CMI Summerschool 2012 / Course Schicho
Definitions

Let K be a field of characteristic zero.

Habitat. A habitat is an equidimensional nonsingular variety W together with a finite sequence of divisors $(E_1, \ldots, E_n)$, such that for each point $p \in W$, there is a system u of regular parameters of the local ring of W at p such that for $i = 1, \ldots, n$, either E_i does not pass through p or there is a parameter in u which serves as a local equation of E_i; in addition, we want that these local equations are pairwise distinct (or equivalently, the E_i' are not allowed to have common components).
For any habitat, we define an operator Δ from the set of ideals on W to itself, as follows. For $I \subset K[W]$, $\Delta(I)$ is the ideal generated by I and all first order partial derivatives of elements in I.

Straight subvariety. A subvariety Z of a habitat as above is called straight iff it is pure-dimensional, and for every point $p \in Z$, there is a system u of regular parameters as above which additionally satisfies the condition that Z is locally defined by a subset of u.

Blowup. Let Z be a straight subvariety of a habitat $(W, (E_1, \ldots, E_n))$. The blowup along Z is the habitat $(W', (E_1', \ldots, E_n', E_{n+1}))$, where W' is the blowup of W along Z, E_i' is the strict transform of E_i for $i = 1, \ldots, n$, and E_{n+1} is the exceptional divisor.

Singularity. A singularity on a habitat as above is a finitely generated Rees algebra $A = \oplus_{i=0}^{\infty} A_i$ over $A_0 = K[W]$, i.e. sequence of ideals $A_i \subset K[W]$ such that $A_0 = K[W]$ and $A_i \cdot A_j \subseteq A_{i+j}$ and equality holds for sufficiently large indices i, j.
We say a singularity is of ideal-type iff there is a $b > 0$ and ideal I such that $A_{nb} = I^n$ for all indices which are multiples of b, and $A_i = (0)$ otherwise. These singularities are denoted by (I, b).

Singular Locus. The singular locus $\mathrm{Sing}(A)$ of a singularity $A = \oplus_{i=0}^{\infty} A_i$ is the intersection of the zero sets of $\Delta^{i-1}(A_i), i > 0$.

Transform. Let Z be a straight subvariety in the singular set of A as above. The transformed singularity on the blowup is $\oplus_{i=0}^{\infty} A_i'$, where A_i' is such that $f^*(A_i) = \mathrm{Ideal}(E_{n+1})^i \cdot A_i'$ for $i > 0$.

Resolution. A resolution of a singularity is a sequence of singularity-habitat pairs, where the next is the transform of the previous under blowup of a straight subvariety in the singular locus, such that the last singularity has empty singular locus.

Monomial Singularity. A monomial singularity is an ideal-type singularity (I, b) such that there exist integers $a_1, \dots, a_n$ such that $I = \prod_{i=0}^{n} \mathrm{Ideal}(E_i)^{a_i}$.

Equivalence. Two singularities A, B are equivalent – $A \approx B$ – iff their singular loci coincide and, for any sequence of blowups in the center in the singular locus, the singular loci of the transforms of A and B coincide.

Differential Closure. A singularity $A = \oplus_{i=0}^{\infty} A_i$ is closed iff $\Delta(A_{i+1}) \subseteq A_i$ for all $i > 0$.
The differential closure of a singularity A is the smallest closed singularity containing A.

Subhabitat. A subhabitat of a habitat $(W, (E_1, \dots, E_n))$ is a locally closed subvariety $V \subset W$ intersecting each E_i transversally, together with the sequence of these transversal intersections.

Restriction. Let $i : V \to W$ the inclusion map of a subhabitat $(V, \_)$ of $(W, \_)$. Let $i^* : K[W] \to K[V]$ be the corresponding homomorphism of function rings. The restriction of a singularity $B = \oplus_{i=0}^{\infty} B_i$ on $(W, \_)$ to $(V, \_)$ is defined as the singularity $A = \oplus_{i=0}^{\infty} A_i$ where $A_i := i^*(B_i) K[V]$.
If $\mathrm{Ideal}(V)^i$ is contained in B_i for all $i > 0$, then we say that B restricts properly to V. (In this case $\mathrm{Sing}(A) = \mathrm{Sing}(B)$.)

Extension. Let W, V, i be as above. Let $r : W \to V$ be a left inverse of i. The extension$_1$ of a singularity $A = \oplus_{i=0}^{\infty} A_i$ on $(V, \_)$ to $(W, \_)$ is defined as $B = \oplus_{i=0}^{\infty} B_i$, where $B_i := r^*(A_i) + \mathrm{Ideal}(V)^i$.
Assume that A is closed. Then the extension$_2$ is defined as the largest closed algebra which is contained in $\oplus_{i=0}^{\infty} (i^*)^{-1}(A_i)$ (it may be constructed by induction on i).