Universidad Autónoma de Madrid

Commutative Algebra for Singular Algebraic Varieties

Spec with capital letter

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Localization

Let A be a ring and

$S \subset A$ a multiplicative set ($1 \in S$ and if $s_1, s_2 \in S$ then $s_1 \cdot s_2 \in S$).

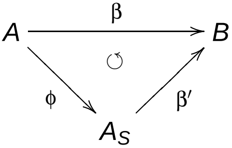
$A_S = \left\{ \frac{a}{s}, a \in A, s \in S \right\}$ where $\frac{a}{s} = \frac{a'}{s'}$ in $A_S \Leftrightarrow t(s'a - a's) = 0$ in A.

The homomorphism $\begin{array}{ccc} A & \xrightarrow{\phi} & A_S \\ a & \longmapsto & \frac{a}{1} \end{array}$ satisfies

a $\phi(S) \subset U(A_S)$,

b **Universal Property of the Localization**

If $A \xrightarrow{\beta} B$ maps S in $U(B)$ then there is a unique β':



Notation

- *If* $S = A \setminus \mathfrak{p}$, $A_S = A_{\mathfrak{p}}$
- *If* $S = \{1, f, \ldots, f^n, \ldots\}$, $A_S = A_f$.

Continuous functions

Let $A \xrightarrow{\beta} B$ be a homomorphism.

$$\mathfrak{q} \subset B \text{ a prime ideal} \Rightarrow \beta^{-1}(\mathfrak{q}) \subset A \text{ is a prime ideal.}$$

We can define

$$\begin{array}{ccc} \mathrm{spec}(A) & \xleftarrow{\pi} & \mathrm{spec}(B) \\ \beta^{-1}(\mathfrak{q}) & \longleftarrow\!\!| & \mathfrak{q} \end{array}$$

The preimage of a closed set is a closed set

$$\begin{array}{ccc} \mathrm{spec}(A) & \xleftarrow{\pi} & \mathrm{spec}(B) \\ \cup & & \cup \\ V(I) & & V(IB) \end{array}$$

A topological space attached to a ring, spec

As a set,

$$\mathrm{spec}(A) = \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } A\}$$

A topology is defined. Let $I \subset A$ an ideal,

$$V(I) := \{\mathfrak{p} \in \mathrm{spec}(A) : \mathfrak{p} \supset I\}$$

is a closed subset in $\mathrm{spec}(A)$.

Example

- $\mathrm{spec}(A/I)$"$=$"$V(I) \subset \mathrm{spec}(A)$,
- $\mathrm{spec}(A_S) \subset \mathrm{spec}(A)$, the set of prime ideals $\mathfrak{p} \subset A$ s. t. $\mathfrak{p} \cap S = \emptyset$.
- $\mathrm{spec}(A_f)$ is an open set of $\mathrm{spec}(A)$.

If $\beta^{-1}(\mathfrak{q}) = \mathfrak{p}$

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ \downarrow & & \downarrow \\ A_{\mathfrak{p}} & \longrightarrow & B_{\mathfrak{q}} \end{array}$$

$$\begin{array}{ccc} \mathrm{spec}(A) & \xleftarrow{\pi} & \mathrm{spec}(B) \\ \cup & & \cup \\ \mathrm{spec}(A_{\mathfrak{p}}) & & \mathrm{spec}(B_{\mathfrak{q}}) \end{array}$$

Schemes

$(X, \mathcal{O}_X)$ where

- X is a topological space
- for each $\mathfrak{p} \in X$ there is a local ring $\mathcal{O}_{X,\mathfrak{p}}$

Morphism of Schemes

$$(X, \mathcal{O}_X) \xleftarrow{\pi} (Y, \mathcal{O}_Y)$$

$$X \xleftarrow{\text{continuous}} Y$$

For each $\mathfrak{q} \in Y$ there exists a homomorphism

$$\mathcal{O}_{X,\pi(\mathfrak{q})} \longrightarrow \mathcal{O}_{Y,\mathfrak{q}}$$

An example: affine schemes

Let A a ring, $\text{Spec}(A) = (X, \mathcal{O}_X)$ is a scheme where

- $X = \text{spec}(A)$,
- for each $\mathfrak{p} \in \text{spec}(A)$, $\mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$.

Morphism of spectra of rings

$$A \xrightarrow{\beta} B$$

$$\text{Spec}(A) \longleftarrow \text{Spec}(B)$$

defined by

$$\text{spec}(A) \longleftarrow \text{spec}(B)$$

and for each $\mathfrak{q} \in B$, there exists a homomorphism

$$A_{\beta^{-1}(\mathfrak{q})} \longrightarrow B_{\mathfrak{q}}$$

There exist open subschemes

Let $(X, \mathcal{O}_X)$ be a scheme.

An open subscheme $(U, \mathcal{O}_U) \subset (X, \mathcal{O}_X)$

- A open subset $U \subset X$,
- $\mathcal{O}_{U,\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}}$ for each $\mathfrak{p} \in U$.

An open affine subscheme

Let $\mathrm{Spec}(A)$ be an affine scheme.

- $U_f = \mathrm{spec}(A_f) \subset \mathrm{spec}(A)$ is an open set,
- $\mathrm{Spec}(A)|_{U_f} =$ the image of $\mathrm{Spec}(A_f)$ in:

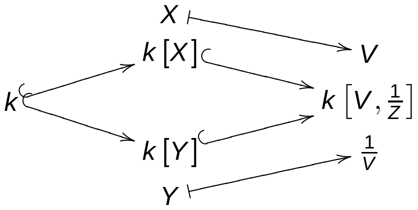
$$\mathrm{Spec}(A) \longleftarrow \mathrm{Spec}(A_f)$$

Thus, $\mathcal{O}_{U_f,\mathfrak{p}} = A_\mathfrak{p}$ $(= (A_f)_\mathfrak{p})$ for each $\mathfrak{p} \in U_f$.

A first example: the projective line $\mathbb{P}^1_k$

Let k be a field. $\mathbb{P}^1_k$ consists in

- two charts:
  - Spec$(k[X])$
  - Spec$(k[Y])$
- glued in a common open set Spec$(k[V, \frac{1}{V}])$,



- And a morphism of schemes

$$\text{Spec}(k) \longleftarrow \mathbb{P}^1$$

A blow up at a point

$$\begin{array}{ccccc} & & \mathbb{C}\left[X, \frac{Y}{X}\right] & & \\ & \nearrow & & \searrow & \\ \mathbb{C}[X,Y] & & & & \mathbb{C}\left[X, \frac{Y}{X}, \frac{X}{Y}\right] = \mathbb{C}\left[\frac{X}{Y}, Y, \frac{Y}{X}\right] \\ & \searrow & & \nearrow & \\ & & \mathbb{C}\left[\frac{X}{Y}, Y\right] & & \end{array}$$

Here the order is the same in the charts...

$$\mathbb{C}[X,Y]_{\langle X,Y\rangle} \hookrightarrow \mathbb{C}\left[X, \frac{Y}{X}\right]_{\langle X\rangle}$$

$$X^2 + Y^3 \longmapsto X^2\left(1 + X\left(\frac{Y}{X}\right)^3\right)$$

A general blow up(I)

Let A be a ring, $I = \langle f_1, \dots, f_r \rangle \subset A$ an ideal.

A chart Spec(B_i)

Consider $A \longrightarrow A_{f_i}$ and $\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i} \in A_{f_i}$.

So, $B_i = A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right] \subset A_{f_i}$ as subring.

$$A \hookrightarrow B_i \hookrightarrow A_{f_i}.$$

A way to glue charts

- $(B_i)_{\frac{f_j}{f_i}} = A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right]\left[\frac{f_i}{f_j}\right] = A\left[\frac{f_1}{f_j}, \dots, \frac{f_r}{f_j}\right]\left[\frac{f_j}{f_i}\right] = (B_j)_{\frac{f_i}{f_j}} \subset A_{f_i f_j}$
- $$\begin{array}{ccc} & B_i \; (\subset A_{f_i}) & \\ A \nearrow & & \searrow (B_i)_{\frac{f_j}{f_i}} = (B_j)_{\frac{f_i}{f_j}} \quad (\subset A_{f_1 f_2}) \\ \searrow & & \nearrow \\ & B_j \; (\subset A_{f_j}) & \end{array}$$

A general blow up(II)

$$
\begin{array}{ccccc}
 & & B_1 & & \\
 & \nearrow & & \searrow & \\
A & & & & (B_1)_{\frac{f_2}{f_1}} = (B_2)_{\frac{f_1}{f_2}} \\
 & \searrow & & \nearrow & \\
 & & B_2 & &
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \mathrm{Spec}(B_1) & & \\
 & \swarrow & & \nwarrow & \\
\mathrm{Spec}(A) & & & & \mathrm{Spec}((B_1)_{\frac{f_2}{f_1}}) = \mathrm{Spec}((B_2)_{\frac{f_1}{f_2}}) \\
 & \nwarrow & & \swarrow & \\
 & & \mathrm{Spec}(B_2) & &
\end{array}
$$