γ.

Definition 12.2. The *transform* Γ' of Γ by π of the above Def.(12.1) is defined to be $\Gamma' \;=\; (\Gamma'_1, \cdots, \Gamma'_s, \Gamma'_{s+1})$ where

(1) Γ'_i is the strict transform of Γ_i by π for every $i, 1 \le i \le s$, ($\Gamma'_i = \emptyset$ if $D = \Gamma_i$)

(2) Γ'_{s+1} is the exceptional divisor $\pi^{-1}(D)$ of π.

Remark 12.1. General agreement (1):
From now on the Γ-*permissibility* is always imposed even when it is not mentioned.
General agreement (2):
The ordering of the components of Γ will be recorded as *the history of their creation*. Thus it is important to note that the new exceptional divisor is placed in the last spot of the sequence Γ'.

Theorem 12.1. *Assume that a NC-data Γ and a smooth subscheme W are given in Z. Then there exists a naturally defined coherent ideal $F(W/\Gamma)$ in $\mathcal{O}_W$ such that*

$$\begin{aligned} &W \quad \textit{is normal crossing with } \Gamma \\ &\Longleftrightarrow \; F(W/\Gamma)_\xi \;=\; \mathcal{O}_{W,\xi} \end{aligned}$$

Proof. Let us pick one generator z_i of $I(\Gamma_i, Z)_\xi$ for each of those Γ_i such that $\xi \in W \cap \Gamma_i \ne W$, $1 \le i \le c(\xi/W)$. Write c for $c(\xi/W)$.

Incidentally we ignore the ordering by the *history of creation* of the Γ_i's just for the sake of notational simplicity in this proof.

$$W \quad \textit{is normal crossing with} \;\; \Gamma$$
$$\Longleftrightarrow$$
$$\exists \textit{ a regular system of parameters } \;(\bar{z}, w)\; \textit{ of } \;\mathcal{O}_{W,\xi}$$
$$\textit{such that } \;\; \textstyle\bigwedge_{i=1}^{d} \delta \bar{z}_i \bigwedge_{j=1}^{d-c} \delta w_j \;\textit{ generates }\; \bigwedge^{d} \Omega_{W,\xi}$$

where $\bar{z} = (\bar{z}_1, \cdots, \bar{z}_c)$ is formed with the images $\bar{z}_i$ of z_i into $\mathcal{O}_{W,\xi}$. The existence of such w of length $d - c$ is equivalent to the condition

$$\bigwedge^{d} \Omega_{W,\xi} \;=\; \Big(\bigwedge^{d-c} \Omega_{W,\xi} \Big) \Big(\bigwedge_{i=1}^{c} \delta_W \; I(\Gamma_i \cap W, W)_\xi \Big)$$

which means

$$F(W/\Gamma)_\xi \;\; \textit{is the unit ideal of} \;\; \mathcal{O}_{W,\xi}.$$

$\square$

Definition 12.3. The ideal $F(W/\Gamma)_\xi$ is the unique ideal satisfying the following equality.

$$F(W/\Gamma)_\xi \left(\textstyle\bigwedge^{d}\Omega_{W,\xi}\right)$$
$$= \left(\textstyle\bigwedge^{d-c(\xi/W)} \Omega_{W,\xi}\right) \left(\textstyle\bigwedge^{c(\xi/W)} \bigoplus_{i:\, \xi \in W \cap \Gamma_i \neq W} \delta_W \big(I(\Gamma_i, Z)\mathcal{O}_{W.\xi}\big)\right)$$

where $d = dim_\xi W$ and $c(\xi/W)$ is the number of the indices i with $\xi \in W \cap \Gamma_i \neq W$.

The following lemma is useful in many inductive steps.

Lemma 12.2. *(called "denominator lifting") Compare* $E = (J, b)$ *with* $\check{E} = (J, m)$ *for some* $m > b$*. Then, after any finite sequence of permissible blowups, their transforms* $E' = (J', b)$ *and* $\check{E}' = (\check{J}', m)$ *by a* Γ'*-monomial factor* Q *in their ideals. Namely* $J' = Q\check{J}'$ *at every point of* $Sing(\check{E}')$*. Here* Γ' *denotes the transform of* Γ

Proof. Straight from the definition of transforms by permissible blowups of Def.(4.2). $\square$

Theorem 12.3. *Assume that* $E = (J, b)$ *has locally* Γ*-monomial* J *everywhere in* Z*. Write* $J = \prod_{1 \le a \le s} J_a^{d_a}$ *where* J_a *is the ideal of* Γ_a *in* $\mathcal{O}_Z$*. Then there exists a canonical sequence of permissible blowups* $\tilde{\pi} : \tilde{Z} \to Z$ *such that* $Sing(\tilde{E}) = \emptyset$ *with the transform* $\tilde{E}$ *of* E *by* $\tilde{\pi}$*.*

The "Canonical Procedure" is as follows.

(1) Let $\Gamma = (\Gamma_1, \cdots, \Gamma_s)$. For each nonempty $A \subset [1, s]$, we denote $D(A) = \cap_{a\in A}\Gamma_a$ and $\sigma(A) = \sum_{a\in A} d_a$

(2) Let $\mathcal{S}_0(E) = \{A \subset [1, s] \bigm| \sigma(A) \geq b \ \ and \ \ D(A) \neq \emptyset \}$.

(3) Let $\mathcal{S}_1(E) = \{A \in \mathcal{S}_0(E) \bigm| |A| = \bar{\lambda}(E)\}$ with $\bar{\lambda}(E) = \min\{ |A| \bigm| A \in \mathcal{S}_0(E) \}$ where $|A|$ is the cardinality of A.

(4) Let $\mathcal{S}_2(E) = \{A \in \mathcal{S}_1(E) \bigm| \sigma(A) = \hat{\sigma}(E)\}$ with $\hat{\sigma}(E) = \max\{\sigma(A)\big| A \in \mathcal{S}_1(E)\}$.

(5) Let $\chi(E)$ denote the cardinality of $\mathcal{S}_2(E)$.

(6) The set $\mathcal{S}_2(E)$ has a lexicographical ordering by means of the given ordering in Γ itself.

Now choose the lexicographically smallest member B in $\mathcal{S}_2(E)$ and take the blowup with center $D(B)$. This process will terminate after a finite number of repeated applications.

Proof. We claim that the following system of three numbers

$$(\ n - \bar{\lambda}(E), \ \hat{\sigma}(E), \ \chi(E) \ )$$

strictly decreases lexicographically after each blowup as above, where $n = dim(Z)$ which is always $\geq \lambda(A)$ with $D(A) \neq \emptyset$. Now, choose the blowup π_1 with center $D(B)$ as above. Let Γ'_a be the strict transform of Γ_a and $\Gamma'_{s+1} = \pi_1^{-1}(D(A)$. We know that $\cap_{a\in B} \Gamma'_a$ is empty. Hence it is only necessary to examine the cases of $A'(i) = (A-\{i\})\cup\{s+1\}, i \in A$, to determine what changes take place in the new

$$(\ n - \bar{\lambda}(E'), \ \hat{\sigma}(E'), \ \chi(E') \ )$$

where E' denotes the transform of E by π_1. Here the point is that for every i we have

$$\begin{aligned} \sigma(A'(i)) &= (-b + \sigma(A)) + \textstyle\sum_{a\in A, \neq i} d_a \\ &= \sigma(A) + (-b + \textstyle\sum_{a\in A,\neq i} d_a) < \sigma(A), \ \forall i \end{aligned} \tag{12.1}$$

which implies the strict lexicographical inequality:

$$(\ n - \bar{\lambda}(E'), \hat{\sigma}(E'), \chi(E') \ ) < (\ n - \bar{\lambda}(E), \hat{\sigma}(E), \chi(E) \ ).$$

In fact the first number can only drop if not equal. If it remains equal then the second number drops or remains equal by Eq,(12.1). Finally the same reasoning applies to the last number also by Eq,(12.1). $\square$

13. Cleaning in the case of $p > 0$

Recall the edge data of $\wp(E)$: $y = (y_1, \cdots, y_r)$, $g = (g_1, \cdots .g_r)$ with $g_i = y_i^{q_i} + \epsilon_i$ and with $q_i = p^{e_i}$ for $1 \le i \le r$ where $e_i \ge 0$. This is our tyical case we want to work "cleaning" of g. Roughly speaking this means to eliminate those monomial terms of every ϵ_i which belong to the ideal $(y_j^{q_j}, 1 \le j \le r)R_\xi$.

Now for the sake of broader applicability we start with a little more general situation than the edge data case as follows.

Remark 13.1. We have

$$g_i = y_i^{d_i} + \epsilon_i \textit{ such that } d_i > 0 \textit{ and } ord_\xi(\epsilon_i) > d_i, \textit{ for all } 1 \le i \le r \tag{13.1}$$

where d_i need not be powers of p. We choose and fix z such that (y, z) is a regular system of parameters of R_ξ

Theorem 13.1. *Let $R(N) = \rho^N(R_\xi)$ with $N \gg 1$. Under the assumption Eq.(13.1) we claim that R_ξ is freely generated as $R(N)$-module by the following g-modified monomials:*

$$\begin{aligned} &y^\alpha g^\beta z^\gamma \quad \textit{with} \quad \alpha \in \mathbb{Z}_0^r,\ \beta \in \mathbb{Z}_0^r \textit{ and } \gamma \in \mathbb{Z}_0^{n-r} \\ &\textit{where } 0 \le \alpha_k < d_k,\ d_k\beta_k + \alpha_k < p^N,\ \forall k, \gamma_j < p^N,\ \forall j. \end{aligned} \tag{13.2}$$

Let $\mathcal{Q}_N(g)$ denote the $R(N)$-module with the generators written in Eq.(13.2). It is naturally an $R(N)[z]$-module.

Proof. Pick any element $h(0) \in R_\xi$ and write it as

$$\sum_{a \in \epsilon^r(p^N)} \in \phi_(a) y^a \textit{ with } \phi_(a) \in R(N)[z] \tag{13.3}$$

Then pick any a and choose $c \in \mathbb{Z}_0^r$ such that $0 \le a_i - c_i d_i < d_i$ for all i. We let $B_a = \phi_(a) y^{a - d*c} g^c$ where $d * c = (d_1 c_1, \cdots, c_r d_r)$. Then observe that

$$ord_\xi(\phi_(a) y^a - A_a) > ord_\xi(\phi_(a) y^a) \ge ord_\xi(h(0))$$

and that $B_a \in \mathcal{Q}_N(g)$. We apply this to every a and we obtain $B(0) = \sum_a B_a \in \mathcal{Q}_N(g)$ such that $ord_\xi(h(0) - B(0)) > ord_\xi(h(0))$. We rewrite $h(1) = h(0) - B(0)$ in the form of Eq.(13.3) and apply the same process as above. We find $B(1) \in \mathcal{Q}_N(g)$ with $ord_\xi(h(1) - B(1)) > ord_\xi(h(1))$ and so on. Therefore for every integer m we can find $B \in \mathcal{Q}_N(g)$ with $ord_\xi(h(1) - B) > m$. By taking m sufficiently large we have $max(R(N))M_\xi^m$ so that $h(0) \in \mathcal{Q}_N(g) + \max(R(N))R_\xi$. Applying this to those $h(0)$ generating $R(N)[z]$-module, we get $R_\xi = \mathcal{Q}_N(g) + max(R(N))R_\xi$. By Nakayama's lemma we get $R_\xi = \mathcal{Q}_N(g)$. □

Definition 13.1. Let $\hat{q} = (q_1, \cdots, q_r)$. For an integer $c > 0$ we define

$$\mathcal{Q}_N(c)^\flat \;=\; \sum_{(13.2)\ \textit{and}\ \beta\cdot\hat{q}<c} y^\alpha g^\beta z^\gamma\, R(N), \quad \textit{and}$$

$$\mathcal{Q}_N(c)^\sharp \;=\; \sum_{(13.2)\ \textit{and}\ \beta\cdot\hat{q}\geq c} y^\alpha g^\beta z^\gamma\, R(N) \;=\; R - \mathcal{Q}_N(c)^\flat$$

Let us now go back to the edge parameters y and the edge equations $g_i, 1 \leq i \leq r$, of $\wp(E)$. In this case we choose the ideal I to be M_ξ

Definition 13.2. Write h as $h^\flat + h^\sharp$ with $h^\flat \in \mathcal{Q}_N(c)^\flat$ and $h^\sharp \in \mathcal{Q}_N(c)^\sharp$ (which are both automatically "belonging to" R_ξ, not only to the completion $\hat{R}_\xi$). We then define $(g, N(c))$-*cleaning* to be the map $h \mapsto h^\flat$.
If $h^\sharp = 0$ then h is said to be $(g, N(c))$-*cleaned* .

Definition 13.3. The g_i is a homogeneous element of degree $q_i = p^{e_i}$ in $\wp(E) \subset gr_M(R)$ for every i. For any homogeneous element h of degree c in $gr_M(R)$ the $(g, N(c))$-*cleaning* of h will be called $(g, N)_z$-*cleaning* or (g, N)-*cleaning* for short. For instance the (g, N)-cleaning of $\epsilon_i = g_i - y_i^{q_i}$ will mean the $(g, N(q_i))$-*cleaning*.

Theorem 13.2. *Any given edge generators g of $\wp(E)$ with $g_i = y_i^{q_i} + \epsilon_i$ can be modified into another edge generators $g^\dagger$ of $\wp(E)$ with $g_i^\dagger = y_i^{q_i} + \epsilon_i^\dagger$ (having the same y) in such a way that $\epsilon_i^\dagger$ is $(g^\dagger, N)$-cleaned for every $i, 1 \leq i \leq r$ in the sense of Def.(13.2).*

Proof. Once the p^N-cleaning of Prop.(**??**) is applied to any given edge generators g and edge decomposition, then any of the $g_i^\flat$ with $q_i = q_1$ is clearly $Q_N(q_1)$-cleaned with respect to g as well as to $g^\flat$ and (y, z)b in the sense of Def.(13.2) thanks to the expression Eq.(**??**) of Prop.(**??**). In fact $\epsilon_i^\flat$ with $q_i = q_1$ does not have no nonzero monomial summands divisible any one of the $g^\flat$. Let $g_i^\dagger = g_i^\flat$ with $q_i = q_1$. Replace g by the $g^\flat$ and rename $g^\flat$ as g again, keeping in mind that $g_i = g_i^\dagger$ for $q_i = q_1$ and any of the ϵ_i with $q_i = q_2$ has no monomial terms divisible by any g_j with $j \geq 2$. Apply Prop.(**??**) again. Then we see that every one of the new $g_i^\flat$ with $q_i = q_2$ is $Q_N(q_1)$-cleaned with respect to $g_i = g_1^\flat = g_i^\dagger$ with $q_i = q_1$ and (y, z). Let $g_i^\flat = g_1^\dagger$ for every i with $q_i = q_2$. Repeat at most stable times and reach the end with $g^\dagger$ is p^N-cleaned with respect to $g^\dagger$ itself and (y, z) □

The modification of the theorem is obtained by repeating r-times cleanings of the kind of Def.(13.3). After the first cleaning the new g_1 stays to be cleaned all the way to the end. After the second the same for the new (g_1, g_2) and so on.

Definition 13.4. We say an edge data $\{y, q, g\}$ is $(N)_z$-cleaned if $\epsilon_i^\dagger = \epsilon_i$ for all i in the sense of Th.(13.2).

Theorem 13.3. *Pick any integer N, say $> \sum_{1 \le i \le r} e_i$. We are given edge data $\{y, q, g\}$ for $\wp(E)$ at ξ which are "$(N)_z$-cleaned". Let $\pi : Z' \to Z$ with center $D \ni \xi$ be permissible for E. Pick a closed point $\xi' \in \pi^{-1}(\xi) \cap Sing(E')$ such that $Inv_{\xi'}(E') = Inv_\xi(E)$. Choose an exceptional parameter $\mathfrak{z}$ at ξ' such that $\mathfrak{z} \in \mathbb{K}[z]$, say $\mathfrak{z} \in z$. Then the transformed edge data*

$$\{\mathfrak{z}^{-1} y_i - c_i, q_i, \mathfrak{z}^{-q_i} g_i, 1 \le i \le r\}$$

of $\wp(E')$ according to Th.(11.2) with Lem.(11.7) is necessarily $(N)_{z'}$-cleaned where z' is an appropriate transform of z by π. For instane z' is a regular system of f parameters of

$$Spec(\mathbb{K}[\mathfrak{z}^{-1}(z \setminus \mathfrak{z}), \mathfrak{z}])$$

at the projection of

> In short the transforms of the "clean" edge data stay to be "clean" at the points where "Edge Invariants" are unchanged by the permissible blowup.

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14. Γ-TRANSVERSALITY

At a closed point $\xi \in Z$, we may be given a specific system of parameters of R_ξ, say $w = (w_1, \cdots, w_r)$. For instance w could be a system of edge parameters y of $\wp(E)$. On the other hand we are given the NC-data Γ created by earlier blowups before the selection of w.

Definition 14.1. We say w is Γ-*transversal* at ξ if w can be extended to a regular system of parameters $x = (w, v)$ of R_ξ in such a way that v contains a generator of the ideal of every one of those members of Γ which go through the point ξ.

In the following theorem we make use of "induction hypothesis" on the dimensions of ambient spaces for the resolution of singularities applied to such ideals as $F(W/\Gamma)$ of Def.(12.3) and Th.(12.1).

Theorem 14.1. *Given parameters w extendable to a regular system of R_ξ, there exists a finite sequence of blowups $\pi : Z' \to Z$, globally successively permissible for E and Γ, such that the transform w' of w is Γ'-transversal at every closed point $\xi' \in \pi^{-1}(\xi)$ where $Inv_{\xi'}(E') = Inv_\xi(E)$. Here Γ' is the transform of Γ by π and E' is that of E. As for the choice of the transform w' of w we make use of exceptional parameters and parameter transformations in the manner of Th.(11.2).*

Proof. Pick any one member of Γ, say Γ_1, and apply the ambient reduction theorem Th.(6.2) on E from Z to Γ_1. Our induction is applicable to this and we obtain a permissible sequence $\pi : Z' \longrightarrow Z$ which eliminate all singularities of the reduction in Γ'. Here what is important to note is that the transform y' of the original y is Γ_{new}-transversal where Γ_{new} is the subsystem of the transform Γ' of Γ by π. This is assured by Lem.(11.4), Lem.(11.5) and Lem.(11.7). It is also assured that if we had a subsystem $\Gamma(*)$ of Γ such that y was $\Gamma(*)$-transversal then we see that y' is $\Gamma(*)' \cup \Gamma_{new}$-transversal where $\Gamma(*)'$ denotes the strict transform of $\Gamma(*)$ by π. Moreover the strict transform of Γ_1 by π is moved away from $Sing(E')$ with the transform E' of E by π. We repeat this process to the transform of Γ_2 by π, and so on until we process all transforms of the members of the original Γ. This completes the proof. □

The locally defined ideal $F(W/\Gamma)$ can be extended globally to Z where we do not concern the loss of its property away from ξ with respect to Γ in the sense of Def.(12.3). The induction hypothesis is used inside the strict transforms of each component of the NC-data, one after another in the order of the history of creation. The point is the transversality becomes automatic with new exceptional divisor after a certain finite number of steps.

Corollary 14.2. *The theorem is applicable to edge paramerters* y *of* $\wp(E)$. *Therefore it is always enough to work with resolution problems under the assumption that the edge parameters* y *is* Γ*-transversal.*

Definition 14.2. Assume that the edge parameters y of an Def.(9.1) of E is Γ-transversal at ξ. This is achieved by Cor.(14.2). We then have a system $x = (y, z)$ in R_ξ such that every member of Γ through ξ is defined by some member of z. Such $x = (y, z)$ will be called *NC-extension* of edge parameters y for E at ξ, or for short *NC-edge* parameters of E at ξ.

Proposition 14.3. *Let* $\pi : Z' \longrightarrow Z$ *with center* D *be a permissible blowup for* E *and* Γ *with a set of edge data in the sense of Def.(9.1) at a closed point* $\xi \in Sing(E)$. *Then there exists NC-edge parameters* $x = (y, z)$ *such that the ideal* I *of* D *at* ξ *is generated by* $y - \phi$ *of Lem.(11.3) and a subsystem* $z0$ *of* z. *Extend* z *so that* $x = (y, z)$ *is a regular system of parameters of* R_ξ *and write* $z = (z0, z(1))$. *Let* K *be the algebraic closure of* $\mathbb{K}$ *in the completion* $\hat{R}$ *of* R_ξ. *Then we can choose* $\hat{\phi}_i \in K[[z(1)]]$ *for each* i *in such a way that* $(y - \hat{\phi}, z(1))$ *is a minimal base of the ideal* $I\hat{R}$.

Proof. The first assertion is due to the assumption that D is permissible not only for E but also for Γ. As for the second assertion we see that $\epsilon_i \equiv y_i^{q_i}$ modulo $I\hat{R}$ and hence the image of ϵ_i in $\hat{R}/I\hat{R} = K[[z(1)]]$ must be a q_i-th power in $K[[z(1)]]$, say $\hat{\phi}_i^{q_i}$ with $\hat{\phi}_i \in K[[z(1)]]$. These $\hat{\phi}_i$ make up the answer. $\square$

15. Γ-MONOMIALIZATION

Let us consider a *set of edge data* in the sense of Def.(9.1) after the edge decomposition of Th.(9.2):

$$E \sim \Big(\bigcap_{i=1}^{r} E_i \Big) \bigcap F \quad \textit{where} \quad E_i = (g_i \mathcal{O}_Z, \, q_i) \tag{15.1}$$

where *edge generators* $g_i = y_i^{q_i} + \epsilon_i$ are written in terms of *edge parameters* $y = (y_1, \cdots, y_r)$ and $F = (I, c), \; ord_\xi(I) > c$ denotes a chosen *remainder component* of E.

Our primary objective of this section is to find a good simplification of the *remainder component* by means of permissible blowups for E and Γ. However, we do it in a more general setup in order to have broader applications which are wanted in later sections.

Remark 15.1. We consider an extra ideal exponent $H = (h, d)$ in addition to the given set of edge data of E in the form of Def.(15.1).

Theorem 15.1. *There exists a finite sequence of proper blowups, say $\hat{\pi} : \hat{Z} \longrightarrow Z$, permissible for E and H (and also for NC-data Γ as always), which has the following property: At every closed point $\hat{\xi} \in Sing(\hat{E}) \subset \hat{Z}$ with the transform $\hat{E}$ of E by $\hat{\pi}$ such that $\hat{E}$ has the same* edge invariant *at $\hat{\xi}$ as that of E at $\xi = \hat{\pi}(\hat{\xi})$,*

(1) there exists a set of edge data of $\hat{E}$ at $\hat{\xi}$ in which we have
- *(a) a regular system of parameters $\hat{x} = (\hat{y}, \hat{z})$ with edge parameters $\hat{y}$, $\hat{\Gamma}$-transversal with the transform $\hat{\Gamma}$ of Γ by $\hat{\pi}$,*
- *(b) while $\hat{z}$ contains the equations of those components of $\hat{\Gamma}$ which pass through $\hat{\xi}$,*
- *(c) edge generators $\hat{g} = (\hat{g}_1, \cdots, \hat{g}_r)$ with $\hat{g}_i = \hat{y}_i^{q_i} + \hat{\epsilon}_i$ and*
- *(d) $\hat{I}$ of a remainder $\hat{F} = (\hat{I}, \hat{c})$ of $\hat{E}$ is a $\hat{\Gamma}$-monomial at $\hat{\xi}$,*

(2) $\hat{I} \neq 0$ at $\hat{\xi}$ and the transform $\hat{H} = (\hat{h}, d)$ of H by π is such that

$$\hat{h} \; \equiv \sum_{\substack{(\beta,\sigma) \in \mathbb{Z}_0^n \\ \sum_i \sigma_i q_i < d}} \hat{y}^{\,\delta} \; \hat{z}^{\,\beta} \; \hat{g}^{\,\sigma} \tag{15.2}$$

modulo $\hat{J}$ with respect to $\hat{E} = (\hat{J}, b)$ locally at $\hat{\xi}$ and

(3) finally every one of those $\hat{z}^\beta$ in Eq.(15.2) are $\hat{\Gamma}$-monomial.

Theorem 15.2. *As a weakened form of the theorem (15.1), we can assert that there exists π with an ideal exponent $E' = (J', d')$ such that $\hat{E} \cap H'$ is equivalent to $\hat{E} \cap \hat{H}$ under the edge invariant condition on E and J' is generated by a Γ-monomial at $\hat{\xi}$.*

Proof. Our proof of the theorems will be done by the following sequence of remarks. □

Remark 15.2. First of all we replace F by $F \cap G_1$, where G_1 is for instance $G_1 = (Der_Z(g_1), q_1 - 1)$ which is clearly nontrivial. Incidentally the change does not affect the equivalence in Eq.(15.1). Hence we assume that $F = F \cap G_1$ from the beginning. Then by any blowup $\hat{\pi}$ for Th.(15.1) or Th.(15.2), the transform $\hat{F}$ of the remainder F does not become equivalent to $(0, \hat{c})$ at $\hat{\xi}$ because the transform of intersection is the intersection of transforms and the transform of G_1 will never become equivalent to $(0, \hat{c})$ as far as the edge invariants of the transforms remain unchanged.

Remark 15.3. The replacement of F done in Rem.(15.2) is much more meritable than just being non-zero. Indeed the new F does not become zero even after the cleaning by means of the system of edge generators $g = (g_1, \cdots, g_r)$ of Th.(15.1) by means of Eq.(**??**) of Lem.(**??**). (See Rem.(15.6) below.)

Remark 15.4. We choose a regular system of parameters (y, z) of R_ξ which includes the edge parameters y of Eq.(15.1). The last condition of the Th.(15.1) can be assumed from the beginning by Th.(**??**). The transversality is upheld by a sequence of permissible blowups at every point of the transforms so long as the edge invariants remain unchanged. This is easily deduced by the lemmas, (11.3), (11.4), (11.5) and (11.6).

Let us first modify the problem data as follows. Let c° be the maximum of the orders of I and d° be the maximum of the orders of J. Then define

$$F^\circ \;=\; (I^{d^\circ} + J^{c^\circ}, c^\circ + d^\circ) \tag{15.3}$$

Then in the notation of Th.(15.1) we replace replace F by F°. After such replacement we consider the case of $J = 0$, that is the case of achieving the monomialization of the remainder F alone.

Remark 15.5. Let us take a maximal Γ-divisors of the ideals I at ξ. For the case of Eq.(15.1) we can choose it to be the z^α of Eq.(15.1) thanks to Rem.(15.4). In any case we will write $I = z^\alpha I^*$ with $I^* \subset R_\xi$.

Remark 15.6. With respect to the chosen g and (y, z), we make use of the cleaning in the manner of Eq.(**??**) of Lem.(**??**) to express a system of generators of I^*. To be precise we choose generators f_i's of I^* such that $ord_\xi(I^*) = \min_i\{ord_\xi(f_i)\}$ and we write each one of the f_i's as a linear combination of g-monomials $\{g^\beta \mid \beta \in \mathbb{Z}_0^r\}$ having coefficients $d_{i\beta}$ to g^β which are elements of

$$\sum_{0\le\delta_j<q_j,\gamma_k<p^N,\forall ij} y^\delta z^\gamma R(N). \tag{15.4}$$

and therefore we have

$$\begin{aligned} f_i \ &= \ \textstyle\sum_\beta d_{i\beta} g^\beta \quad where \\ d_{i\beta} \ &= \ \textstyle\sum_{0\le\delta_j<q_j,\gamma_k<p^N,\forall ij} b_{i\beta\delta\gamma} y^\delta z^\gamma \quad with \ \ b_{i\beta\delta\gamma} \in R(N). \end{aligned} \tag{15.5}$$

We then delete from each f_i those summands with $\beta\cdot\hat{q} \ge c$ with $\hat{q} = (q_1,\cdots,q_r)$. Incidentally as for I of the remainder F this deletion does not affect the equivalence in the edge decomposition of Th.(9.2).

Remark 15.7. When a permissible blowup $\pi : Z' \longrightarrow Z$ with center D is chosen, we replace y by $\mathfrak{y}$ according to Lem.(11.3). This must make changes in the above expression in Eq.(15.4) but the expressions of the initial term $in_\xi(f_i)$ does not change for i, because we have the same g and $ord_\xi(y_i - \mathfrak{y}_i) \ge 2$ for all i, Our inductive reasoning to be employed in the following remarks will be only concerned with $in_\xi(f_i)$.

Remark 15.8. Let

$$m = ord_\xi(I^*) = \min_{\forall i,\forall\beta}\{ord_\xi(d_{i\beta}) + \beta\cdot\hat{q}\}.$$

If we have at least one β such that for all i

$$ord_\xi(d_{i\beta}) > 0 \quad while \quad ord_\xi(d_{i\beta}) + \sum_j \beta_j q_j \ = m, \tag{15.6}$$

then we go on to the next Rem.(15.9) in which we apply the induction on edge invariants after the denominator lifting from c to m. Here let us consider the case in which there is no β having Eq.(15.6). In the expression of f_i by Eq.(15.5), we may replace any g_k by $g_l^{p^{e_k-e_l}}, l < k$, without affecting our problem at all. The reason is that we are considering only blowups with centers D such that $ord_D(g_j) = q_j = p^{e_j}$ and only those points $\tilde{\xi}$ where the transforms $\tilde{g}_i$ have $ord_{\tilde{\xi}}(\tilde{g}_j) = q_j$ for all j. Now, making such replacement if necessary, we may assume that for every i we can write

$$f_i \ = \ \sum_\beta d_{i\beta} g^{\beta(1)} g^{\beta(2)} \quad where \ \ \beta(2)\cdot\hat{q} = m \ \forall\beta \tag{15.7}$$

Then without affecting our problem at all we can replace every one of those f_i by $f_i^\flat = \sum_\beta d_{i\beta} g^{\beta(1)}$, I by $I^\flat = (f_i^\flat, \forall i)$ and accordingly $F = (I,c)$ by $F^\flat = (I^\flat, c-m)$. After this replacement if necessary, we go on to the remark next Rem.(15.9).

Remark 15.9. We then take up the denominator lifting $F^{\dagger} = (I^*, m)$ according to Lem.(12.2) and we see that the induction on edge invariants of Strategy I is effectively applicable to $E \cap F^{\dagger}$ because the initial $in_{\xi} I$ is not generated by the initials of g thanks to Rm.(15.8). Here we should keep in mind that our interest in the Strategy I is strictly restricted to those singular points in the transforms where "*the edge invariants*" remain unchanged. Refer to the inductive strategy Def.(9.2), Rem.(9.2) and Th.(11.1).

Remark 15.10. The effect of the sequence of blowups of Rm.(15.9) in the transform $\tilde{F} = (\tilde{I}, c)$ of the original $F = (I, c)$ is as follows. With the transform $\tilde{\Gamma}$ of Γ we factor out the $\tilde{\Gamma}$-maximal factor of $\tilde{I}$ according to Rm.(15.5) the resulting ideal $\tilde{I}^*$ has a lower order at $\tilde{\xi}$ than that of I^* at ξ. We repeat this process of "*denominator lifting*" followed by the induction of Strategy I if necessary. After a finite number of repetition we reach the point at which the final ideal $(\hat{I})^*$ turns into a unit ideal. This means that the final transform $\hat{F} = (\hat{I}, c)$ is of the claimed form of Th.(15.1). The theorem Th.(15.1) is now proven.

Let us now consider an arbitrary ideal $J \subset R_{\xi}$ which may not come from the remainder of Eq.(15.1). Note that with any positive integer $d < ord_{\xi}(J)$ we may replace F of the Th.(15.1) by the ideal exponent (J, d) while the edge parameters y and edge generators g are kept the same. This way we will find a better applications of our Γ-monomialization in later sections. In other words,

Theorem 15.3. *The edge decomposition with y and g of Eq.(15.1) being given, we take any ideal exponent (J, d) with $d < ord_{\xi}(J)$. We then have the existence of $\hat{\pi}$ of the Th.(15.1) applied to (J, d) as well as to (I, c). To be precise we can choose $\hat{\pi}$, permissible for both E and (J, d), such that*

(1) *$\hat{\pi}$ has all the properties of Th.(15.1), for the edge decomposition Eq.(15.1) of E and*
(2) *the same is also true to the replacement of (I, c) by (J, d) in Eq.(15.1).*