CMI Summerschool 2012 / Course Schicho
Definitions

Let K be a field of characteristic zero.

Habitat. A habitat is an equidimensional nonsingular variety W together with a finite sequence of nonsingular divisors $(E_1, \ldots, E_n)$ such that no two have a common component, and such that their sum is a normal crossing divisor.
For any habitat, we define an operator Δ from the set of ideals on W to itself, as follows. For $I \subset K[W]$, $\Delta(I)$ is the ideal generated by I and all first order partial derivatives of elements in I.

Straight subvariety. A subvariety Z of a habitat as above is called straight iff it is pure-dimensional, and for every point $p \in Z$, there is a system u of regular parameters as above which additionally satisfies the condition that Z is locally defined by a subset of u.

Blowup. Let Z be a straight subvariety of a habitat $(W, (E_1, \ldots, E_n))$. The blowup along Z is the habitat $(W', (E'_1, \ldots, E'_n, E_{n+1}))$, where W' is the blowup of W along Z, E'_i is the strict transform of E_i for $i = 1, \ldots, n$, and E_{n+1} is the exceptional divisor.

Singularity. A singularity on a habitat as above is a finitely generated Rees algebra $A = \oplus_{i=0}^{\infty} A_i$ over $A_0 = K[W]$, i.e. sequence of ideals $A_i \subset K[W]$ such that $A_0 = K[W]$ and $A_i \cdot A_j \subseteq A_{i+j}$ and equality holds for sufficiently large indices i, j.
We say a singularity is of ideal-type iff there is a $b > 0$ and ideal I such that $A_{nb} = I^n$ for all indices which are multiples of b, and $A_i = (0)$ otherwise. These singularities are denoted by (I, b).

Singular Locus. The singular locus $\mathrm{Sing}(A)$ of a singularity $A = \oplus_{i=0}^{\infty} A_i$ is the intersection of the zero sets of $\Delta^{i-1}(A_i), i > 0$.

Transform. Let Z be a straight subvariety in the singular set of A as above. The transformed singularity on the blowup is $\oplus_{i=0}^{\infty} A'_i$, where A'_i is such that $f^*(A_i) = \mathrm{Ideal}(E_{n+1})^i \cdot A'_i$ for $i > 0$.

Resolution. A resolution of a singularity is a sequence of singularity-habitat pairs, where the next is the transform of the previous under blowup of a straight subvariety in the singular locus, such that the last singularity has empty singular locus.

Differential Closure. A singularity $A = \oplus_{i=0}^{\infty} A_i$ is closed iff $\Delta(A_{i+1}) \subseteq A_i$ for all $i > 0$.
The differential closure of a singularity A is the smallest closed singularity containing A.

Equivalence. Two singularities A, B on the same habitat are equivalent iff there exists $N > 0$ such that $\mathrm{Closure}(A)_k N = \mathrm{Closure}(B)_k N$ for all $k \in \mathbb{Z}_+$.

Generating Degree. A number $b > 0$ is a generating degree of a singularity A iff A is equivalent to the ideal-type singularity (A_b, b).

Subhabitat. A subhabitat of a habitat $(W, (E_1, \ldots, E_n))$ is a locally closed straight subvariety $V \subset W$ which is not contained in any E_i, together with the sequence of these intersections $(V \cap E_1, \ldots, V \cap E_n)$.

Restriction. Let $i : V \to W$ the inclusion map of a subhabitat $(V, \_)$ of $(W, \_)$. Let $i^* : K[W] \to K[V]$ be the corresponding homomorphism of function rings. The restriction of a singularity $B = \oplus_{i=0}^{\infty} B_i$ on $(W, \_)$ to $(V, \_)$ is defined as the singularity $A = \oplus_{i=0}^{\infty} A_i$ where $A_i := i^*(B_i) K[V]$. If $\mathrm{Ideal}(V)^i$ is contained in B_i for all $i > 0$, then we say that B restricts properly to V. (In this case $\mathrm{Sing}(A) = \mathrm{Sing}(B)$.)

Extension. Let W, V, i be as above. Let $r : W \to V$ be a left inverse of i. The extension$_1$ of a singularity $A = \oplus_{i=0}^{\infty} A_i$ on $(V, \_)$ to $(W, \_)$ is defined as $B = \oplus_{i=0}^{\infty} B_i$, where $B_i := r^*(A_i) + \mathrm{Ideal}(V)^i$.
Assume that A is closed. Then the extension$_2$ is defined as the largest closed algebra which is contained in $\oplus_{i=0}^{\infty}(i^*)^{-1}(A_i)$ (it may be constructed by induction on i).

Gallimaufry. A gallimaufry is a closed singularity A on a habitat $(W, \_)$ such that there exists a collection of subhabitats $(V_p, \_)$ to which A restricts properly and which cover the singular locus.

Monomial Factor and Order. The monomial factor of an ideal-type singularity (I, b) is the sequence $\left(\frac{e_1}{b}, \ldots, \frac{e_n}{b}\right)$ such that $I = \tilde{I}\mathrm{Ideal}(E_1)^{e_1} \ldots \mathrm{Ideal} E_n{}^{e_n}$ with e_i chosen as big as possible. The monomial factor is not defined if there are divisors in components of W where I is locally zero.
The order of an ideal-type singularity as above is defined as $\frac{\min\{a | \Delta^a(\tilde{I}) = \langle 1 \rangle\}}{b}$.
For general singularities, the monomial factor and the order are defined by passing to an equivalent ideal-type singularity.
For gallimaufries, the monomial factor and the order are defined by restricting to a subhabitat of correct dimension.
A singularity or gallimaufry is called monomial iff it has order 0.

Tightness and tightification. A singularity/gallimaufry is called tight iff it has order 1 and monomial factor $a(i) = 0$ for all i such that $E_i \cap S \neq \emptyset$.
The tightification of an ideal-type singularity (I, b) as in the paragraph above is defined as $(\tilde{I}, c) + (I, b)$ where $c = \mathrm{ord}(\tilde{I}, 1)$.
The tightification of a general singularity is defined by passing to an equivalent ideal type singularity.
The tightification of a gallimaufry is defined by restriction, singularity tightification, and extension.

Descent. Let A be a singularity on a habitat $(W, \_)$. A proper restriction of A to a subhabitat $(V, \_)$ of codimension 1 is called a descent singularity.
Let (A, d) be a gallimaufry. If $(A, d-1)$ is also a gallimaufry, then it is the descent gallimaufry of (A, d).
Intersection. Let $(W, (E_1, \ldots, E_n))$ be a habitat. Let $j \in [n]$. The intersection gallimaufry B has habitat $(W, (E_1, \ldots, E_{j-1}, \emptyset, E_{j+1}, \ldots, E_n))$ and is defined by $B_i = A_i + \mathrm{Ideal}(E_j)^i$ for $i > 0$.