PROJECTIVE SCHEMES AND BLOW-UPS

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Contents

Introduction

The objective of these notes is to discuss projective morphisms, with particular interest in the case of the blow-up of an ideal in a ring.

A precise approach to this topic requires some acquaintance with *scheme theory.* Yet the aim of this presentation is precisely to discuss this subject avoiding the notions of sheave or scheme theory. So at some point we will state some properties of schemes, which the reader should accept, and that should be enough for carrying on with our discussion.

Roughly speaking, the blow-up of a ring at an ideal, is an object obtained by *patching* a finite number of new rings. The aim here is to focus on the precise meaning of patching. Schemes in general are conceived as objects obtained by patching rings in some prescribed way, and we aim to clarify this point.

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Here is a brief summary of the contents. In Section 1 we discuss about the meaning of patching, a point of particular interest for our discussion. Section 2 is devoted to the notion of locally ringed spaces. Affine schemes are presented as a particular example of ringed spaces in Section 3. In Section 4 we discuss about patching locally ringed spaces and present the example of patching affine schemes in Section 5. This is a first step towards the concept of scheme.

Some properties of schemes are presented in Sections 6 and 9, and coherent modules and ideals in Section 7. Projective schemes and graded rings are addressed in Section 8.

Blow-ups of ideals, and the study of their universal properties are presented in Section 10. Transformations of ideals by blow-ups are explained in Section 11. The special case of blow-ups at regular center is treated in Section 12.

All rings are supposed to be commutative, with unity and noetherian. We do not do not restrict our attention to the case of rings of functions on a variety.

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Part I. **Patching affine schemes**

1. On the notion of patching

1.1. Patching sets

It is usual in geometry to produce new objects by patching others. Consider, for simplicity, a surface $\mathcal{C}$, such as a sphere or a torus, that can be entirely covered by finitely many pieces of cloth, say U_1, U_2, $\dots U_r$. We will assume that each U_i, interpreted as a set of points, is mapped bijectively into its image, say $V_i \subset \mathcal{C}$. In other words, assume that points of U_i do not overlap when they are patched on $\mathcal{C}$, and that $\mathcal{C} = \cup V_i$.

Although U_i and V_i can be identified we consider them separately. Clearly $\mathcal{C}$ can be reconstructed by *patching* the sets U_i. The task is to extract the essential information needed to make this reconstruction possible.

This fact is intuitively clear, however some formalism is necessary in order to make this assertion precise.

There is, of course, a natural surjective map, say

$$\pi : \bigsqcup U_l \to \mathcal{C},$$

where the left hand side is the disjoint union. So $\mathcal{C}$ can be obtained as the set of equivalence classes when we consider the equivalence relation on $\bigsqcup U_l$ defined by this function. The drawback of this approach is that it makes use of the existence of $\mathcal{C}$, whereas the question, as it arises in geometry, is to *reconstruct* $\mathcal{C}$. However, this equivalence already contains the clue to our question, as we shall see below.

There are some observations that grow from the previous map π:

A) Since each U_i can be identified with its image V_i, a subset U_{ij} of U_i is defined by considering the subset $V_i \cap V_j$ in V_i. Note here that with this definition $U_{ii} = U_i$.

B) For any two indices $1 \leq i, j \leq r$ there is a naturally defined bijection

$$\alpha_{ij} : U_{ij} \to U_{ji}.$$

Moreover, the following *properties* hold:

B1) $\alpha_{ji} = \alpha_{ij}^{-1}$, and

B2) $\alpha_{ii} = id_{U_i}$ (the identity map on U_i).

C) Given indices $1 \leq i, j, k \leq r$, and points $x_l \in U_l$, with $l \in \{i, j, k\}$,

$$(x_i \in U_{ij} \text{ and } \alpha_{ij}(x_i) = x_j) \wedge (x_j \in U_{jk} \text{ and } \alpha_{jk}(x_j) = x_k) \Rightarrow (x_i \in U_{ik} \text{ and } \alpha_{ik}(x_i) = x_k).$$

The following lemma will settle our question. In fact, it shows that the previous data and the properties in A), B), and C), are all we need to reconstruct the set $\mathcal{C}$.

Lemma 1.2. (Patching Lemma) *Assume we are given subsets U_1, U_2, $\dots U_r$, together with the following information:*

A) *For each $i, j \in \{1, \dots, r\}$, a collection of subsets $U_{ij} \subset U_i$, with $U_{ii} = U_i$;*

B) *For any two indices $1 \leq i, j \leq r$ a bijection*

$$\alpha_{ij} : U_{ij} \to U_{ji},$$

such that:

B1) *$\alpha_{ji} = \alpha_{ij}^{-1}$, and*

B2) *$\alpha_{ii} = id_{U_i}$ (the identity map on U_i).*

C) *Given indices $1 \leq i, j, k \leq r$, and points $x_l \in U_l$, with $l \in \{i, j, k\}$,*

$$(x_i \in U_{ij} \text{ and } \alpha_{ij}(x_i) = x_j) \wedge (x_j \in U_{jk} \text{ and } \alpha_{jk}(x_j) = x_k) \Rightarrow (x_i \in U_{ik} \text{ and } \alpha_{ik}(x_i) = x_k).$$

Then:

1) An equivalence relation is defined on the disjoint union $\bigsqcup U_l$ by setting, for $x_i \in U_i$ and $y_j \in U_j$:

$$x_i R y_j \text{ if } x_i \in U_{ij},\ y_j \in U_{ji},\ \text{and } \alpha_{ij}(x_i) = y_j. \tag{1.2.1}$$

2) If $\mathcal{C}$ denotes the quotient set of $\bigsqcup U_l$ by R, then the natural map

$$U_i \to \mathcal{C}$$

is injective for each index i.

The Lemma sorts out the precise information needed to construct a set $\mathcal{C}$ by patching the sets U_i, where now $\mathcal{C}$ denotes the quotient set of $\bigsqcup U_l$ by R, the equivalence relation in 1). The proof of the Patching Lemma is left as an exercise. Note that the transitivity property of the relation is given by property C). On the other hand 2) follows from property B2).

Remark 1.3. 1) Let $\Lambda = \{1, 2, \dots, r\}$ be the set of indices in the Patching Lemma. The data involved therein are

$$\text{(Sets): } \{U_i, i \in \Lambda;\ U_{ij}, (i, j) \in \Lambda \times \Lambda\}$$

$$\text{(Bijections): } \{\alpha_{ij} : U_{ij} \to U_{ji}, (i,j) \in \Lambda \times \Lambda\},$$

under the conditions given by A), B) and C). The sets $\{U_i, i \in \Lambda\}$ are said to *cover* $\mathcal{C}$, when the images of these maps cover $\mathcal{C}$. If $\Gamma \subset \Lambda$ is a subset and the images of $\{U_i, i \in \Gamma\}$ cover $\mathcal{C}$, then $\{U_i, i \in \Lambda;\ U_{ij}, (i,j) \in \Gamma \times \Gamma\}$ and $\{\alpha_{ij} : U_{ij} \to U_{ji}, (i,j) \in \Gamma \times \Gamma\}$, also fulfill A),B), and C), and define the same set $\mathcal{C}$.

2) There is also a **Topological Patching Lemma**, in which each U_i is a topological space. In this case one requires that each subset U_{ij} be an open subset of U_i, and that each $\alpha_{ij} : U_{ij} \to U_{ji}$ be a homeomorphism. Under these conditions the set $\mathcal{C}$ can be endowed with a topology characterized by the following two properties:

i) Each U_i is an open subset;

ii) The restriction of the topology on U_i coincides with that already defined on this set.

Example 1.4. An illustrative example is that of the projective line over $\mathbb{C}$, $\mathbb{P}^1_{\mathbb{C}}$, which can be realized as the topological space obtained by a quotient space of the circle. This can be covered by open subsets of the complex line with convenient identifications. However, if we want to study $\mathbb{P}^1_{\mathbb{C}}$ from the algebraic geometric point of view, we will want to consider open covers by affine complex lines that respect the underlying algebraic structure. In this case, not every identification between the two open subsets of the affine lines will be allowed, since it will have to be compatible with the ring structure attached to each affine open piece. This motivates the content of the upcoming sections.

2. Locally ringed spaces

Definition 2.1. A *locally ringed space* $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is a topological space $\mathcal{C}$ in which a local ring, say $\mathcal{O}_{\mathcal{C},x}$, is assigned to each point $x \in \mathcal{C}$. A *morphism of locally ringed spaces*

$$\delta_{\mathcal{C},\mathcal{D}} : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$$

is a continuous map of the underlying topological spaces,

$$\delta_{\mathcal{C},\mathcal{D}} : \mathcal{C} \to \mathcal{D},$$

together with a homomorphism of local rings for each $x \in \mathcal{C}$,

$$\delta^*_{\mathcal{C},\mathcal{D}}(x) : \mathcal{O}_{\mathcal{D},\delta_{\mathcal{C},\mathcal{D}}(x)} \to \mathcal{O}_{\mathcal{C},x}.$$

Example 2.2. Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ be a locally ringed space, and let $U \subset \mathcal{C}$ be an open subset. Then the inclusion

$$i : U \hookrightarrow \mathcal{C}$$

induces a morphism of locally ringed spaces in a natural way,

$$(U, \mathcal{O}_U) \to (\mathcal{C}, \mathcal{O}_{\mathcal{C}}).$$

This is usually referred to as a *restriction.*

It follows readily from the definition that a composition of morphisms is a morphism.

Definition 2.3. A morphism of locally ringed spaces

$$\delta_{\mathcal{C},\mathcal{D}} : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$$

is an *isomorphism* if $\delta_{\mathcal{C},\mathcal{D}} : \mathcal{C} \to \mathcal{D}$ is an homeomorphism, and $\delta^*_{\mathcal{C},\mathcal{D}}(x) : \mathcal{O}_{\mathcal{D},\delta_{\mathcal{C},\mathcal{D}}(x)} \to \mathcal{O}_{\mathcal{C},x}$ is an isomorphism of rings for all $x \in \mathcal{C}$. We say that two isomorphic locally ringed spaces, $(\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1})$ and $(\mathcal{C}_2, \mathcal{O}_{\mathcal{C}_2})$, are *identified*, when we fix an isomorphism between them.

3. Affine schemes

In algebraic geometry there is a class of locally ringed spaces called *schemes*. Schemes are locally ringed spaces that satisfy some important extra conditions. Some of these conditions will be mentioned in forthcoming sections. There is also a notion of *morphism of schemes*, which is, in particular, a morphism of locally ringed spaces. We begin by discussing the notion of *affine scheme* and that of *morphism of affine schemes*.

Affine schemes

Let A be a ring. Then $spec(A)$ is a set endowed with a topology[1]. Moreover, a local ring ring A_p can be assigned to each $p \in spec(A)$. Hence A determines a locally ringed space which we denote by $(spec(A), \mathcal{O}_{spec(A)})$. This is an *affine scheme*. The notation $(spec(A), \mathcal{O}_{spec(A)})$ will sometimes be shortened writing $Spec(A)$ instead. However, the reader must be warned of the abuse of notation as $Spec(A)$ is normally equipped with a structure of sheaf, which we will not discuss here.

Morphisms of affine schemes

A homomorphism of rings, say $B \to A$, defines a continuous map

$$f : spec(A) \to spec(B),$$

and it also defines, for each $p \in A$, a local homomorphism of local rings

$$B_p \to A_{f(p)}.$$

So $B \to A$ defines a morphism of affine schemes (of locally ringed spaces)

$$Spec(A) \to Spec(B).$$

All morphisms of affine schemes considered through these notes, say $Spec(A) \to Spec(B)$, will be defined by a ring homomorphism $B \to A$.

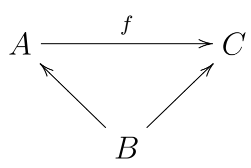
Two homomorphisms, say $C \to B$ and $B \to A$, define morphisms

$$Spec(A) \to Spec(B) \to Spec(C),$$

and the composition is the morphism defined by $C \to A$.

A morphism between two B-algebras is denoted by a commutative diagram:

(3.0.1)



[1] Recall that $spec(A)$ is the set of prime ideals in A with the Zariski topology. The closed sets are collections of primes that contain some ideal $I \subset A$; in particular the subsets $spec(A_f) \subset spec(A)$, with $f \in A$, form a basis of open sets in $spec(A)$.

This induces a commutative diagram of morphisms of affine schemes:

$$\begin{array}{ccc} Spec(A) & \longleftarrow & Spec(C) \\ & \searrow \quad \swarrow & \\ & Spec(B) & \end{array} \tag{3.0.2}$$

3.1. Some illustrative examples: open restrictions and closed immersions

Let A be a ring. There are two basic algebraic constructions with strong geometrical meaning:
a) The localization of A with respect to a multiplicative set S.
b) The quotient of A by an ideal I, say $A \to A/I$.

Localizations. Given a homomorphism $A \to B$ and a multiplicative set S in A, then $A_S \to B_S$ defines a diagram

$$\begin{array}{ccc} Spec(B) & & Spec(B_S) \\ \downarrow & & \downarrow \\ Spec(A) & \longleftarrow & Spec(A_S). \end{array} \tag{3.1.1}$$

Observe that a prime ideal Q in B_S is a prime ideal in B mapping to prime ideals in A_S. In addition, $(B_S)_Q = B_Q$.

The following is an interesting setting within this framework. Fix an element $f \in A$. Then the morphism $A \to A_f$ induces an injective continuous map $spec(A_f) \to spec(A)$. The image is an open set in $spec(A)$, and

$$(A_f)_p = A_p$$

for any $p \in spec(A_f)$. So $Spec(A_f)$ is the natural restriction of $Spec(A)$ to the open set $spec(A_f)$. In this particular case the open restriction of the affine scheme is again an affine scheme. Restrictions of an affine scheme to arbitrary open sets are not affine in general.

Morphisms can also be restricted in the class of affine schemes. Fix an element $f \in A$. A ring homomorphism $A \to B$ induces, say $A_f \to B_f$, by localization. This defines a diagram

$$\begin{array}{ccc} Spec(B) & & Spec(B_f) \\ \downarrow & & \downarrow \\ Spec(A) & \longleftarrow & Spec(A_f) \end{array} \tag{3.1.2}$$

where the second vertical arrow is interpreted as the restriction of the first to the open set $spec(A_f)$ (and its pull-back, which is also an affine scheme). To clarify this point just note that a prime in B maps to a prime in $spec(A_f)$ if, and only if, its image in $spec(A)$ is a prime ideal not containing f.

Another case of interest is that in which $S = A \setminus p$ for some prime p, namely the localization at p, denoted by A_p. Here, prime ideals in $B \otimes_A A_P$ are those mapping to primes included in p.

Quotients. Given an ideal I in A and a homomorphism $A \to B$, then the extended ideal in B, namely IB, is called here *the total transform* of I to $Spec(B)$. In this case, there is a natural diagram

$$\begin{array}{ccc} Spec(B) & \longleftarrow & Spec(B/IB) \\ \downarrow & & \downarrow \\ Spec(A) & \longleftarrow & Spec(A/I) \end{array} \tag{3.1.3}$$

where both horizontal morphisms are closed immersions. So a prime Q in B is in the closed subscheme if and only if it maps to a prime in A that contains I. From a set theoretical point of view, points in $Spec(B/I)$ are the prime ideals in B mapping to the closed set $V(I)$.

Fibers. Finally, and as a combination of (3.1.1) and (3.1.3) one obtains the notion of fiber over a prime ideal p in A. Let $k(p) = A_p/pA_p$, then there is a natural commutative diagram

$$\begin{array}{ccc} Spec(B) & \longleftarrow & Spec(B \otimes_A k(p)) \\ \downarrow & & \downarrow \\ Spec(A) & \longleftarrow & Spec(k(p)) \end{array} \tag{3.1.4}$$

where the points in $Spec(B \otimes_A k(p))$ are identified with the prime ideals in B mapping to the prime p.

4. Patching locally ringed spaces

4.1. On the identification of rings

The notion of identification has appeared in a set theoretical level in Section 1. There is also a notion of *identification of rings* that we will discuss in the following lines. Of course two rings, say B_1 and B_2, that are isomorphic can be identified, in the sense that a property expressed in the language of rings will hold on B_1 if and only if it holds on B_2. In what follows, whenever we say that we *identify* B_1 *with* B_2, or say

$$B_1 = B_2$$

what we really mean is that we prescribe a (unique) isomorphism between them. In other words, that we have fixed an isomorphism, say

$$\beta_{1,2} : B_1 \to B_2,$$

which should be clearly expressed in the context. One obtains the same identification by using the isomorphism $\beta_{2,1} : B_2 \to B_1$ where

$$\beta_{2,1} = \beta_{1,2}^{-1}.$$

An example of identification occurs when considering two multiplicative sets, say S and T in B, so that $S \subset T$. Here we will say that

$$B_T = (B_S)_T.$$

In this particular case the isomorphism to be considered is the unique isomorphism of B-algebras arising from the universal property of localization.

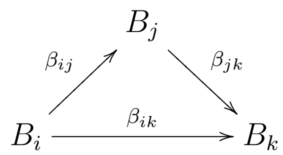
We shall also consider a notion of *simultaneous identification* of several rings, $\{B_1, \dots, B_r\}$. Consider the set of indices $\Lambda = \{1, \dots, r\}$. An identification is defined by fixing, for each pair $(i, j) \in \Lambda \times \Lambda$, an isomorphism

$$\beta_{ij} : B_i \to B_j$$

with the following conditions

1) $\beta_{ii} = id_{B_i}$,
2) $\beta_{ji} = \beta_{ij}^{-1}$, and
3) $\beta_{jk}\beta_{ij} = \beta_{ik}$. Namely, we require the commutativity of all diagrams of the form:

(4.1.1)



Observe that 1), 2), and 3), enable us to define an equivalence relation, say R, on the disjoint union $\coprod_{i \in I} B_i$, which defines a set of classes,

$$D = (\coprod B_i)/R. \tag{4.1.2}$$

In this way, given an element $a_i \in B_i$, one obtains an element $a_j \in B_j$ for any $1 \leq j \leq r$. Moreover, if we fix an index i and two elements, $a_i, b_i \in B_i$, then $a_i + b_i \in B_i$ is naturally identified with the element $a_j + b_j \in B_j$, and the product $a_i b_i$ is naturally identified with the element $a_j b_j \in B_j$ for any index $j \in \Lambda$. So D has a natural structure of ring, and D can be identified with any ring B_i, say

$$D = B_i. \tag{4.1.3}$$

The ring D will be our canonical choice of a representative, or say the *the ring defined by the equivalence relation* on $\{B_1, \dots, B_r\}$. This allows us to reduce the identification of several rings, to the case of two rings.

Sometimes the data $\{B_i, i \in \Lambda;\ \beta_{ij}, (i, j) \in \Lambda \times \Lambda\}$ will be expressed here simply by:

$$B_1 = B_2 = \cdots = B_r.$$

Note that if Γ is an non-empty subset of Λ, then the data $\{B_i, i \in \Gamma, \beta_{ij}, (i, j) \in \Gamma \times \Gamma\}$ also fulfill properties 1), 2), and 3). The corresponding equivalence relation defines a quotient set say D' which is also a ring as indicated above. Clearly D' can be *identified* with D. In other words, there is a naturally defined isomorphism between both rings.

4.2. On the Patching of locally ringed spaces

Let $(U_1, \mathcal{O}_{U_1})$, $(U_2, \mathcal{O}_{U_2}), \dots (U_r, \mathcal{O}_{U_r})$, be locally ringed spaces, and let $\Lambda = \{1, \dots, r\}$. Assume that the following data, consisting of subsets and isomorphism, are given:

A) A collection of open subsets, $U_{ij} \subset U_i$, for $1 \leq i, j \leq r$, with $U_{ii} = U_i$.

B) Setting $(U_{ij}, \mathcal{O}_{U_{ij}})$ as the restriction of $(U_i, \mathcal{O}_{U_i})$, an isomorphism of locally ringed spaces,

$$\alpha_{ij} : (U_{ij}, \mathcal{O}_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{U_{ji}})$$

for all pairs $(i, j) \in \Lambda \times \Lambda$.

Assume, in addition, that:

i) These data define an equivalence relation on the disjoint union $\bigsqcup U_l$ (of the underlying topological spaces) as in Lemma 1.2, so as to define a topological space, say $\mathcal{C}$.

ii) For each $x \in \mathcal{C}$, if $\{x_{i_1}, \dots, x_{i_s}\}$ is the fiber of x defined by

$$\bigsqcup U_l \to \mathcal{C},$$

and if $\Lambda_x = \{i_1, \dots, i_s\}(\subset \Lambda)$, then the corresponding family of local rings

$$\{\mathcal{O}_{U_{i_1}, x_{i_1}}, \dots, \mathcal{O}_{U_{i_r}, x_{i_r}}\} \tag{4.2.1}$$

together with the isomorphisms defined among these local rings,

$$\{\alpha^*_{i_n i_m}(x_{i_n}) : \mathcal{O}_{U_{i_m}, x_{i_m}} \to \mathcal{O}_{U_{i_n}, x_{i_n}};\ (i_n, i_m) \in \Lambda_x \times \Lambda_x\},$$

fulfill conditions 1), 2) and 3) in 4.1.

Then a locally ringed space is defined on $\mathcal{C}$, say

$$(\mathcal{C}, \mathcal{O}_{\mathcal{C}}),$$

where, for $x \in \mathcal{C}$, $\mathcal{O}_{\mathcal{C},x}$, is the local ring obtained from the equivalence relation, i.e.,

$$\mathcal{O}_{\mathcal{C},x} = (\coprod_{l \in \Lambda_x} \mathcal{O}_{U_l, x_l})/R$$

(see 4.1.2).

Remark 4.3. The previous construction provides a natural identification of the locally ringed space $(U_i, \mathcal{O}_{U_i})$, with that defined by the restriction of $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ to U_i for every $i \in \Lambda$.

5. Patching affine schemes

The locally ringed spaces that arise in algebraic and arithmetical geometry are those obtained by patching finitely many affine schemes. In other words, we will focus on locally ringed spaces obtained by patching spaces $(U_1, \mathcal{O}_{U_1})$, $(U_2, \mathcal{O}_{U_2}), \dots (U_r, \mathcal{O}_{U_r})$, where for $i = 1, \dots, r$,

$$(U_i, \mathcal{O}_{U_i}) = Spec(A_i).$$

So implicit in the presentation is the collection of rings $\{A_1, \dots, A_r\}$. However, more information is required in order to make this patching possible. A first step in this direction will be given with the notion of Local-Global data in 5.2.

Definition 5.1. Fix a ring A. We say that an open subset U in $spec(A)$ is an *affine open subset* if there is a ring B, and a ring homomorphism

$$A \to B$$

so that:

1) The induced map $f : spec(B) \to spec(A)$ is injective with image U;

2) The naturally induced homomorphism $A_{f(p)} \to B_p$ is an isomorphism for any $p \in spec(B)$.

The previous definition can be reformulated by saying that $U \subset spec(A)$ is an affine open subset if there is a morphism of affine schemes, say $Spec(B) \to Spec(A)$, so that the image of the underlying spaces is the open set U, and $Spec(B)$ is naturally identified with the restriction of $Spec(A)$ to U. For instance, this occurs when one takes $B = A_g$ for some element $g \in A$, and the homomorphism is that defined by the localization: $A \to A_g$.

5.2. Local-Global Data of rings

Let $\Lambda = \{1, \dots, r\}$. Given a collection of rings, homomorphisms, and isomorphisms:

$$\begin{array}{c} \text{(Rings): } \{A_i, i \in \Lambda;\ A_{ij}, (i,j) \in \Lambda \times \Lambda\} \\ \text{(Ring homomorphisms): } \{A_i \to A_{ij} : i, j \in \Lambda\} \\ \text{(Isomorphisms): } \{\beta_{ij} : A_{ij} \to A_{ji},\ (i,j) \in \Lambda \times \Lambda\}. \end{array} \tag{5.2.1}$$

We say that $U_1 = Spec(A_1), \dots, U_r = Spec(A_r)$ are *patched by the previous data* if:

A*) For each pair $(i,j) \in \Lambda \times \Lambda$, the ring homomorphism

$$A_i \to A_{ij}$$

defines an affine open subset $U_{ij} \subset U_i = spec(A_i)$; with $A_{ii} = A_i$ for $1 \leq i \leq r$.

B*) For each pair $(i,j) \in \Lambda \times \Lambda$, the isomorphism

$$\beta_{ij} : A_{ij} \to A_{ji},$$

is so that:

B_1^*) $\beta_{ji} = \beta_{ij}^{-1}$, and

B_2^*) $\beta_{ii} = id_{A_i}$

C*) The sets U_i, U_{ij} and the isomorphisms $\alpha_{ij} : U_{ij} \to U_{ij}$, induced by $\beta_{ij} : A_{ij} \to A_{ji}$, fulfill condition C) as in the Patching Lemma 1.2;

C**) Given $x \in \mathcal{C}$, and setting $\pi^{-1}(x) = \{x_{i_1}, \dots, x_{i_s}\}$ and $\Lambda_x = \{i_1, \dots, i_s\}$, the local rings

$$\{(A_{i_a})_{x_{i_a}},\ i_a \in \Lambda_x\},$$

and the isomorphisms

$$\{\beta_{i_a, i_b}(x) : (A_{i_a})_{x_{i_a}} \to (A_{i_b})_{x_{i_b}},\ (i_a, i_b) \in \Lambda_x \times \Lambda_x\}, \tag{5.2.2}$$

fulfill conditions 1) 2) and 3) in 4.1.

Under conditions A*-C** an underlying topological space $\mathcal{C}$, together with a map can be defined,

$$\pi : \coprod U_i \to \mathcal{C};$$

and local rings can be constructed by identification,

$$\mathcal{O}_{\mathcal{C},x} = \left(\coprod_{i_a \in \Lambda_x} (A_{i_a})_{x_{i_a}} \right) / R.$$

Here

$$(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \tag{5.2.3}$$

is called the *locally ringed space defined by the local-global data in (5.2.1).* Note that, by construction, the restriction of $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ to the open set U_i is $Spec(A_i)$. In particular, given $x \in U_i \subset \mathcal{C}$, then

$$\mathcal{O}_{\mathcal{C},x} = (A_i)_{x_i}$$

for some prime ideal x_i in A_i (see (4.1.3)).

Remark 5.3. With the same notation as in 5.2, consider a subset Γ of $\Lambda = \{1, \ldots, r\}$. Then the data

$$\{A_i, i \in \Gamma, \beta_{ij} : A_{ij} \to A_{ji}, (i,j) \in \Gamma \times \Gamma\}$$

also fulfill the required conditions from Definition 5.2. Moreover, if the open sets $spec(A_j)$ with $j \in \Gamma$ cover $\mathcal{C}$, then they also define the same locally ringed space $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$.

Remark 5.4. A) In all examples to be considered here, the data $\Lambda = \{1, \ldots, r\}$ and

$$\{A_i, i \in \Lambda;\ \beta_{ij} : A_{ij} \to A_{ji}, (i,j) \in \Lambda \times \Lambda\},$$

in the conditions of 5.2, will arise with the following additional properties:

i) For each index $i \in \Lambda$, there is a set of r elements

$$\{a_{i1}, \ldots, a_{ir}\} \subset A_i,$$

with $a_{ii} = 1$ and such that:

ii) $A_{ij} = (A_i)_{a_{ij}}$ (so $U_{ij} = spec(A_i)_{a_{ij}} \subset spec(A_i)$), and

$$A_i \to A_{ij} = (A_i)_{a_{ij}}$$

is the localization. In other words, for each (i,j), the ring A_{ij} is the localization of A_i in some element $a_{ij} \in A_i$.

B) One can check that the ideal $\langle a_{r,1}, \ldots, a_{r,r-1} \rangle = A_r$ if and only if $U_r = spec(A_r)$ is included in the union of the other open sets, when viewed as open subsets of $\mathcal{C}$. If this is the case, then, fixing $\Gamma = \{1, \ldots, r-1\}$ as set of indices, the rings and isomorphisms

$$\{A_i, i \in \Gamma;\ \beta_{ij} : A_{ij} \to A_{ji}, (i,j) \in \Gamma \times \Gamma\}$$

define the same locally ringed space.

5.5. An illustrative example: Open restrictions

Fix a ring A and an open set U in $spec(A)$. We claim that the restriction of $Spec(A)$ to the open set U can be endowed of local-global data as in 5.2. To clarify this claim we will exhibit a family of rings and homomorphisms as in 5.2.

In the first place, since U is open in $spec(A)$, the complement is a closed set, say $V(I)$, for some ideal $I \subset A$. Assume that $\langle f_1, \ldots f_r \rangle = I$. Then note that U is the union of the affine open subsets $U_1 = spec(A_{f_1}), \ldots, U_r = spec(A_{f_r})$.

Now, given a pair (i,j), $1 \leq i, j \leq r$, define

$$A_{ij} = (A_{f_i})_{f_j}$$

and set

$$\beta_{ij} : A_{ij} \to A_{ji}$$

as the unique A-algebra homomorphism between them extracted from the universal property of localization over A.

Finally check that these collection of rings, homomorphisms and isomorphisms fulfill the conditions stated in 5.2. The outcome is a locally ringed space, denoted by $(U, \mathcal{O}_U)$, which is naturally identified with the restriction of $Spec(A)$ to U.

Now observe that if Γ is a non empty subset of Λ, and if

$$\cup_{j\in\Gamma} spec(A_{f_j}) = U$$

then the local-global data

$$\{A_{f_i}, i \in \Gamma, \beta_{ij} : A_{ij} \to A_{ji}, (i,j) \in \Gamma \times \Gamma\}$$

also patch and define the same locally ringed space $(U, \mathcal{O}_U)$. It follows that $(U, \mathcal{O}_U)$ can also be defined by elements $g_1, \dots g_s$ of A, as long as

$$\cup spec(A_{g_i}) = U.$$

5.6. Affine A-schemes and local-global data of A-algebras

Let $\Lambda = \{1, \dots, r\}$, and consider a collection of rings, homomorphisms and isomorphisms as in 5.2,

$$\text{(Rings): } \{A_i, i \in \Lambda; A_{ij},\ (i,j) \in \Lambda \times \Lambda\}$$

$$\text{(Homomorphisms): } \{A_i \to A_{ij}, i, j \in \Lambda\}$$

$$\text{(Isomorphisms): } \{\beta_{ij} : A_{ij} \to A_{ji},\ (i,j) \in \Lambda \times \Lambda\}.$$

Let A be a ring, and assume that for each index $1 \leq i \leq r$, there is a ring homomorphism

$$\delta_i : A \to A_i.$$

Then, for each pair (i,j), a ring homomorphism

$$A \to A_{ij}$$

is obtained by composition. If *in addition* the isomorphisms

$$\beta_{ij} : A_{ij} \to A_{ji}$$

are compatible with the A-algebra structure $A \to A_{ij}$, for each $i, j \in \Lambda$, then the different morphisms of affine schemes

$$Spec(A_i) \to Spec(A)$$

define a morphism

$$(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to Spec(A).$$

In this case, the locally ringed space $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is said to be an A-*scheme*. This is the first example of an A-scheme, as we will see in Section 6.

Example 5.7. 1)Let A be a ring, and let $U \subset spec(A)$ be an open subset. Then, using the same notation as in 5.5, note that the morphisms $A \to A_{f_i}$ patch to define a morphism:

$$(U, \mathcal{O}_U) \to Spec(A).$$

2) Let A and C be rings. Then a ring homomorphism, $C \to A$, defines a morphism $Spec(C) \to Spec(A)$, and an open restriction $(U, \mathcal{O}_U) \to Spec(A)$ induces a restriction of the first.

Part II. Projective schemes and projective morphisms

6. Properties of A-schemes (I)

Schemes are locally ringed spaces with and additional structure, namely that of a sheaf, which we will not discuss here. Roughly speaking, a scheme is a locally ringed space obtained by patching affine schemes, i.e., spaces of the form $Spec(B)$, where B is a ring. Morphisms among schemes, or say, morphisms of schemes, are those obtained by gluing morphisms of affine schemes. Now, instead of giving the formal definition of scheme and that of morphism of schemes, in these notes we choose to describe them by giving some reasonable properties that they satisfy.

About schemes

A first example of a locally ringed space obtained by patching affine schemes appears in 5.2. But not all schemes are obtained as in. In general, a scheme can be defined by giving an open cover of affine schemes, that do not necessarily patch along open sets that correspond to affine schemes. Consider, for instance, the scheme obtained by patching two copies of the affine plane along the open subset obtained after removing the origin in both of them. However, one of the properties of schemes is that it is always possible to give an open cover by a set of affine schemes with the properties stated in 5.2 (see **Property (C)** below).

About morphisms of schemes

We shall sometimes fix a ring A, and discuss about a subclass in the class of schemes, which we will refer to as *A-schemes*. A first example of morphism defined by patching affine morphisms appears in 5.6, where an example of A-scheme is presented. But not all morphisms of A-schemes arise patching morphisms of affine A-schemes. This fact already appears in the notion of morphism of schemes between two affine schemes (see Property (A) below).

Just to have some intuition...

A scheme $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ will be said to be an *A-scheme* if there is a morphism of locally ringed spaces

$$(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to Spec(A)$$

satisfying certain additional properties (see **Properties (A)** and **(C)** below, see also Remark 6.1).

Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ and $(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$ be A-schemes. A morphism of locally ringed spaces,

$$(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to (\mathcal{D}, \mathcal{O}_{\mathcal{D}}),$$

will be a *morphism of A-schemes* if the diagram of morphisms of locally ringed spaces

$$\begin{array}{ccc} (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) & \longrightarrow & (\mathcal{D}, \mathcal{O}_{\mathcal{D}}) \\ & \searrow \quad \swarrow & \\ & Spec(A) & \end{array} \qquad (6.0.1)$$

commutes, and some additional conditions are satisfied. In this section we will describe morphisms of A-schemes, $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$, when $(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$ is affine (see **Property (D)**). For the general definition we refer to Section 9.

Properties of A-schemes

In the following lines we list some (natural) properties that are required for a locally ringed space to be an A-scheme, and for a morphism of locally ringed spaces to be a morphism of A-schemes.

Property (A): Affine A-schemes and morphisms of affine A-schemes
Within the class of affine schemes, *affine A-schemes* will be the affine schemes defined by the A-algebras. A *morphism of affine A-schemes* is simply a morphism defined by a homomorphisms of A-algebras. So a morphism of affine A-schemes, say

(6.0.2)
$$\begin{array}{ccc} Spec(C) & \longleftarrow & Spec(B) \\ & \searrow \quad \swarrow & \\ & Spec(A) & \end{array}$$

is given by giving a commutative diagram of ring homomorphisms

(6.0.3)
$$\begin{array}{ccc} C & \xrightarrow{f} & B \\ & \nwarrow \quad \nearrow & \\ & A & \end{array}$$

Note that any affine scheme is a $\mathbb{Z}$-scheme. The role of the ring A will be significant for the formulation of some of the further properties to be discussed, particularly **Property (C)**, ii), below. An affine A-scheme of finite type will be one defined by an A-algebra of finite type.

Property (B): Compositions and restrictions
Although we have not defined yet what a morphism of A-schemes is, it is quite natural to ask that the following conditions hold:

1) If $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to (\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1})$ and $(\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1}) \to (\mathcal{C}_2, \mathcal{O}_{\mathcal{C}_2})$ are two morphisms of A-schemes, then the composition $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to (\mathcal{C}_2, \mathcal{O}_{\mathcal{C}_2})$ is also a morphism of A-schemes.

2) The restriction of an A-scheme to an open set is also an A-scheme, and the inclusion is a morphism of A-schemes.

Property (C): Local-global data for A-schemes
If $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is an A-scheme, then there is an open cover $\{U_i, i \in \Lambda\}$ of $\mathcal{C}$, so that:

(i) Each open restriction $(U_i, \mathcal{O}_{U_i})$ is an affine A-scheme, say $(U_i, \mathcal{O}_{U_i}) = Spec(A_i)$ for some A-algebra A_i.

(ii) Each restriction to an open set $U_i \cap U_j$, $(i, j) \in \Lambda \times \Lambda$, is also affine, say

$$(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j}) = Spec(A_{ij}),$$

and the restrictions $Spec(A_{ij}) \to Spec(A_i)$ and $Spec(A_{ij}) \to Spec(A_j)$ are morphisms of affine A-schemes.

Remark 6.1. We stress here that a given A-scheme may admit different open coverings as in **Property (C)**. However, **Property (C)** (ii) is rather a property of the so called *separated schemes*, a notion not discussed here. Formally is not required in the definition of scheme. Not every A-scheme given by local-global data is a separated A-scheme, but those treated in these notes will be within this class.

Property (D): Morphisms of A-schemes
Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ be an A-scheme, let $Spec(B)$ be an affine A-scheme, and let $\{U_i, i \in \Lambda\}$ be an open cover of $\mathcal{C}$ satisfying properties (i) and (ii) from **Property (C)**. Then a *morphism of A-schemes*

$$(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to Spec(B),$$

induces, by composition (see **Property (B)**) morphisms of affine A-schemes,

$$f_i : Spec(A_i) \to Spec(B),$$

so that the diagrams

(6.1.1)

$$\begin{array}{ccccc} & & Spec(A_i) = (U_i, \mathcal{O}_{U_i}) & & \\ & \nearrow & & \searrow & \\ Spec(A_{ij}) = (U_i \cap U_j, \mathcal{O}_{U_i \cap U_j}) & & & & Spec(B) \\ & \searrow & & \nearrow & \\ & & Spec(A_j) = (U_j, \mathcal{O}_{U_j}) & & \end{array}$$

commute for each pair $(i, j) \in \Lambda \times \Lambda$. Thus the construction of a morphism of A-schemes

$$(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to Spec(B),$$

is *equivalent* to the definition of morphisms of affine A-schemes

$$f_i : Spec(A_i) \to Spec(B),$$

so that the diagrams as (6.1.1) commute for all $(i, j) \in \Lambda \times \Lambda$.

By **Property (A)**, this is equivalent to saying that giving a morphism of A-schemes $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to Spec(B)$ is the same as specifying homomorphisms $f_i : B \to A_i$, of A-algebras for each index $i \in \Lambda$, producing commutative diagrams

(6.1.2)

$$\begin{array}{ccccc} & & A_i & & \\ & \swarrow & & \nwarrow & \\ A_{ij} & & & & B \\ & \nwarrow & & \swarrow & \\ & & A_j & & \end{array}$$

for each pair $(i, j) \in \Lambda \times \Lambda$.

6.2. Some illustrative examples: localizations and open restrictions

Open restrictions are important for the study of local properties. Suppose that a morphism of A-schemes, $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to Spec(B)$, is defined by local-global data of A-algebras and homomorphisms of A-algebras as in 5.6 (see also **Property (D)** above). Fix an open restriction, $(U, \mathcal{O}_U)$, of $Spec(B)$, and consider the diagram,

$$\begin{array}{ccc} (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) & & \\ \downarrow & & \\ Spec(B) & \longleftarrow & (U, \mathcal{O}_U). \end{array} \tag{6.2.1}$$

This induces, by taking the pull-back, an open restriction of $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ to the inverse image of U in $\mathcal{C}$. Recall that open restrictions of A-schemes are again within the class (see **Property (B)** (2)).

Let S be a multiplicative set in B. Observe that the localization $B \to B_S$ applied to the local-global data defines a local-global data of A_S-algebras

$$\{(A_i)_S, i \in \Lambda; (A_{ij})_S, (i,j) \in \Lambda \times \Lambda; \beta_{ij} : (A_{ij})_S \to (A_{ji})_S, (ij) \in \Lambda \times \Lambda\}$$

and a morphism of schemes, say

$$(\mathcal{C}_S, \mathcal{O}_{\mathcal{C}_S}) \to Spec(B_S).$$

Of particular interest is the case when S is the multiplicative set defined by the powers of an element $a \in B$. Notice that open sets of the form $spec(B_a)$ form a basis of the topology on $spec(B)$. Observe that $(U, \mathcal{O}_U) = Spec(B_a)$ is an open restriction of $Spec(B)$, and there is a commutative diagram

$$\begin{array}{ccc} (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) & \longleftarrow & (\mathcal{C}_a, \mathcal{O}_{\mathcal{C}_a}) \\ \downarrow & & \downarrow \\ Spec(B) & \longleftarrow & Spec(B_a) \end{array} \tag{6.2.2}$$

in which $(\mathcal{C}_a, \mathcal{O}_{\mathcal{C}_a})$ is obtained, as above, by localization on the local-global data, and both horizontal morphisms are open restrictions. In fact, one can easily check that the open restriction of $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ to the inverse image of $spec(B_a)$ is given by local-global data of A-algebras. This holds because all homomorphisms are assumed to be of A-algebras.

7. Coherent modules

7.1. Identifications of modules

The identification previously discussed for rings (see Section 4), has a natural extension to modules. Assume that two rings A and B have been identified, i.e., that an isomorphism $\beta : A \to B$ has been fixed. We want to define now an *identification of modules*, which in some natural way is compatible with this identification of the rings. If N is an A-module, and M is a B-module, then both are, in particular, abelian groups. We will say that an isomorphism of abelian groups, say

$$\delta : N \to M$$

is compatible with $\beta : A \to B$ if for any $a \in A$ and $n \in N$,

$$\delta(a \cdot n) = \beta(a)\delta(n).$$

This can be reformulated by saying that $\delta : N \to M$ is an isomorphism of A-modules, where M is endowed with an A-module structure via $\beta : A \to B$. Note that $\delta^{-1} : M \to N$ is compatible with $\beta^{-1} : B \to A$.

We will *identify* N with M by fixing a group isomorphism $\gamma : N \to M$ compatible with $\beta : A \to B$.

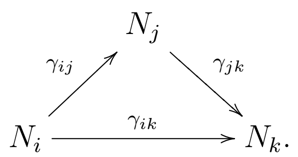
Suppose that an identification of several rings has been fixed: Set $\Lambda = \{1, \dots, r\}$, rings B_i, with $i \in \Lambda$, and an isomorphism $\beta_{ij} : B_i \to B_j$ for each pair $(i, j) \in \Lambda \times \Lambda$, so that conditions 1), 2), and 3) from 4.1 hold.

Given now a B_i-module N_i, we define an *identification* of $\{N_1, \dots, N_r\}$, compatible with the previous identification of rings, by fixing an isomorphism of abelian groups, say

$$\gamma_{ij} : N_i \to N_j,$$

compatible with $\beta_{ij} : B_i \to B_j$, for any pair $(i, j) \in \Lambda \times \Lambda$, and we require that:
1) $\gamma_{ii} = id_{N_i}$,
2) $\gamma_{ji} = \gamma_{ij}^{-1}$, and
3) $\gamma_{jk} \circ \gamma_{ij} = \gamma_{ik}$, i.e., we require the commutativity of the diagrams

(7.1.1)



These isomorphisms of abelian groups define an equivalence relation on their disjoint union, $\coprod_{i\in\Lambda} N_i$, say R, and a quotient set,

$$N = (\coprod N_i)/R.$$

Note that N is an abelian group, and that it has a structure of D-module, where D is the ring obtained by the equivalence relation on the rings $\{B_1, \dots, B_r\}$. Note also that there is now a natural identification of each B_i-module N_i with the D-module N.

Example 7.2. An example arises naturally when defining ideals. Following the notation introduced in 4.1 and 7.1, let I_i be an ideal in B_i for $i = 1, \dots, r$, and assume that $\beta_{ij}(I_i) = I_j$. Then set $N_i = I_i$ and let $\gamma_{ij} : I_i \to I_j$ be the isomorphism of abelian groups obtained by restriction of β_{ij}. Then $N = (\coprod N_i)/R$ defines an ideal on D.

7.3. Modules over locally ringed spaces

Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ be a locally ringed space. An $\mathcal{O}_{\mathcal{C}}$*-module*, say $(\mathcal{C}, \mathcal{N})$, is defined by setting, for each $x \in \mathcal{C}$, an $\mathcal{O}_{\mathcal{C},x}$-module, $\mathcal{N}_x$.

For example, given a ring A and an A-module N, then a $Spec(A)$-module is naturally defined by setting N_p (localization at p) for any $p \in spec(A)$.

Given two isomorphic locally ringed spaces, say $(\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1})$ and $(\mathcal{C}_2, \mathcal{O}_{\mathcal{C}_2})$, an identification is defined by fixing an isomorphism, say

$$\delta_{\mathcal{C}_1,\mathcal{C}_2} : (\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1}) \to (\mathcal{C}_2, \mathcal{O}_{\mathcal{C}_2}).$$

Given a $\mathcal{O}_{\mathcal{C}_1}$-module $\mathcal{N}_1$, and a $\mathcal{O}_{\mathcal{C}_2}$-module $\mathcal{N}_2$, we define an *identification compatible with* $\delta_{\mathcal{C}_1,\mathcal{C}_2}$, say

$$\gamma_{\mathcal{N}_1,\mathcal{N}_2} : \mathcal{N}_1 \to \mathcal{N}_2,$$

to be an isomorphism of abelian groups for each $x \in \mathcal{C}_1$,

$$\gamma_{\mathcal{N}_1,\mathcal{N}_2}(x) : (\mathcal{N}_2)_{(\delta_{\mathcal{C}_1,\mathcal{C}_2})(x))} \to (\mathcal{N}_1)_x,$$

which is compatible with the isomorphism of local rings $\delta_{\mathcal{C}_1,\mathcal{C}_2}(x) : \mathcal{O}_{\mathcal{C}_2,\delta_{\mathcal{C}_1,\mathcal{C}_2}(x)} \to \mathcal{O}_{\mathcal{C}_1,x}$ in the sense of 7.1.

Example 7.4. As an example, fix an isomorphism of rings $\beta : A \to B$, an A-module N, a B-module M, and an isomorphism of abelian groups, say $\delta : N \to M$, compatible with β as in 7.1. Here $\beta : A \to B$ defines an isomorphism $Spec(B) \to Spec(A)$, and $\delta : N \to M$ defines an identification, compatible with this isomorphism, between the $Spec(B)$-module $\mathcal{M}$ defined by M, and the $Spec(A)$-module $\mathcal{N}$ defined by N.

7.5. Coherent modules

Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ be an A-scheme. An $\mathcal{O}_{\mathcal{C}}$-module $\mathcal{N}$ is said to be a *coherent $\mathcal{O}_{\mathcal{C}}$-module* if there is an open cover $\{U_1, \dots, U_r\}$ of $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ together with A-algebras $\{A_1, \dots, A_r\}$, and modules $\{M_1, \dots, M_r\}$, so that:

I) $\{U_1, \dots, U_r\}$ and $\{A_1, \dots, A_r\}$ fulfill the conditions in **Property (C)** from Section 6. In particular, there is an identification of $(U_i, \mathcal{O}_{U_i})$ with $Spec(A_i)$.

II) Each M_i is a finitely generated A_i-module. In particular, M_i defines a $Spec(A_i)$-module, say $\mathcal{M}_i$.

III) There is an identification of the restriction of $\mathcal{N}$ to each U_i, say $(U_i, \mathcal{N}_{U_i})$, with $\mathcal{M}_i$, which is compatible with the identification in (I).

7.6. Three remarks on coherent modules

Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ be an A-scheme, and let $\{U_i, i \in \Lambda\}$ be an open cover as in **Property (C)** from Section 6. So we are assuming here that $(U_i, \mathcal{O}_{U_i}) = Spec(A_i)$, and that $(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j}) = Spec(A_{ij})$ for some rings A_i, A_{ij} for all $i, j \in \Lambda$. Note that if $x \in \mathcal{C}$ is a point in U_i, then $\mathcal{O}_{\mathcal{C},x} = (A_i)_{\mathfrak{p}}$ for a suitable prime ideal $\mathfrak{p}$ in A_i. If $x \in U_i \cap U_j$, then $\mathcal{O}_{\mathcal{C},x} = (A_i)_{\mathfrak{p}} = (A_{ij})_{\mathfrak{p}}$. Therefore the homomorphism $A_i \to A_{ij}$ is such that $spec(A_{ij}) \to spec(A_i)$ is an open inclusion on $spec(A_i)$, and $(A_i)_{\mathfrak{p}} = (A_{ij})_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ in $spec(A_{ij})$. There are elements $g_1, \dots, g_l$ in A_i so that $\{spec((A_i)_{g_1}), \dots, spec((A_i)_{g_l})\}$ is an open cover of $spec(A_{ij})$. A homomorphism $(A_i)_{g_s} \to (A_{ij})_{g_s}$ is defined by localization. The previous observations show that $(A_i)_{g_s} \to (A_{ij})_{g_s}$ is an isomorphism for every index s.

Observe that:

1) The homomorphism $A_i \to A_{ij}$ makes of A_{ij} a flat A_i-algebra. This means that for all short exact sequence of A-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

the corresponding sequence

$$0 \to M_1 \otimes_{A_i} A_{ij} \to M_2 \otimes_{A_i} A_{ij} \to M_3 \otimes_{A_i} A_{ij} \to 0$$

is also exact. The claim follows from the fact that localizations of the form $A_i \to (A_i)_{g_s}$ have this property.

2) A coherent $\mathcal{O}_{\mathcal{C}}$-module as in Definition 7.5 will be given by an A_i-module M_i, $i = 1, \dots, r$, so that

$$M_i \otimes_{A_i} A_{ij} = M_j \otimes_{A_j} A_{ij}$$

for $1 \leq i, j \leq r$. This equality, or say, identification, will appear clearly on the examples, in particular for the case of coherent modules over projective schemes, to be discussed in the next section.

3) An $\mathcal{O}_{\mathcal{C}}$*-ideal* will be presented by an ideal J_i in A_i, for $i = 1, \dots, r$, so that the two extended ideals coincide, i.e.,

$$J_i A_{ij} = J_j A_{ij} \tag{7.6.1}$$

where the left hand side is the extension defined by $A_i \to A_{ij}$, and the other by $A_j \to A_{ij}$.

7.7. Closed subschemes

A closed subscheme of an affine scheme $Spec(A)$ is an affine scheme of the form $Spec(A/J)$ for an ideal J in A. Now let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ be an A-scheme, and let $\{U_i, i \in \Lambda\}$ be an open cover as in **Property (C)** in Section 6. The notion of $\mathcal{O}_{\mathcal{C}}$-ideal leads to the definition of *closed subscheme*. In fact, a new scheme can be constructed by patching the affine schemes $Spec(A_i/J_i)$. The equality in (7.6.1) enables us to replace $A_i \to A_{ij}$ by $A_i/J_i \to A_{ij}/J_iA_{ij}$. In this way an $\mathcal{O}_{\mathcal{C}}$-ideal defines a *closed subscheme* of $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$.

7.8. Coherent modules and local-global data

Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ be an A-scheme given by the following local-global data for a given set of indices $\Lambda = \{1, \dots, r\}$:

$$\text{(A-algebras)} : \{A_i, i \in \Lambda;\ A_{ij};\ (i,j) \in \Lambda \times \Lambda\}$$

$$\text{(Homomorphisms of A-algebras)} : \{A_i \to A_{ij};\ i,j \in \Lambda\}$$

$$\text{(Isomorphisms of A-algebras)} : \{\beta_{ij} : A_{ij} \to A_{ji};\ (i,j) \in \Lambda \times \Lambda\}.$$

Then a coherent $\mathcal{O}_{\mathcal{C}}$-module, say $\mathcal{O}_{\mathcal{N}}$ will be presented by giving a finitely generated A_i-module N_i for each $i \in \Lambda$, and an isomorphism of abelian groups

$$\gamma_{ij} : A_{ij} \otimes N_i \to A_{ji} \otimes N_j$$

for each $(i,j) \in \Lambda \times \Lambda$, with the following properties:

i) γ_{ij} is compatible with $\beta_{ij} : A_{ij} \to A_{ji}$.

Now let $\mathcal{N}_i$ be the $Spec(A_i)$-module obtained from N_i. Note that γ_{ij} provides an identification of the restriction $\mathcal{N}_i$ with the restriction of $\mathcal{N}_j$ along the open set $U_i \cap U_j$. Fix $x \in \mathcal{C}$ and consider the set of indices $\Lambda_x = \{i, 1 \leq i \leq r,$ and $x \in U_i\}$. Note that (i) ensures that for each pair $(i,j) \in \Lambda_x \times \Lambda_x$, the isomorphism

$$\gamma^*_{ij}(x) : (N_i)_x \to (N_j)_x,$$

is compatible with the isomorphism of rings $\beta_{ij}(x) : (A_i)_x \to (A_j)_x$ (see (5.2.2)).

ii) The isomorphisms γ_{ij}, defined for each pair $(i, j) \in \Lambda_x \times \Lambda_x$, fulfill the conditions of equivalence for morphisms of abelian groups in 7.1.

A particular example of coherent module over an A-scheme will be that of an ideal. An ideal will be constructed by fixing an ideal I_i in each ring A_i, so that $\beta_{ij} : A_{ij} \to A_{ji}$ maps I_iA_{ij} to I_jA_{ji}.

Remark 7.9. In most examples to be considered A_{ij} will be the localization of A_i at an element, say $a_{ij} \in A_i$. So $A_i \to A_{ij}$ will be

$$A_i \to (A_i)_{a_{ij}}.$$

In this case a coherent module is presented by giving a finitely generated A_i-module N_i, and an isomorphism of abelian groups

$$\gamma_{ij} : (N_i)_{a_{ij}} \to (N_j)_{a_{ji}}$$

for each $(i, j) \in \Lambda \times \Lambda$, with the prescribed conditions. Moreover, an $\mathcal{O}_{\mathcal{C}}$-ideal will be given by ideals J_i in A_i, $i \in I$, so that

$$\beta_{ij} : (A_i)_{a_{ij}} \to (A_j)_{a_{ji}}$$

maps $(J_i)_{a_{ij}}$ to $(J_j)_{a_{ji}}$.

7.10. Total transforms of ideals

Suppose that a morphism of schemes $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to Spec(A)$ is defined by local-global data of A-algebras and A-homomorphisms as in 5.6. Let $J \subset A$ be an ideal and consider the extended ideal $J_i = JA_i$ in each ring A_i. Then an $\mathcal{O}_{\mathcal{C}}$-ideal is defined by setting $N_i = J_i$. This $\mathcal{O}_{\mathcal{C}}$ -ideal is called the *total transform* of the ideal $J \subset A$ by the morphism $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to Spec(A)$.

Note, in addition, that the local-global data

$$\{A_i/J_i, i \in \Lambda; A_{ij}/J_iA_{ij}, (ij) \in \Lambda \times \Lambda; \beta_{ij} : A_{ij}/J_iA_{ij} \to A_{ji}/J_iA_{ji}, (i, j) \in \Lambda \times \Lambda\}$$

define a scheme, say $(\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1})$ and a morphism

$$(\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1}) \to Spec(A/J).$$

8. Projective Schemes and projective morphisms

In this section we introduce projective schemes, presented here in terms of local-global data. We begin by recalling some properties of graded rings and graded morphisms.

8.1. Graded rings and graded modules

We shall fix a totally ordered semi-group $(T, +)$, typically T will be $\mathbb{Z}$, or $\mathbb{Z}_{\geq N}$, for some integer N, or simply the natural numbers $\mathbb{N}$. A *T-graded ring* R is a ring which is a direct sum of abelian subgroups, say

$$R = \oplus_{i \in T} R_i,$$

where $R_iR_j \subset R_{i+j}$ for all $i, j \in T$.

An R-module M is said to by T-*graded*, or simply *graded*, or *homogeneous*, if it is a direct sum of abelian subgroups, say

$$M = \oplus_{i\in T} M_i$$

and $R_i M_j \subset M_{i+j}$ for all $i, j \in T$.

A non-zero element $m \in M_i$ is said to be *homogeneous of degree* i. An R-submodule of M is also T-graded if it is generated by homogeneous elements.

A morphism between two graded R-modules, say

$$M = \oplus_{i\in T} M_i \to N = \oplus_{i\in T} N_i$$

is said to be *homogeneous*, or *graded*, if it maps homogeneous elements to homogeneous elements of the same degree.

If R is T-graded and if $r \in R$ is homogeneous, the localization R_r is also endowed with a natural graded structure, and

$$R \to R_r$$

is a (homogeneous) homomorphism of graded rings. The same holds, more generally, when R is localized in a multiplicative set S in which all elements are homogeneous. Namely, R_S is graded, and

$$R \to R_S$$

is a homomorphism of graded rings.

Note here that if $T = \mathbb{Z}_{\geq N}$, or if $T = \mathbb{N}$, then R_r is $\mathbb{Z}$-graded. Note also that a morphism of graded R-modules, say

$$M = \oplus_{i\in T} M_i \to N = \oplus_{i\in T} N_i,$$

induces, by localization, a morphism of graded R_S-modules

$$M_S = \oplus_{i\in T} M_i' \to N_S = \oplus_{i\in T} N_i'.$$

We shall fix some conventions to ease the notion: If $M = \oplus_{i\in T} M_i$ is a graded module, then

$$[M]_i = M_i,$$

will denote here the homogeneous part of degree $i \in T$.

8.2. Simple graded rings and the triviality of graded modules

A polynomial ring in one variable over a ring B, say $B[X]$, is graded by $\mathbb{N}$ (by the powers of X).

However, the simplest graded structure is that which is obtained when localizing at the element X. In this case we get $B[X, X^{-1}]$, which is $\mathbb{Z}$-graded, say

$$B[X, X^{-1}] = \oplus_{i\in\mathbb{Z}} BX^i.$$

Th e simplicity of this structure will be justified in Lemmas 8.3 and 8.4.

Lemma 8.3. *Let B be a ring and let X be an indeterminate. Then:*

1) There is a natural correspondence of graded $B[X, X^{-1}]$-modules with B-modules that allows us to identify $B[X, X^{-1}]$-modules and graded morphisms with B-modules and morphisms of B-modules. With this identification exact sequences on one side correspond to exact sequences on the other.

2) There is a natural identification of homogeneous ideals in $B[X, X^{-1}]$ with ideals in B.

3) In the previous correspondence, prime ideals in B are identified with the homogeneous prime ideals in $B[X, X^{-1}]$.

Proof: There is a natural homomorphism $B \to B[X, X^{-1}]$, which defines, for any B-module M, the graded $B[X, X^{-1}]$-module:

$$M \otimes_B B[X, X^{-1}] = \oplus_{i\in\mathbb{Z}} MX^i$$

Also, a morphism of B-modules,

$$f : N \to M$$

induces a homogeneous morphism of graded modules,

$$F : \oplus_{i\in\mathbb{Z}} NX^i \to \oplus_{i\in\mathbb{Z}} MX^i.$$

Moreover, a short exact sequence of B-modules,

$$0 \to N \to M \to P \to 0$$

induces a short exact sequence of homogeneous morphisms:

$$0 \to \oplus_{i\in\mathbb{Z}} NX^i \to \oplus_{i\in\mathbb{Z}} MX^i \to \oplus_{i\in\mathbb{Z}} PX^i \to 0.$$

The point is that any graded $B[X, X^{-1}]$-module, and any homogeneous morphism arises in this way. In fact, one readily checks that if $M = \oplus_{i\in I} M_i$ is a graded $B[X, X^{-1}]$-module, then $M_i = M_0 X^i$, so $M = M_0 \otimes_B B[X, X^{-1}]$. In this way, graded modules, and homogeneous morphisms, over the graded ring $B[X, X^{-1}]$, are naturally identified with the modules and morphisms over the ring B. ⥀

Lemma 8.4. *Let B be a ring and let X be an indeterminate. Then:*

1) If S is a multiplicative set of homogeneous elements in $B[X, X^{-1}]$, then there is a multiplicative set S' of B such that

$$B[X, X^{-1}]_S = (B_{S'})[X, X^{-1}]$$

as graded rings.

2) If bX^k is homogeneous of degree k, then $B[X, X^{-1}]_{bX^k} = (B_b)[X, X^{-1}]$

Proof. It suffices to check the claim when S is defined by the powers of a homogeneous element. A homogeneous element is of the form bX^k, for some $b \in B$ and $k \in \mathbb{Z}$. As X is a unit, so is X^k, in particular the localization at bX^k can be identified with localization at b. Note finally that $B[X, X^{-1}]_b$ is naturally identified with $B_b[X, X^{-1}]$. ⥀

8.5. Graded algebras generated in degree one

Let A be a ring, and consider a graded A-algebra of the form:

$$R = A[x_1, x_2, \ldots, x_l] = \oplus_{i\in\mathbb{N}} R_i$$

where:

i) $A = R_0$; and

ii) Each x_i is homogeneous of degree one.

A graded A-algebra with these properties is said to be *(finitely) generated in degree one.*

An example is that of the polynomials with coefficients in A with the usual grading, $R = A[X_1, X_2, \ldots, X_l]$. Any graded A-algebra generated in degree one, can be expressed as a quotient of $A[X_1, X_2, \ldots, X_l]$ by a homogeneous ideal.

Claim. *If $L \in R = A[x_1, x_2, \ldots, x_l]$ is a homogeneous element of degree one, the localization R_L is a $\mathbb{Z}$-graded ring of the form $B[X, X^{-1}]$ with $B = [R_L]_0$.*

Proof of the claim: Observe that the localization of the $\mathbb{N}$-graded ring $R = A[x_1, x_2, \ldots, x_l]$ at a homogeneous element is graded. Now, set $B = [R_L]_0$. If $a_k \in [R_L]_k$ then

$$b_0 = (a_k)L^{-k} \in [R_L]_0,$$

and of course $a_k = (b_0)L^k$. In other words,

$$R_L = B[L, L^{-1}]$$

as graded rings. Since L is homogeneous of degree one, L^{-1} is homogeneous of degree minus one, and

$$[B[L, L^{-1}]]_k = BL^k$$

for any $k \in \mathbb{Z}$. Therefore L is transcendental over B, and hence

$$R_L = B[L, L^{-1}] = B[X, X^{-1}]$$

as graded rings. ⟲

8.6. Introducing $Proj(R)$ via local-global data

Consider a graded A-algebra finitely generated in degree one,

$$R = A[x_1, x_2, \ldots, x_l] = \oplus_{i \in \mathbb{N}} R_i.$$

Our goal is to attach a locally ringed space to R, which we refer to as the *projective scheme defined by* $R = A[x_1, \ldots, x_l]$, and will be denoted by $Proj(R)$. This space will be an A-scheme, and it will be defined together with a morphism

$$Proj(R) \to Spec(A).$$

When $R = A[X_1, X_2, \ldots, X_l]$ is a polynomial ring, then the underlying topological space of $Proj(R)$ is denoted by $\mathbb{P}_A^{l-1}$, so

$$Proj(R) = (\mathbb{P}_A^{l-1}, \mathcal{O}_{\mathbb{P}^{l-1}}).$$

The scheme $Proj(R)$ will be presented by local-global data of A-algebras as in 5.6.

To start the construction, consider the homogeneous ideal spanned by all elements of degree one:

$$\mathcal{I} = R_1 A[x_1, x_2, \ldots, x_l],$$

and choose a finite set of generators, $\{L_1, \ldots, L_s\} \subset R_1$ so that

$$\mathcal{I} = \langle L_1, \ldots, L_s \rangle. \tag{8.6.1}$$

Define the index set $\Lambda = \{1, \ldots, s\}$, together with the following rings and homomorphisms

$$\{A_i, i \in \Lambda; A_{ij}, (i,j) \in \Lambda \times \Lambda\}$$

$$\{A_i \to A_{ij};\ \beta_{ij} : A_{ij} \to A_{ji},\ (i,j) \in \Lambda \times \Lambda\},$$

as in 5.2, where:

i) $A_i = [R_{L_i}]_0$,

ii) $A_{ij} = [R_{L_iL_j}]_0$, and

iii) $\beta_{ij} : A_{ij} \to A_{ji}$ is obtained by restriction to degree zero of the graded isomorphism

$$(R_{L_i})_{L_j} \to (R_{L_j})_{L_i}.$$

In the following paragraphs we will show that the local-global data given in (i), (ii) and (iii) define a scheme of A-algebras, by proving that all the conditions in 5.2 hold, and that each $\beta_{ij} : A_{ij} \to A_{ji}$ is a homomorphism of A-algebras. The proof will be presented in different steps: 8.7-8.10.

8.7. On the construction of $Proj(R)$: patching two affine schemes

Let $R = A[x_1, x_2, \dots, x_l]$ be as in 8.6, and let L_1 and L_2 be two homogeneous elements of degree one. Let $[R_{L_i}]_0 = A_i$, and accordingly, let $R_{L_i} = A_i[L_i, L_i^{-1}]$ for $i = 1, 2$. We now define, from these data, two elements:

$$a_1 \in [R_{L_1}]_0 = A_1, \quad \text{and} \quad a_2 \in [R_{L_2}]_0 = A_2,$$

and an isomorphism between the localizations, say

$$\beta_{1,2} : ([R_{L_1}]_0)_{a_1} = (A_1)_{a_1} \to ([R_{L_2}]_0)_{a_2} = (A_2)_{a_2}.$$

Observe that $spec((A_1)_{a_1})$ is an open subset in $spec(A_1)$, and that $spec((A_2)_{a_2})$ is an open subset in $spec(A_2)$. The isomorphism $\beta_{1,2}$ will enable us to patch $Spec(A_1)$ with $Spec(A_2)$ along these open sets, as it defines an identification of the two restrictions. We proceed to define this isomorphism in two steps. First consider the homomorphisms defined by localization:

$$R \to R_{L_i}, i = 1, 2.$$

The image of L_1 in R_{L_2} is homogeneous of degree one, and $L_1 = a_2L_2 \in A_2[L_2, L_2^{-1}]$ for some element $a_2 \in A_2$. Similarly, $L_2 = a_1L_1 \in A_1[L_1, L_1^{-1}]$ for some $a_1 \in A_1$. Therefore using Lemma 8.4,

$$(R_{L_1})_{L_2} = (A_1[L_1, L_1^{-1}])_{a_1L_1} = (A_1)_{a_1}[L_1, L_1^{-1}].$$

In the same way,

$$(R_{L_2})_{L_1} = (A_2[L_2, L_2^{-1}])_{a_2L_2} = (A_2)_{a_2}[L_2, L_2^{-1}].$$

Fix the unique isomorphism of R algebras obtained from the universal property of localization,

$$\tilde{\beta}_{1,2} : (R_{L_1})_{L_2} \to (R_{L_2})_{L_1}. \tag{8.7.1}$$

Observe that it is an isomorphism of graded rings. Finally, let

$$\beta_{1,2} : (A_1)_{a_1} \to (A_2)_{a_2} \tag{8.7.2}$$

be the isomorphism obtained by restriction of $\tilde{\beta}_{1,2}$ to degree zero.

Remark 8.8. Let $\mathfrak{q}_1 \subset (A_1)_{a_1}$ be a prime ideal. This can be identified with a prime $\mathfrak{q}_1 \subset A_1$. Let $\mathfrak{q}_2 = \beta_{1,2}(\mathfrak{q}_1)$. Then a homomorphism of local rings is obtained by localization in (8.7.2),

$$\beta_{1,2} : (A_1)_{\mathfrak{q}_1} \to (A_2)_{\mathfrak{q}_2}. \tag{8.8.1}$$

The identification defined by (8.7.2) can be expressed as follows: There are natural identifications of $(R_{L_1})_{L_2}$ and $(R_{L_2})_{L_1}$ with $R_{L_1L_2}$. The localizations $R_{L_1} \to R_{L_1L_2}$, $R_{L_2}. \to R_{L_1L_2}$, and the expressions:

i) $L_2 = a_1 L_1$ in R_{L_1},
ii) $L_1 = a_2 L_2$ in R_{L_2},
define identifications of graded R-algebras:

$$R_{L_1L_2} = (R_{L_1})_{a_1} = (R_{L_2})_{a_2} \tag{8.8.2}$$

Each a_i is of degree zero, and taking restriction to degree zero:

$$(A_1)_{a_1} = [R_{L_1L_2}]_0 = (A_2)_{a_2}. \tag{8.8.3}$$

Notice that $A_{1,2} = (A_1)_{a_1} = [R_{L_1L_2}]_0 = (A_2)_{a_2} = A_{2,1}$. Finally observe that (8.8.1) can be interpreted as a localization of the last equalities at the same prime ideal.

Remark 8.9. By Lemma 8.3 we can identify:
1) Prime ideals of $A_1 = [R_{L_1}]_0$ with homogeneous primes in R_{L_1}.
2) Prime ideals of $(A_1)_{a_1}$, with homogeneous primes in $R_{L_1L_2}$.
3) Prime ideals of $(A_2)_{a_2}$, also correspond to homogeneous primes in $R_{L_1L_2}$.
In addition,
4) $R_{L_1L_2}$ has two structures of simple graded ring:

$$(A_1)_{a_1}[L_1, L_1^{-1}] = R_{L_1L_2} = (A_2)_{a_2}[L_2, L_2^{-1}],$$

and a prime in $(A_1)_{a_1}$ is in correspondence with a prime in $(A_2)_{a_2}$ if and only if both correspond to the same homogeneous prime in $R_{L_1L_2}$.

In other words, a prime $\mathfrak{p}_1 \in spec(A_1)$ is in correspondence with a prime $\mathfrak{p}_2 \in spec(A_2)$, and

$$(A_1)_{\mathfrak{p}_1} = (A_2)_{\mathfrak{p}_2}, \tag{8.9.1}$$

as indicated after (8.8.3), if and only if both primes ideals define the same homogeneous prime in $R_{L_1L_2}$.

8.10. On the construction of $Proj(R)$: patching three affine schemes

Suppose given three elements $L_1, L_2, L_3 \in R = A[x_1, \dots, x_l]$, all homogeneous of degree one. Let

$$A_i = [R_{L_i}]_0,\ i = 1, 2, 3.$$

We will patch now the three affine schemes: $Spec(A_1)$, $Spec(A_2)$, and $Spec(A_3)$.

As in 8.7, consider the localization

$$R \to R_{L_1},$$

and attach to the images of L_2 and L_3 some elements $a_1^{(2)}, a_1^{(3)} \in A_1 = [R_{L_1}]_0$ so that $L_2 = a_1^{(2)} L_1$ and $L_3 = a_1^{(3)} L_1$ in R_{L_1}. Since $1L_1 = L_1$, we set $a_1^{(1)} = 1$, and consider $\{a_1^{(1)}, a_1^{(2)}, a_1^{(3)}\} \subset [R_{L_1}]_0 = A_1$. These elements define, by localization, the rings $(A_1)_{a_1^{(i)}} = [R_{L_1L_i}]_0 = A_{1,i}$ for $i = 1, 2, 3$.

More generally, the images of L_1, L_2, and L_3, via

$$R \to R_{L_i},\ i = 1, 2, 3,$$

define elements $a_i^{(1)}$, $a_i^{(2)}$, $a_i^{(3)} \in [R_{L_i}]_0 = A_i$, so that $L_j = a_i^{(j)} L_i$ in R_{L_i}. This way, we obtain, by localization, the rings $(A_j)_{a_j^{(i)}} = [R_{L_jL_i}]_0 = A_{j,i}$ for $i, j = 1, 2, 3$.

In order to patch three topological spaces $U_1 = spec(A_1)$, $U_2 = spec(A_2)$, and $U_3 = spec(A_3)$, we must specify open sets $U_{ij} (\subset U_i)$, for $1 \leq i, j \leq 3$, and homeomorphisms, $\alpha_{ij} : U_{ij} \to U_{ji}$ that fulfill condition C) from the Patching Lemma 1.2: i.e., if α_{ij} maps a point $\mathfrak{p}_1 \in U_{1,2}$ to, say $\mathfrak{p}_2 = \alpha_{(1,2)}(\mathfrak{p}_1) \in U_2$, and if $\mathfrak{p}_2$ is in $U_{2,3}$, then it is required that:
i) $\mathfrak{p}_1 \in U_{1,3}$, and
ii) $\alpha_{2,3}\alpha_{1,2}(\mathfrak{p}_1) = \alpha_{1,3}(\mathfrak{p}_1)$.

We will check that this condition holds with $U_{i,j} = spec(A_{ij}) = spec([R_{L_i L_j}]_0)$, and $\alpha_{i,j}$ the map induced by the natural ring homomorphisms, $\beta_{i,j} : A_{i,j} \to A_{j,i}$, for $i, j = 1, 2, 3$. In addition, if $\alpha_{2,3}\alpha_{1,2}(\mathfrak{p}_1) = \alpha_{1,3}(\mathfrak{p}_1) = \mathfrak{p}_3$, then there is a commutative diagram of isomorphisms:

(8.10.1)
$$\begin{array}{ccccc} & & (A_2)_{\mathfrak{p}_2} & & \\ & \overset{\beta_{1,2}}{\nearrow} & & \overset{\beta_{2,3}}{\searrow} & \\ (A_1)_{\mathfrak{p}_1} & & \xrightarrow{\beta_{1,3}} & & (A_3)_{\mathfrak{p}_3} \end{array}$$

Assumption 1. *Suppose that $\mathfrak{p}_1$ is a prime of A_1 and that $a_1^{(2)}$ is a unit at the localization $(A_1)_{\mathfrak{p}_1}$ (or equivalently, that $a_1^{(2)} \notin \mathfrak{p}_1$).*

Since $R_{L_1} = A_1[L_1, L_1^{-1}]$, $\mathfrak{p}_1$ defines the homogeneous prime $\mathfrak{p}_1 R_{L_1}$ (see Remark 8.9). Clearly $a_1^{(2)}$ is not included in such homogeneous prime, so $\mathfrak{p}_1 R_{L_1}$ can be identified with a homogenous prime at the localization $R_{L_1L_2}$, say $\mathfrak{p}_1 R_{L_1L_2}$. In fact:

$$R_{L_1} \to R_{L_1L_2} = (R_{L_1})_{L_2} = (R_{L_1})_{a_1^{(2)}},$$

and restricting to degree zero we get:

$$A_1 = [R_{L_1}]_0 \to [R_{L_1L_2}]_0 = (A_1)_{a_1^{(2)}}.$$

Since $R_{L_1L_2} = (R_{L_2})_{L_1}$, the previous arguments applied now to

$$R_{L_2} \to R_{L_1L_2},$$

enable us to define, by restriction to degree zero:

$$A_2 = [R_{L_2}]_0 \to [R_{L_2L_1}]_0 = (A_2)_{a_2^{(1)}}.$$

Briefly, by Assumption 1, the prime $\mathfrak{p}_1$ can be viewed as a homogeneous prime in both R_{L_1}, and $R_{L_1L_2}$. Since there are localization maps,

(8.10.2)
$$\begin{array}{ccc} R_{L_1} & & \\ & \searrow & \\ & & R_{L_1L_2} \\ & \nearrow & \\ R_{L_2} & & \end{array}$$

the prime $\mathfrak{p}_1 R_{L_1L_2}$ contracts to a homogeneous prime in R_{L_2}. Finally, by taking restriction to degree zero, $\mathfrak{p}_1$ can be identified with a prime, $\mathfrak{p}_2 \in A_2 = [R_{L_2}]_0$ which does not contain

$a_2^{(1)}$ (see Remark 8.9). This is how we have defined the identification,

$$\alpha_{1,2} : spec((A_1)_{a_1^2}) = spec(A_{1,2}) \to spec((A_2)_{a_2^{(1)}}) = spec(A_{2,1}).$$

Now we have to show that if $\alpha_{1,2}(\mathfrak{p}_1) \in spec((A_2)_{a_2^{(3)}}) = spec(A_{2,3})$, then $\mathfrak{p}_1 \in spec((A_1)_{a_1^{(3)}}) = spec(A_{1,3})$, and moreover

$$\alpha_{2,3}\alpha_{1,2}(\mathfrak{p}_1) = \alpha_{1,3}(\mathfrak{p}_1).$$

Assumption 2. *Suppose that the previous prime $\mathfrak{p}_2 \subset A_2$ does not contain $a_2^{(3)}$.*

Then arguing as before, $\mathfrak{p}_2$ can be identified with a homogeneous prime at the localization $R_{L_2L_3}$ via

$$R_{L_2} \to R_{L_2L_3}.$$

So, summarizing, using Assumption 1, we identified $\mathfrak{p}_1$ with an homogeneous prime in $R_{L_1L_2}$, and using Assumption 2, we have identified $\mathfrak{p}_2$ with a homogeneous prime in $R_{L_2L_3}$. Both rings $R_{L_1L_2}$ and $R_{L_2L_3}$ have a common homogeneous localization:

(8.10.3)
$$\begin{array}{ccc} R_{L_1L_2} & & \\ & \searrow & \\ & & R_{L_1L_2L_3} \\ & \nearrow & \\ R_{L_2L_3} & & \end{array}$$

Claim 1. *Both homogeneous primes, $\mathfrak{p}_1R_{L_1L_2}$, and $\mathfrak{p}_2R_{L_2L_3}$, extend to homogeneous prime ideals in the localization $R_{L_1L_2L_3}$.*

Proof: By Assumption 1, $\mathfrak{p}_1$ can be viewed as a prime in $R_{L_1L_2}$ because it did not contain $a_1^{(2)}$. There is a natural map from A_2 to $R_{L_1L_2}$, and Assumption 2 says that $a_2^{(3)}$ is not an element of $\mathfrak{p}_1R_{L_1L_2}$.

Set formally, at $R_{L_1L_2}$:

$$a_2^{(3)} = \frac{L_3}{L_2} = \frac{L_3}{L_1}\frac{L_1}{L_2} = a_1^{(3)}(a_1^{(2)})^{-1},$$

to conclude that $\mathfrak{p}_1R_{L_1L_2}$ does not contain $a_1^{(3)}$. So $\mathfrak{p}_1$ extends to a (proper) homogeneous prime in $R_{L_1L_2L_3} = (R_{L_1L_2})_{a_1^{(3)}}$.

Since $\mathfrak{p}_2$ was defined by contraction of $\mathfrak{p}_1R_{L_1L_2}$ via $A_2 \to R_{L_1L_2}$, it follows that

$$\mathfrak{p}_1R_{L_1L_2L_3} = \mathfrak{p}_2R_{L_1L_2L_3}. \quad \circlearrowleft$$

On the other hand $R_{L_1L_2L_3}$ is also a homogeneous localization of $R_{L_1L_3}$, so $\mathfrak{p}_1R_{L_1L_2L_3}$ induces a homogeneous prime in $R_{L_1L_3}$, and hence a homogeneous prime, say $Q_3^{(1)}$ in R_{L_3}. Similarly , $p_2R_{L_1L_2L_3}$ induces an homogeneous prime in $R_{L_2L_3}$, and hence an homogeneous prime, say $Q_3^{(2)}$ in R_{L_3}.

Claim 2. $Q_3^{(1)} = Q_3^{(2)}$.

Proof: The claim follows from the construction since

$$Q_3^{(1)} R_{L_1L_2L_3} = \mathfrak{p}_1 R_{L_1L_2L_3} = \mathfrak{p}_2 R_{L_1L_2L_3} = Q_3^{(2)} R_{L_1L_2L_3}. \quad \circlearrowleft$$

Claim 2 implies that:

$$\alpha_{2,3}\alpha_{1,2}(\mathfrak{p}_1) = \alpha_{1,3}(\mathfrak{p}_1),$$

by taking restriction to degree zero. Moreover, one can check that the conditions in 4.1.1 hold, as all three rings coincide when viewed as a localization of the ring $[R_{L_1L_2L_3}]_0$.

Remark 8.11. 1) The additional conditions stated in Remark 5.4, A) hold in this context, where now $\{a_i^{(1)}, \dots, a_i^{(r)}\} \subset A_i$, is defined so that $a_i^{(j)} L_i = L_j$ at R_{L_i} for $I = 1, \dots, r$ (see 8.7).

2) The observation in Remark 5.4, B) shows that the scheme obtained from the previous data is independent of the choice of generators in (8.6.1). In fact if $\{L_1, \dots, L_s\} \subset R_1$ and $\{L'_1, \dots, L'_r\} \subset R_1$ are two different families of generators of $\mathcal{I} = R_1 A[x_1, x_2, \dots, x_l]$, then so is the union. If we consider the definitions in 8.6 by taking as set of generators $\{L_1, \dots, L_s, L'_1, \dots, L'_r\}$, then Remark 5.4, B) says that such scheme coincides with that defined by $\{L_1, \dots, L_s\} \subset R_1$, and with that defined by $\{L'_1, \dots, L'_r\} \subset R_1$.

8.12. On the underlying topological space of $Proj(R)$

For the graded ring $R = A[x_1, x_2, \dots, x_l] = \oplus_{i \in \mathbb{N}} R_i$, there is a natural identification of the underlying set (topological space) of $Proj(R)$ with a subset of prime ideals in R. We claim that this set is naturally identified with the subset of homogeneous prime ideals not containing the homogeneous ideal

$$\mathcal{I} = R_1 A[x_1, x_2, \dots, x_l].$$

We have fixed generators $\{L_1, \dots, L_s\} \subset R_1$ of this ideal, to provide local-global data used to construct this scheme. In this construction we have made use of the simplicity of a localization of the form R_{L_i}, in the sense that it is a graded ring of the form studied in 8.2.

Homogeneous primes in R_{L_i} are identified with homogeneous primes in R not containing the element L_i. Our construction shows that homogeneous primes in R_{L_i} and in R_{L_j} are identified if and only if they coincide as primes in the localization $R_{L_iL_j}$. In particular, both prime ideals arise from a same homogeneous prime in R, and this prime does not contain the product L_iL_j. This observation already proves our claim.

8.13. Graded R-modules and Coherent modules on $Proj(R)$

Let $R = A[x_1, x_2, \dots, x_l] = \oplus_{i \in \mathbb{N}} R_i$ and let M be a finitely generated R-graded module. We will indicate how to define a coherent $Proj(R)$-module, say $\mathcal{M}$, in terms of M. The construction will show that a homogeneous morphism of graded R-modules, say $M \to N$, will define a morphism of $\mathcal{O}_{Proj(R)}$-modules, say $\mathcal{M} \to \mathcal{N}$. Moreover, a short exact sequence of graded R-modules, say

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

will define a short exact sequence

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0,$$

in the sense that for any $\mathfrak{p} \in Proj(R)$, there is a short exact sequence $0 \to (\mathcal{M}_1)_\mathfrak{p} \to (\mathcal{M}_2)_\mathfrak{p} \to (\mathcal{M}_3)_\mathfrak{p} \to 0$.

Before we start the construction of the $Proj$-sheaf of modules $\mathcal{M}$ defined by the R-module M, observe that given a non-zero element $L \in R_1$, the localization M_L is a graded module over $R_L = [R_L]_0[L, L^{-1}]$. In particular M_L can be canonically identified with the $[R_L]_0$-module $[M_L]_0$. We will follow the notation introduced in 7.5.

We start by describing $Proj(R)$ with local global data as in 8.6. So assume that $\{L_1, \ldots, L_s\} \subset R_1$ is a finite set of generators of

$$\mathcal{I} = R_1 A[x_1, x_2, \ldots, x_l].$$

Set $\Lambda = \{1, \ldots, s\}$, and the following rings and homomorphisms

$$\{A_i = [R_{L_i}]_0; \quad i \in \Lambda; A_{ij} = [R_{L_iL_j}]_0; \quad (i,j) \in \Lambda \times \Lambda\}$$

$$\{A_i \to A_{ij}; \ \beta_{ij} : A_{ij} \to A_{ji}, \ (i,j) \in \Lambda \times \Lambda\},$$

where $A_i = [R_{L_i}]_0 \to A_{ij} = [R_{L_iL_j}]_0$ is a localization of A_i (in fact $[R_{L_iL_j}]_0 = (A_i)_{a_i^{(j)}}$ for a suitable degree zero element $a_i^{(j)} \in A_i$ defined by the equation $a_i^{(j)} L_i = L_j$), and $\beta_{ij} : A_{ij} \to A_{ji}$ is obtained by restriction to degree zero of the graded isomorphism

$$(R_{L_i})_{L_j} \to (R_{L_j})_{L_i}.$$

Let M_i be the finite A_i-module $[M_{L_i}]_0$. The localization $R_{L_i} \to R_{L_iL_j}$ defines a localization of modules, say $N_{L_i} \to N_{L_iL_j}$, and

$$M_i = [M_{L_i}]_0 \to [M_{L_iL_j}]_0$$

is the corresponding localization $M_i \to (M_i)_{a_i^{(j)}}$.

The isomorphism $(R_{L_i})_{L_j} \to (R_{L_j})_{L_i}$ induces an isomorphism of graded abelian groups, say

$$(M_{L_i})_{L_j} \to (M_{L_j})_{L_i}.$$

And the restriction to the degree zero part defines

$$(M_i)_{a_i^{(j)}} \to (M_j)_{a_j^{(i)}},$$

which is naturally compatible with the isomorphism

$$A_{ij} = (A_i)_{a_i^{(j)}} \to (A_j)_{a_j^{(i)}} = A_{ji}.$$

If $\mathfrak{p}_1$ denotes a prime in A_{ij} mapping to $\mathfrak{p}_2$ in A_{ji}, then set

$$\gamma_{ij} : ((M_i)_{a_i^{(j)}})_{\mathfrak{p}_1} \to ((M_j)_{a_j^{(i)}})_{\mathfrak{p}_2}$$

or equivalently

$$\gamma_{ij} : (M_i)_{\mathfrak{p}_1} \to (M_j)_{\mathfrak{p}_2}$$

as the naturally induced map.

Lemma 8.14. *1) A graded R-module M defines a $Proj(R)$-module, say $\mathcal{M}$.*

2) A short exact sequence of graded R-modules, say

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

defines an exact sequence of $Proj(R)$-modules:

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0.$$

Proof: The statements are a consequence of the construction described in 8.13, and the fact that exact sequences of modules are preserved by localization. ↺

Corollary 8.15. *Let $I \subset R$ be a homogenous ideal, and consider the exact sequence of graded R-modules*

$$0 \to I \to R \to \overline{R} \to 0,$$

then:

1) The $Proj(R)$-module, defined by the middle term, is simply $Proj(R)$.

2) The $Proj(R)$-module defined by I is an ideal in $Proj(R)$.

3) $Proj(\overline{R})$ can be identified with the $Proj(R)$-module defined by $\overline{R}$. Therefore $Proj(\overline{R})$ is a closed sub-scheme in $Proj(R)$ defined by the $Proj(R)$-ideal from 2).

Proof: For 2) and 3) observe that the homogeneous ideal I, gives rise to an ideal by the local-global data that we have introduced to define $Proj(R)$ (see 7.8 and 7.10). ↺

Remark 8.16. Given a graded ring generated in degree one, $R = A[x_1, x_2, \dots, x_l]$, we have defined $Proj(R)$ in terms of data consisting of rings: A_i, A_{ij}, and isomorphisms $\beta_{ij} : A_{ij} \to A_{ji}$. Recall here that in this construction all rings are endowed with an A-algebra structure, and the β_{ij} are homomorphisms of A-algebras:

1) An ideal J in A defines an ideal JA_i for each index i, and these ideals define an ideal in $Proj(R)$.

On the other hand, as $J \subset A$, is a set of homogeneous elements of degree zero in $R = A[x_1, x_2, \dots, x_l]$,

$$I = JA[x_1, x_2, \dots, x_l]$$

is a homogeneous ideal in R. So I also defines an ideal in $Proj(R)$, as was indicated in Corollary 8.15. One readily checks, by looking at each A_i, that both $Proj(R)$-ideals coincide.

2) If S is a multiplicative set in A, then $R_S = A_S[x_1, x_2, \dots, x_l]$ is a graded A_S-algebra generated in degree one (8.5). Note here that

$$Proj(R_S) \to Spec(A_S)$$

is defined by localization of the local-global data of A-algebras and A-homomorphisms in 8.6.

8.17. Veronese rings

Let $R = A[x_1, x_2, \dots, x_l] = \oplus R_i$ be a graded ring generated in degree one, and let $N \geq 1$ be a positive integer. Then, a new graded ring can be obtained from R,

$$V^{(N)}(R) = \oplus_{i \in \mathbb{N}} R'_i$$

where $R'_i = R_i$ if i is a multiple of N, and otherwise $R'_i = 0$. $V^{(N)}(R)$ is called the *N-th Veronese ring of R*. It is clearly a subring of R, and the inclusion

$$V^{(N)}(R) \subset R$$

is a finite extension as the N-th power of an homogeneous element in R is in the Veronese subring.

Let

$$V^{(N)}(R) = \oplus_{i\in\mathbb{N}} R_i^{**}$$

where $R_i^{**} = R_{i\cdot N}$. This provides an expression of $V^{(N)}(R)$ as an A-algebra, which is finitely generated in degree one. Therefore, one can also define the A-scheme, say

$$Proj(V^{(N)}(R)) \to Spec(A).$$

Repeating the arguments in 8.6-8.10, local-global data defining $Proj(V^{(N)}(R))$ can obtained by fixing a set of homogenous elements $\{H_1, \dots, H_{l_N}\} \subset R_N$, which generate the A-module R_N. In fact, in this case

$$V^{(N)}(R) = A[H_1, \dots, H_{l_N}].$$

The main property of the Veronese subrings, discussed below, is that they all define the the same projective schemes, i.e., $Proj(V^{(N)}(R)) = Proj(R)$ as A-schemes for all $N \geq 1$. On the other hand the affine covers, defined by the local-global data, are different. The point is that as N grows, one obtains different affine covers. An interesting feature of Veronese rings is the following property: Any open cover of $Proj(R)$ can be refined by the affine cover obtained by a set $\{H_1, \dots, H_{l_N}\} \subset R_N$, for all N big enough.

Claim: *$Proj(V^{(N)}(R)) = Proj(R)$ for any positive integer $N \geq 1$.*

Proof: Let $\{H_1, \dots, H_{l_N}\}$ be the set of all monomial expressions in $\{x_1, \dots, x_s\}$ of degree N. So $V^{(N)}(R) = A[H_1, \dots, H_{l_N}]$. Express the localization of $V^{(N)}(R)$ at H_i in the form

$$(V^{(N)}(R))_{H_i} = B_i[W, W^{-1}]$$

where B_i denotes the subring of elements of degree zero.

One can also localize the inclusion $V^{(N)}(R) \subset R$, say

$$(V^{(N)}(R))_{H_i} = B_{i_0}[W, W^{-1}] \subset R_{H_i}$$

which is a finite extension of graded rings. The N-th powers $\{x_1^N, \dots, x_l^N\}$ are among the monomials $\{H_1, \dots, H_{l_N}\}$, and there is a natural identification:

$$R_{x_i^N} = R_{x_i}$$

for $i = 1, \dots, l$. For those $H_i = x_i^N$, there is an inclusion

$$B_i[W, W^{-1}] \subset R_{x_i} = B_i[T, T^{-1}]$$

where $W = T^N$.

The affine charts obtained from $\{H_1, \dots, H_{l_N}\}$, define $Proj(V^{(N)}(R))$. Among those charts, the ones obtained from $\{x_1^N, \dots, x_l^N\}$, are the same affine charts as that obtained from $\{x_1, \dots, x_l\}$, which define $Proj(R)$.

Finally, consider, in the graded ring $V^{(N)}(R)$, the inclusion of ideals

$$\langle x_1^N, \dots, x_l^N \rangle \subset \langle H_1, \dots, H_{l_N} \rangle,$$

and check that

$$\langle x_1^N, \dots, x_l^N \rangle \supset \langle H_1, \dots, H_{l_N} \rangle^N.$$

The discussion in 8.12 shows now that $Proj(V^{(N)}(R))$ can be covered by the charts (local-global data) defined by the elements $\{x_1^N, \dots, x_l^N\}$. Therefore $Proj(V^{(N)}(R)) = Proj(R)$, and the identification extends to $Proj(V^{(N)}(R)) \to Spec(A)$. Summarizing, among the charts defined by $\{H_1, \dots, H_{l_N}\}$, those corresponding to the N-th powers $\{x_1^N, \dots, x_l^N\}$, already cover the projective scheme. ◯

Note, in particular, that all other affine charts defined by $\{H_1, \dots, H_{l_N}\}$ are localizations of the latter. Take, for example $N = 2$ and $H_i = x_1 x_2$. In this case one can express formally

$$[V^{(2)}(R)_{M_i}]_0 = A\left[\frac{M_1}{x_1x_2}, \dots, \frac{M_{l_2}}{x_1x_2}\right].$$

Check, for example, that

$$\frac{x_1^2}{x_1x_2} = \frac{x_1}{x_2}; \quad \frac{x_2^2}{x_1x_2} = \frac{x_2}{x_1},$$

and furthermore, that this ring is a localization of $[V^{(2)}(R)_{x_1^2}]_0 = [(R)_{x_1}]_0$. In fact, if we set formally

$$[(R)_{x_1}]_0 = A\left[\frac{x_2}{x_1}, \dots, \frac{x_l}{x_1}\right],$$

then

$$[V^{(2)}(R)_{M_i}]_0 = A\left[\frac{x_2}{x_1}, \dots, \frac{x_l}{x_1}\right]_{\frac{x_2}{x_1}}.$$

9. Properties of A-schemes (II)

Up to this point we have introduced an A-scheme $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ by fixing an open cover $\{U_i, i \in \Lambda\}$ of $\mathcal{C}$, and affine A-schemes $Spec(A_i)$, so that $(U_i, \mathcal{O}_{U_i}) = Spec(A_i)$. This was done in in Section 6, where A-schemes were presented by local-global data (see **Property (C)** in Section 6). However, this relation of the A-scheme with the open cover may be relaxed. The point is that the same A-scheme admits different covers by affine open sets. A first step in the clarification of this point is to discuss the notion of *affine open restriction of an A-scheme.*

9.1. On affine open restrictions

Assume that the A-scheme $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ is presented by fixing an affine open cover $\{U_i, i \in \Lambda\}$ of $\mathcal{C}$, where $(U_i, \mathcal{O}_{U_i}) = Spec(A_i)$ (U_i is $spec(A_i)$), and A_i is an A-algebra (see Section 6, **Property (C)**). In this case

$$Spec(A_i) \to (\mathcal{C}, \mathcal{O}_\mathcal{C})$$

is an example of an open affine restriction, and hence it is a morphism of A-schemes (see Section 6, **Property (B)**). In this case we say that $Spec(A_i)(= (\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i}))$ is an *affine chart* of the given affine cover of $(\mathcal{C}, \mathcal{O}_\mathcal{C})$.

In general, an open restriction $(\mathcal{U}, \mathcal{O}_\mathcal{U})$ is said to be an *affine restriction* if $(\mathcal{U}, \mathcal{O}_\mathcal{U}) = Spec(D)$ for some A-algebra D, and the morphism

$$Spec(D) \to (\mathcal{C}, \mathcal{O}_\mathcal{C}) \tag{9.1.1}$$

is a morphism of A-schemes (see Section 6, **Property (D)**). This last condition is very strong, and deserves some clarification.

A particular property of A-schemes, and more precisely, of *separated A-schemes*, is that the intersection of two open affine restrictions is again an affine restriction. In particular $\mathcal{U} \cap U_i$ is affine both in $Spec(A_i)$ and in $Spec(D)$. Moreover, there is an A-algebra, say D_i, and a diagram

(9.1.2)
$$\begin{array}{ccc} & & Spec(A_i) \\ & \nearrow & \\ Spec(D_i) & & \\ & \searrow & \\ & & Spec(D) \end{array}$$

defining the open restrictions in the class of affine A-algebras. This imposes a condition of compatibility of (9.1.1) with the patching of affine schemes from **Property (C)** in Section 6. Not every identification of the form $(\mathcal{U}, \mathcal{O}_{\mathcal{U}}) = Spec(D)$ will fulfill these conditions.

An affine cover of an A-scheme is defined by a cover by affine restrictions. Each such affine restriction is called a *chart (or an affine chart)* of the cover.

Let us mention that if the open affine cover in **Property (C)** from Section 6 is replaced the by another affine cover, then one obtains the same underlying structure of A-scheme.

Property (E): Given an A-scheme together with an arbitrary open cover of the underlying topological space (not necessarily by affine open sets), there is a refinement of this cover by an affine open cover of $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$, by affine open sets $\{U_i, i \in \Lambda\}$ of $\mathcal{C}$.

Property (F): Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ and $(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$ be two A-schemes. A morphism of locally ringed spaces

(9.1.3)
$$(\mathcal{D}, \mathcal{O}_{\mathcal{D}}) \to (\mathcal{C}, \mathcal{O}_{\mathcal{C}})$$

is a morphism of A-schemes if there is an affine cover of $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ by affine open sets $\{U_i, i \in \Lambda\}$ as in **Property (C)** from Section 6, so that taking $V_i \subset \mathcal{D}$ as the pull back of U_i in $\mathcal{C}$, and $(V_i, \mathcal{O}_{V_i})$ as the restriction of $(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$, the restriction of the morphism, say

$$(V_i, \mathcal{O}_{V_i}) \to (U_i, \mathcal{O}_{U_i}) = Spec(A_i)$$

is a morphism of A-schemes as defined in **Property (D)** in Section 6, for each index $i \in \Lambda$.

Here we have fixed an affine cover $\{U_i, i \in \Lambda\}$ of $\mathcal{C}$, and then we have defined an open cover of $\mathcal{D}$ by setting $\{V_i = f^{-1}(U_i), i \in \Lambda\}$, where $f : \mathcal{D} \to \mathcal{C}$ is the continuos map on the underlying topological spaces.

Recall that the restriction of an A-scheme to an open set is an A-scheme (see **Property (B)** in Section 6). So a morphism of A-schemes is obtained by patching morphisms of affine

A-schemes, or say by commutative diagrams of the form:

$$\begin{array}{ccccc} & & (V_i,\mathcal{O}_{V_i}) \longrightarrow Spec(A_i) & & \\ & \nearrow & & \nwarrow & \\ (V_i\cap V_j,\mathcal{O}_{V_i\cap V_j}) & & \longrightarrow & & Spec(A_{ij}) \\ & \searrow & & \swarrow & \\ & & (V_j,\mathcal{O}_{V_j}) \longrightarrow Spec(A_j) & & \end{array} \tag{9.1.4}$$

Suppose that W_{i_0} is an affine chart in V_i, say $(W_{i_0},\mathcal{O}_{W_{i_0}}) = Spec(B_{i_0})$, and that W_{j_0} is an affine chart in V_j, and let $(W_{j_0},\mathcal{O}_{W_{j_0}}) = Spec(B_{j_0})$. This situation arises, as above, when we choose an affine cover of $\mathcal{D}$ which refines the open cover $\{V_i = f^{-1}(U_i), i \in I\}$ (by **Property (E)**). In this latter case the morphism is characterized by commutative diagrams of the form (9.1.5)

$$\begin{array}{ccccc} & & (W_{i_0},\mathcal{O}_{W_{i_0}}) = Spec(B_{i_0}) \longrightarrow Spec(A_i) & & \\ & \nearrow & & \nwarrow & \\ (W_{i_0}\cap W_{j_0},\mathcal{O}_{W_{i_0}\cap W_{j_0}},) = Spec(B_{i,j}) & & \longrightarrow & & Spec(A_{ij}) \\ & \searrow & & \swarrow & \\ & & (W_{j_0},\mathcal{O}_{W_{j_0}}) = Spec(B_{j_0}) \longrightarrow Spec(A_j) & & \end{array}$$

So from this point of view, every A-scheme is obtained by patching affine A-schemes, and morphisms of A-schemes are obtained by patching morphisms of affine A-schemes.

Remark 9.2. In the formulation of **Property (F)** we have fixed a morphism of locally ringed spaces between two A-schemes. However this condition can also be relaxed. This is due to the fact that morphisms of A-schemes can be glued. Suppose that $(\mathcal{D},\mathcal{O}_{\mathcal{D}})$ and $(\mathcal{C},\mathcal{O}_{\mathcal{C}})$ are two A-schemes, and suppose given two covers, on $\mathcal{C}$ and $\mathcal{D}$ respectively, with the same index set, say $\{U_i, i \in \Lambda\}$ of $\mathcal{C}$, and $\{V_i, i \in \Lambda\}$ of $\mathcal{D}$. Assume, for simplicity, that $\{U_i, i \in \Lambda\}$ is an affine cover, say $(U_i,\mathcal{O}_{U_i}) = Spec(A_i)$ and that morphisms of A-schemes, are defined for each $i \in \Lambda$,

$$(V_i,\mathcal{O}_{V_i}) \xrightarrow{f_i} Spec(A_i). \tag{9.2.1}$$

The morphisms f_i are said to glue when all diagrams (9.1.4) commute. If such is the case, there is a morphisms of A-schemes, say

$$(\mathcal{D},\mathcal{O}_{\mathcal{D}}) \xrightarrow{f} (\mathcal{C},\mathcal{O}_{\mathcal{C}}) \tag{9.2.2}$$

inducing the morphisms f_i by restrictions.

Let us stress that given morphisms as in (9.2.1), the criterion used to ensure that theses morphisms glue is given by the commutativity of all diagrams (9.1.5). This is possible once we fix a refinement of the cover $\{V_i, i \in I\}$ by an affine cover of $\mathcal{D}$.

Part III. **Blow-ups**

10. The blow-up of an ideal

10.1. The construction of the blow-up

Let A be a ring, let $I \subset A$ be an ideal, and consider the ring $R = \oplus_{k\in\mathbb{N}} I^k$, where $I^0 = A$. Note that R is a graded ring. We will usually write

$$R = \oplus_{k\in\mathbb{N}} I^k W^k$$

where W is a variable over A used just to recall the grading. Observe that there is an inclusion of graded rings:

$$R = \oplus_{k\in\mathbb{N}} I^k W^k \subset A[W].$$

Now suppose that $I = \langle f_1, \ldots, f_r\rangle$. Then

(10.1.1) $$R = A[f_1W, \ldots, f_rW](\subset A[W]).$$

Define $\Lambda = \{1, \cdots, r\}$, and set $L_i = f_iW$ for $i \in \Lambda$. Arguing as in 8.6, the following local-global data can be considered:

i) $A_i = [R_{L_i}]_0$;
ii) $A_{ij} = [(R_{L_i})_{L_j}]_0$;
iii) and the isomorphisms $\beta_{ij} : A_{ij} \to A_{ji}$.

This local-global data describes the A-scheme, $ProjR$ as in 8.6, which in this particular case is referred to as *the blow-up of A at the ideal I* and denoted by $Bl_I(A)$. The purpose of this section is to study some properties of this projective scheme.

10.2. A closer look at the rings $A_i = [R_{L_i}]_0$

Note that $L_i = f_iW$ is not necessarily a product of two elements in the graded ring R. However L_i is the product of $f_i \in A$ with W when viewed in $A[W]$, and therefore

$$R_{L_i} \subset A[W]_{L_i} = A_{f_i}[W, W^{-1}]$$

as graded rings. In particular

(10.2.1) $$A_i = [R_{L_i}]_0 \subset [A_{f_i}[W, W^{-1}]]_0 = A_{f_i}$$

Rewrite (10.1.1) as $R = A[L_1, \cdots, L_r]$, where each L_i is homogeneous of degree one. Here $R_{L_i} = A[L_1, \cdots, L_r]_{L_i}$, and set, formally,

$$[R_{L_i}]_0 = A\left[\frac{L_1}{L_i}, \ldots, \frac{L_1}{L_i}\right].$$

Finally, the inclusion (10.2.1) says that

(10.2.2) $$A_i = [R_{L_i}]_0 = A\left[\frac{f_1}{f_i}, \ldots \frac{f_r}{f_i}\right] \subset A_{f_i}.$$

This expression provides each A_i with a structure of A-algebra together with a precise presentation of A_i as a subring of the localization A_{f_i},

(10.2.3) $$A \to A_i = A\left[\frac{f_1}{f_i}, \ldots \frac{f_r}{f_i}\right] (\subset A_{f_i}).$$

10.3. A closer look at the rings A_{ij}

Recall that $B_{i,j} = [(R_{L_i})_{L_j}]_0$. Now

(10.3.1) $$R_{L_iL_j} \subset A[W]_{L_iL_j} = A_{f_if_j}[W, W^{-1}]$$

¿From the expression $R = A[L_1, \cdots, L_r]$, we get $R_{L_iL_j} = A[L_1, \cdots, L_r]_{L_iL_j}$ and

$$[R_{L_iL_j}]_0 = A\left[\frac{L_1}{L_i}, \ldots, \frac{L_1}{L_i}\right]\left[\frac{L_i}{L_j}\right] = A\left[\frac{L_1}{L_j}, \ldots, \frac{L_1}{L_j}\right]\left[\frac{L_j}{L_i}\right]$$

Finally, the inclusion (10.3.1) gives a precise formulation of the formal expression:

(10.3.2) $$A_{ij} = [R_{L_iL_j}]_0 = A\left[\frac{f_1}{f_i}, \ldots \frac{f_r}{f_i}\right]\left[\frac{f_i}{f_j}\right]\left(= A\left[\frac{f_1}{f_j}, \ldots \frac{f_r}{f_j}\right]\left[\frac{f_j}{f_i}\right]\right) \subset A_{f_if_j}.$$

Here $A\left[\frac{f_1}{f_i}, \ldots \frac{f_r}{f_i}\right]\left[\frac{f_i}{f_j}\right]$ expresses the localization of $A_i = A\left[\frac{f_1}{f_i}, \ldots \frac{f_r}{f_i}\right]$ at the element $\frac{f_j}{f_i}$, say

(10.3.3) $$A_{ij} = (A_i)_{\frac{f_j}{f_i}}$$

10.4. A closer look at the isomorphisms β_{ij}

The equality $A\left[\frac{f_1}{f_i}, \ldots \frac{f_r}{f_i}\right]\left[\frac{f_i}{f_j}\right] = A\left[\frac{f_1}{f_j}, \ldots \frac{f_r}{f_j}\right]\left[\frac{f_j}{f_i}\right]$ can be easily checked in the ring $A_{f_if_j}$. It says that

$$A_{ij} = (A_i)_{\frac{f_j}{f_i}} = A_{ji} = (A_j)_{\frac{f_i}{f_j}}$$

when viewed as subrings of $A_{f_if_j}$. Here $\beta_{ij} : A_{ij} \to A_{ji}$ is the isomorphism that defines this identification, namely:

$$\beta_{ij}\left(\frac{f_s}{f_i}\right) = \frac{f_s}{f_j}\left(\frac{f_i}{f_j}\right)^{-1} \quad s = 1, \ldots r.$$

10.5. Towards the Universal Property of the blow-up I

A particular but significant example is that in which the ideal $I = \langle f_1, \ldots, f_r\rangle$ of A is a free A-module, and generated by f_1. Then A is a subring of A_{f_1}, as f_1 is a non-zero divisor, and

$$A_1 = A\left[\frac{f_2}{f_1}, \ldots \frac{f_r}{f_1}\right] = A.$$

For any other index $i \neq 1$, the ring A_i in 10.2.3, can be expressed now as:

$$A\left[\frac{f_1}{f_i}, \ldots \frac{f_r}{f_i}\right] = A\left[\frac{f_2}{f_1}, \ldots \frac{f_r}{f_1}\right]\left[\frac{f_1}{f_i}, \ldots \frac{f_r}{f_i}\right] (\subset A_{f_i}).$$

Note that

$$A\left[\frac{f_2}{f_1}, \ldots \frac{f_r}{f_1}\right]\left[\frac{f_1}{f_i}, \ldots \frac{f_r}{f_i}\right] = A\left[\frac{f_2}{f_1}, \ldots \frac{f_r}{f_1}\right]\left[\frac{f_1}{f_i}\right] (\subset A_{f_i}).$$

So for each index $i \neq 1$, A_i can be identified with the localization of A at the element $\frac{f_i}{f_1} \in A$. Therefore, in this particular case $Proj(R) = Spec(A)$, or equivalently, $Proj(R) \to Spec(A)$ is an isomorphism.

10.6. Towards the Universal Property of the blow-up II

Let (B, M) be a local ring, and an A-algebra. Assume that IB is free.

Claim 1. *If IB generated by f_1, then there is a unique morphism of A-algebras*

$$A_1 = A\left[\frac{f_2}{f_1}, \dots \frac{f_r}{f_1}\right] \to B.$$

Proof: Since f_1 is a non-zero divisor in B, there is an inclusion of B in the localization B_{f_1}. We claim that the morphism $A_{f_1} \to B_{f_1}$, defined by localization, maps the subring $A\left[\frac{f_2}{f_1}, \dots \frac{f_r}{f_1}\right]$ into the subring B. To this end we write a system of r equations:

(10.6.1) $$f_i = X_i f_1,\ i = 1, 2, \dots, r.$$

Since the rings A_1, B, and B_{f_1} are A-algebras, the equations can be formulated on each of these three rings. In each case the system has a solution, and moreover, such solution is unique. On the ring $A\left[\frac{f_2}{f_1}, \dots \frac{f_r}{f_1}\right]$, the unique solution is given by $X_i = \frac{f_i}{f_1}$ for $i = 1, \dots, r$, so the morphism $A_{f_1} \to B_{f_1}$ maps each $\frac{f_i}{f_1}$ to the (unique) solution of equation $f_i = X_i f_1$ in B. Let $\{c_1^{(1)}, \dots, c_r^{(1)}\}$ be the uniquely defined elements in B which fulfill the equations (10.6.2), namely,

(10.6.2) $$f_i = c_i^{(1)} f_1,\ i = 1, \dots, r.$$

Then the unique morphism of A-algebras is defined as:

(10.6.3) $$\begin{array}{ccc} A\left[\frac{f_2}{f_1}, \dots \frac{f_r}{f_1}\right] & \longrightarrow & B \\ \frac{f_i}{f_1} & \longmapsto & c_i^{(1)}. \end{array}$$

This discussion proves the existence of the morphism, and also its uniqueness.

In the previous definition of the morphism (10.6.3) we have assumed that IB is a free B-module, and generated by f_1. Conversely, if IB is free over B, and the morphism is defined, then $\{f_1\}$ is a basis of IB. In brief, if IB is free, the existence of (10.6.3) is *equivalent* to the requirement that $\{f_1\}$ be a basis of IB.

Let $\mathfrak{q}_1 \in spec\left(A\left[\frac{f_2}{f_1}, \dots \frac{f_r}{f_1}\right]\right)$ denote the image of M, the maximal ideal of B. Suppose now that the free module IB is also generated by f_2, or equivalently, that $c_2^{(1)}$ is a unit in B. As $\frac{f_2}{f_1}$ maps to $c_2^{(1)}$, this will occur if and only if $\frac{f_2}{f_1}$ is not an element in $\mathfrak{q}_1$. Therefore, $\{f_2\}$ is also a basis of the free B-module IB if and only if

$$\mathfrak{q}_1 \in spec\left(A\left[\frac{f_2}{f_1}, \dots \frac{f_r}{f_1}\right]_{\frac{f_2}{f_1}}\right)\left(\subset spec(A\left[\frac{f_2}{f_1}, \dots \frac{f_r}{f_1}\right]\right),$$

and in this case a morphism

(10.6.4) $$A\left[\frac{f_1}{f_2}, \dots \frac{f_r}{f_2}\right] \to B$$

is also defined. Let $\mathfrak{q}_2 \in spec\left(A\left[\frac{f_1}{f_2},\dots\frac{f_r}{f_2}\right]\right)$ denote the image of the maximal ideal M. The same argument used above says that the system of equations

$$f_j = Y_i f_2,\ i = 1, 2, \dots, r, \tag{10.6.5}$$

has a unique solution in B, say $Y_i = c_i^{(2)}$ for $i = 1, \dots, r$. Thus the morphism in (10.6.4) is defined by mapping $\frac{f_j}{f_2}$ to $c_2^{(j)}$, and

$$q_2 \in spec\left(A\left[\frac{f_1}{f_2},\dots\frac{f_r}{f_2}\right]_{\frac{f_1}{f_2}}\right)\left(\subset spec\left(A\left[\frac{f_2}{f_1},\dots\frac{f_r}{f_1}\right]\right)\right).$$

So if IB is free, and the two sets, say $\{f_1\}$ and $\{f_2\}$, are both bases of IB, then the morphism from (10.6.3) is defined and factors through

$$\begin{array}{ccc} & A\left[\frac{f_2}{f_1},\dots\frac{f_r}{f_1}\right] & \\ \swarrow & & \searrow \\ A\left[\frac{f_2}{f_1},\dots\frac{f_r}{f_1}\right]_{\frac{f_2}{f_1}} & \longrightarrow & B \end{array} \tag{10.6.6}$$

and the morphism from (10.6.4) is also defined, and induces a commutative diagram

$$\begin{array}{ccc} A\left[\frac{f_1}{f_2},\dots\frac{f_r}{f_2}\right]_{\frac{f_1}{f_2}} & \longrightarrow & B \\ \nwarrow & & \nearrow \\ & A\left[\frac{f_1}{f_2},\dots\frac{f_r}{f_2}\right]. & \end{array} \tag{10.6.7}$$

Claim 2. *If $\{f_1\}$ and $\{f_2\}$ are two bases of IB, then the morphisms (10.6.6) and (10.6.7) are compatible with the identification*

$$A\left[\frac{f_2}{f_1},\dots\frac{f_r}{f_1}\right]_{\frac{f_2}{f_1}} = A\left[\frac{f_1}{f_2},\dots\frac{f_r}{f_2}\right]_{\frac{f_1}{f_2}}. \tag{10.6.8}$$

Proof: First notice that the identification (10.6.8) can be expressed by the equalities:

$$\frac{f_j}{f_1} = \frac{f_j}{f_2}\left(\frac{f_1}{f_2}\right)^{-1} \quad j = 1, 2, \dots, r \tag{10.6.9}$$

In fact the isomorphism of A-algebras

$$\beta_{1,2} : A\left[\frac{f_2}{f_1},\dots\frac{f_r}{f_1}\right]_{\frac{f_2}{f_1}} \to A\left[\frac{f_1}{f_2},\dots\frac{f_r}{f_2}\right]_{\frac{f_1}{f_2}}$$

is defined by

$$\beta_{1,2}\left(\frac{f_j}{f_1}\right) = \frac{f_j}{f_2}\left(\frac{f_1}{f_2}\right)^{-1} \quad j = 1, 2, \dots, r \tag{10.6.10}$$

We claim that the equalities

$$c_j^{(1)} = c_j^{(2)}(c_1^{(2)})^{-1} \quad j = 1, 2, \dots, r \tag{10.6.11}$$

hold in B, and, in particular, that the horizontal morphisms in (10.6.6) and in (10.6.7) are compatible with the identification.

Recall that $\{c_1^{(1)}, \dots, c_r^{(1)}\}$ are the elements in B which fulfill the equations

(10.6.12) $$f_j = c_j^{(1)} f_1,\ j = 1, \dots, r,$$

and $\{c_1^{(2)}, \dots, c_r^{(2)}\}$ fulfill the equations:

(10.6.13) $$f_j = c_j^{(2)} f_2,\ i = 1, \dots, r.$$

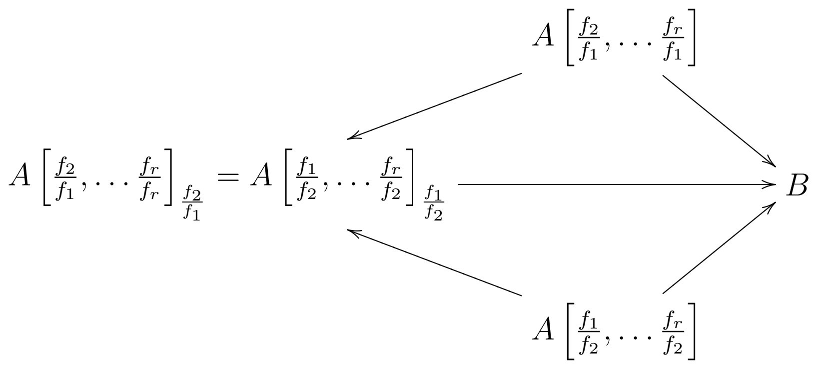
In particular $f_1 = c_1^{(2)} f_2$ in B. Recall also that $c_1^{(2)}$ is assumed to be a unit. Finally check that (10.6.11) follow from:

$$f_j = c_j^{(1)} f_1 = c_j^{(1)} c_1^{(2)} f_2$$

and (10.6.13).

The commutativity of the diagram

(10.6.14)



ensures that $\mathfrak{q}_1 = \mathfrak{q}_2$ as primes in $A\left[\frac{f_2}{f_1}, \dots \frac{f_r}{f_1}\right]_{\frac{f_2}{f_1}} = A\left[\frac{f_1}{f_2}, \dots \frac{f_r}{f_2}\right]_{\frac{f_1}{f_2}}$. Or equivalently, the isomorphism

$$\beta_{1,2} : A\left[\frac{f_2}{f_1}, \dots \frac{f_r}{f_1}\right]_{\frac{f_2}{f_1}} \to A\left[\frac{f_1}{f_2}, \dots \frac{f_r}{f_2}\right]_{\frac{f_1}{f_2}}$$

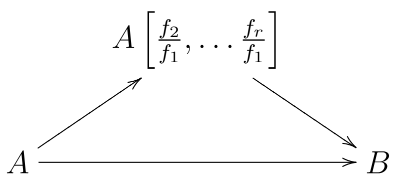
maps $\mathfrak{q}_1$ to $\mathfrak{q}_2$, and therefore induces an isomorphism

$$\beta_{1,2} : A\left[\frac{f_2}{f_1}, \dots \frac{f_r}{f_1}\right]_{\mathfrak{q}_1} \to A\left[\frac{f_1}{f_2}, \dots \frac{f_r}{f_2}\right]_{\mathfrak{q}_2}.$$

10.7. Towards the Universal Property of the blow-up III

Let B be a (non-necessarily local) A-algebra, and let $I \subset A$ be the ideal spanned by elements $\{f_1, \dots, f_r\}$. If the extended ideal IB is free of rank one, and if $IB = f_1 B$, then the previous discussion ensures that there are unique morphisms defining the commutative diagram

(10.7.1)



In addition, if $IB = f_2B$ we also get a commutative diagram as in (10.6.14). Recall that in the study of the commutativity of such diagram, addressed under the assumption that B was local, a role is played by the fact that $c_1^{(2)}$ is a unit at the local ring. Note here that $c_1^{(2)}$ is a unit in any localization of B at a prime ideal, so $c_1^{(2)}$ must be a unit in B.

Theorem 10.8. The Universal Property of the blow-up. *Let A be a ring, let $I \subset A$ be an ideal, and let $Bl_A(I)$ be the blow-up of A at I. Then:*

1) The total transform of I via

$$Bl_I(A) \to Spec(A)$$

is an invertible $Bl_I(A)$-ideal.

2) Let $(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$ be an A-scheme. If the total transform of I in $(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$ is an invertible $\mathcal{O}_{\mathcal{D}}$-ideal, then there is a factorization

(10.8.1)
$$\begin{array}{ccc} & Bl_I(A) & \\ {}^{g}\nearrow & & \searrow^{i} \\ (\mathcal{D}, \mathcal{O}_{\mathcal{D}}) & \longrightarrow & Spec(A) \end{array}$$

for a unique morphism g of A-algebras.

Proof: 1) The scheme $Bl_I(A)$ is covered by affine charts of the form $Spec(A_i)$, where $A_i = A\left[\frac{f_1}{f_i}, \dots \frac{f_r}{f_i}\right]$, and the restricted morphism $Spec(A_i) \to Spec(A)$ is defined by the homomorphism $A \to A\left[\frac{f_1}{f_i}, \dots \frac{f_r}{f_i}\right]$. Now $IA_i = \langle f_i \rangle$, and the inclusion of A_i in A_{f_i} ensures that f_i is a non-zero divisor in A_i. So IA_i is a free A_i-module of rank one with basis $\{f_1\}$. Since $Bl_I(A)$ is covered by the open affine charts $Spec(A_i)$, $i = 1, \dots, r$, it follows that the $Bl_I(A)$-ideal defined by I is invertible.

2) We will show that $\mathcal{D}$ can be covered by r open sets, say $\{V_1, \dots, V_r\}$, and that a morphism of schemes:

$$(\mathcal{O}_{V_i}, V_i) \to Spec(A_i)$$

is defined, where $(\mathcal{O}_{V_i}, V_i)$ denotes the restriction of $(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$, for $i = 1, \dots, r$. Then we will prove that these morphisms glue to define a morphism of A-schemes $(\mathcal{D}, \mathcal{O}_{\mathcal{D}}) \to Bl_I(A)$. The uniqueness will follow from the construction.

Let $\mathfrak{p} \in \mathcal{D}$. Since $I\mathcal{O}_{\mathcal{D},\mathfrak{p}}$ is a free $\mathcal{O}_{\mathcal{D},\mathfrak{p}}$-module, $I\mathcal{O}_{\mathcal{D},\mathfrak{p}} = f_i\mathcal{O}_{\mathcal{D},\mathfrak{p}}$ for some index $i \in \{1, \dots, r\}$. Note that $I\mathcal{O}_{\mathcal{D},\mathfrak{p}} = f_i\mathcal{O}_{\mathcal{D},\mathfrak{p}}$ is an open condition. Define

$$V_i = \{\mathfrak{p} \in \mathcal{D},\ I\mathcal{O}_{\mathcal{D},\mathfrak{p}} = f_i\mathcal{O}_{\mathcal{D},\mathfrak{p}}\}.$$

The hypothesis on the total transform of I ensures that $\{V_1, \dots, V_r\}$ is an open cover of $\mathcal{D}$. Let D_{i_0} be an A-algebra so that $Spec(D_{i_0})$ be an affine open set included in V_i. Then there is a unique morphism of A-schemes $Spec(D_{i_0}) \to Spec(A_i)$ (see 10.7). This uniqueness ensures that there is a (unique) morphism of schemes defining a commutative diagram

(10.8.2)
$$\begin{array}{ccc} (V_i, \mathcal{O}_{V_i}) & \longrightarrow & Spec(A_i) \\ & \searrow \quad \swarrow & \\ & Spec(A) & \end{array}$$

for $i = 1, \dots, r$.

Now we use the same notation as in (9.1.5), and argue as there to prove that these morphisms of schemes glue to define a morphism $(\mathcal{D}, \mathcal{O}_{\mathcal{D}}) \to Bl_I(A)$ (see also 9.2). Let $(W_{i_0}, \mathcal{O}_{W_{i_0}}) = Spec(D_{i_0})$ be an affine chart in V_i, let $(W_{j_0}, \mathcal{O}_{W_{j_0}}) = Spec(D_{j_0})$ be an affine chart in V_j, and let

$$(W_{i_0} \cap W_{j_0}, \mathcal{O}_{W_{i_0} \cap W_{j_0}},) = Spec(D_{i_0,j_0}).$$

We claim that ID_{i_0,j_0} is free of rank one, and that

$$ID_{i_0,j_0} = f_i D_{i_0,j_0} = f_j D_{i_0,j_0}.$$

This can be checked easily by localizing at every point $\mathfrak{p} \in W_{i_0} \cap W_{j_0}$. The commutativity of diagrams as in (9.1.5) follow now from the commutativity of homomorphisms

(10.8.3)
$$\begin{array}{ccccc} & D_{i_0} & \longleftarrow & A_i & \\ \swarrow & & & & \searrow \\ D_{i_0,j_0} & & \longleftarrow & & A_{ij} \\ \nwarrow & & & & \nearrow \\ & D_{j_0} & \longleftarrow & A_j & \end{array}$$

(see (10.6.14)). ⟲

10.9. Further properties of blow-ups

Let A be a ring, let $I \subset A$ be an ideal, and consider he blow-up of A at I, $Bl_I(A)$.

1. The scheme $Bl_I(A)$ does not depend on the choice of the generators of I. Since $Bl_I(A) = Proj(R)$ for $R = A \oplus I \oplus I^2 \oplus \dots$ the claim follows from the argument given in 8.11.

2. Let $S \subset A$ be a multiplicative subset. Then S is a multiplicative set on R, and the localization is given by

$$R_S = A_S \oplus I_S \oplus I_S^2 \oplus \dots.$$

Note that I_S^k is an ideal in A_S, and it is the k-th power of I_S. In particular, the localization at S of the blow-up, is the blow-up of A_S at I_S, and this defines

(10.9.1)
$$\begin{array}{ccc} Bl_I(A) & \longleftarrow & Bl_{I_S}(A_S) \\ \downarrow & & \downarrow \\ Spec(A) & \longleftarrow & Spec(A_S) \end{array}$$

Of particular interest is the case in which S consists on the powers of an element $a \in A$:

$$Spec(A) \leftarrow Spec(A_a),$$

which is an open restriction, and the right column of

$$
\begin{array}{ccc}
Bl_I(A) & \longleftarrow & Bl_{I_a}(A_a) \\
\downarrow & & \downarrow \\
Spec(A) & \longleftarrow & Spec(A_a)
\end{array}
\tag{10.9.2}
$$

is viewed as the restriction of the blow-up at the open set. This notion of restriction is a natural extension of that discussed in (3.1.2) for morphisms of affine schemes.

Recall that $Bl_I(A)$ can be presented by by local-global data. If I is the ideal in A generated by elements $\{f_1, \dots, f_r\}$, then $Bl_I(A)$ is defined by patching the affine schemes $Spec(A_i)$, where $A_i = A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right]$ $(\subset A_{f_i})$. All rings involved in the local-global data are A-algebras, and the homomorphisms are also of A-algebras. So the localization applies naturally to the local-global data. Here we obtain,

$$
\begin{array}{ccc}
Spec(A_i) & & Spec((A_i)_a) \\
\downarrow & & \downarrow \\
Spec(A) & \longleftarrow & Spec(A_a)
\end{array}
\tag{10.9.3}
$$

and $Bl_{I_a}(A_a)$ is obtained by patching the affine schemes $Spec((A_i)_a)$.

3. If I is an invertible ideal in A, then $Bl_I(A) \to Spec(A)$ is the identity.

This is a consequence of the universal property of the blow-up. One can check this property directly by taking an open cover of *spec*(A) by sets of the form *spec*(A_a), so that IA_a is a free A_a-module of rank one. If I is generated by elements $f_1, \dots, f_r$, we may assume that $IA_a = f_iA_a$ for some index i, $1 \leq i \leq r$. In this case $Bl_I(A_a) \to Spec(A_a)$ is an isomorphism (see 10.5). As this holds for an open cover of $Spec(A)$, $Bl_I(A) \to Spec(A)$ is an isomorphism.

4. The morphism $Bl_I(A) \to Spec(A)$ defines an isomorphism when restricted to the open set *spec*$(A) \setminus V(I)$.

Let $f_1, \dots, f_r$ is a set of generators of I. Then *spec*$(A) \setminus V(I)$ is covered by the open sets *spec*(A_{f_i}), $i = 1, \dots, r$. Since $IA_{f_i} = A_{f_i}$, each $Bl_{IA_{f_i}}(A_{f_i}) \to Spec(A_{f_i})$ is an isomorphism. Therefore local rings at points of $Bl_I(A)$ are isomorphic to local rings at points of $Spec(A)$ except, perhaps, for those points mapping to $V(I) \subset$ *spec*(A).

Proposition 10.10. *Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ be a k-scheme with an affine open cover U_i, $i = 1, \dots, r$, and restrictions $(U_i, \mathcal{O}_{U_i}) = Spec(A_i)$. Consider a coherent $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$-ideal, say $\mathcal{J}$, with restrictions given by ideals J_i in A_i, $i = 1, \dots, r$. Then the blow-ups*

$$BL_{J_i}(A_i) \to Spec(A_i)$$

glue so as to define a morphism of k-schemes, say

$$BL_{\mathcal{J}}(\mathcal{C}) \to (\mathcal{C}, \mathcal{O}_{\mathcal{C}}).$$

Proof: We will use the same notation as in **Property (C)** from Section 6, where $(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j})$ is $Spec(A_{ij})$, and there are homomorphism $A_i \to A_{ij}$ and $A_j \to A_{ij}$. The remarks in 7.6 show that the restriction of $Bl_{J_i}(A_i) \to Spec(A_i)$ to the open set $Spec(A_{ij})$, is $Bl_{J_i}(A_{ij})$. The claim follows now from 7.6.1. ◌

Definition 10.11. Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ be a scheme and let $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a closed subscheme defined by an $\mathcal{O}_{\mathcal{C}}$-ideal J. Then *blow-up of* $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ *at* $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is the blow-up of $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ at J. This is also usually referred to as the *monoidal transformation of* $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ with center $\mathcal{Y}$.

11. On blow-ups and transforms of ideals

Let A be a ring, let $I \subset A$ be an ideal, and consider the blow-up of A at I, $Bl_I(A)$. If $I = \langle f_1, \dots, f_r \rangle$ then we have seen that

$$Bl_I(A) \to Spec(A)$$

is defined by patching the affine morphisms

$$Spec(A_i) \to Spec(A),$$

where $A_i = A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right]$ $(\subset A_{f_i})$. Let $J \subset A$ be another ideal. We can define two different transforms of J in $Bl_I(A)$.

11.1. The total transform of an ideal $J \subset A$ in $Bl_I(A)$

The *total transform of an ideal* $J \subset A$ is the coherent $Bl_I(A)$-ideal defined by patching the extended ideals ideals JA_i in A_i. There is yet another interpretation of the total transform. In fact this ideal is also defined by the homogeneous ideal JR (see 8.16, 1)). In other words, the inclusion of graded modules

$$0 \to JR \to R$$

defines an ideal on $Bl_I(A)$, which, in each affine chart $Spec(A_i)$, is also the ideal JA_i.

11.2. The strict transform of an ideal $J \subset A$ in $Bl_I(A)$

Let $J \subset A$ be an ideal, and let $\overline{f}_1, \dots, \overline{f}_r$ denote the image of of $f_1, \dots, f_r$ in $B = A/J$. So $\overline{f}_1, \dots, \overline{f}_r$ are generators of $\overline{I} = IB$. Consider

$$\overline{R} = B \oplus \overline{I} \oplus \overline{I}^2 \oplus \dots$$

and note that there is an exact sequence:

$$0 \to H \to R \to \overline{R} = B \oplus \overline{I} \oplus \overline{I}^2 \oplus \dots \to 0, \tag{11.2.1}$$

for some homogeneous ideal H, and check that

$$[H]_i = I^i \cap J \text{ for all } \; i \geq 0.$$

The ideal H defines a $Bl_I(A)$-ideal which we refer to as *the strict transform of* J *in* $Bl_I(A)$. The restriction of the strict transform to the affine chart $Spec(A_i)$ is given by an ideal, say J_i, in A_i. There is an inclusion of homogeneous ideals:

$$JR \subset H$$

so JA_i is contained in J_i, as ideals in A_i, for each index i.

The restriction to the affine chart $Spec(A_i)$ of the exact sequence of $Bl_I(A)$-modules obtained from (11.2.1), is given by:

$$0 \to J_i \to A_i \to B_i \to 0 \tag{11.2.2}$$

where
$$B_i = B\left[\frac{\overline{f}_1}{\overline{f}_i}, \ldots, \frac{\overline{f}_r}{\overline{f}_i}\right] (\subset B_{\overline{f}_i}).$$

The relation between JA_i and J_i becomes clear when we localize: recall that A_i is a subring of A_{f_i} and note that $(A_i)_{f_i} = A_{f_i}$. Similarly, B_i is a subring of $B_{\overline{f}_i}$, and $(B_i)_{\overline{f}_i} = B_{\overline{f}_i}$.

Localizing $0 \to J \to A \to B \to 0$ at f_i, one obtains
$$0 \to JA_{f_i} \to A_{f_i} \to B_{\overline{f}_i} \to 0.$$

It follows now that,
$$J_i = JA_{f_i} \cap A_i,$$
or, equivalently, that

$$J_i = (JA_i)_{f_i} \cap A_i. \tag{11.2.3}$$

So the strict transform of J in A_i, namely J_i, is the biggest ideal containing the total transform JA_i, and with the added condition that both coincide at the localization $(A_i)_{f_i}$.

12. On monoidal transformations on regular schemes

A scheme is said to be *regular* when it can be covered by affine regular schemes, i.e., by affine schemes defined by regular rings (the definition and further properties of regular rings can be found in [1]).

Proposition 12.1. *The blow-up of a regular scheme $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ at a closed and regular subscheme, $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is again a regular scheme.*

Proof: Consider the blow-up of $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ with center $\mathcal{Y}$,
$$(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \xleftarrow{\pi} (\mathcal{C}_1, \mathcal{O}_{\mathcal{O}_1}).$$

To prove the regularity of $(\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1})$, it suffices to consider the case in which $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is affine. So suppose that $\mathcal{C} = spec(A)$ where A is a regular ring, and let $P \subset A$ be the defining ideal of the closed subscheme $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, i.e., $\mathcal{Y} = spec(A/P)$. Also, $\mathcal{Y}$ can be assumed to be irreducible, in which case P is a prime ideal in A, and the quotient A/P is again a regular ring.

Let $R = A \oplus P \oplus P^2 \ldots$, and consider the blow-up of A at P,
$$Spec(A) \longleftarrow Bl_P(A) = Proj R.$$

Let Q be a point in $spec(A)$. Recall that the blow-up is an isomorphism locally over every point $Q \notin spec(A/P)$, so in order to prove the regularity of $Bl_P(A)$ it suffices to show that $Bl_P(A)$ is regular at any point $Q' \in Bl_P(A)$ mapping to a point, say $Q \in spec(A/P)$, i.e., a prime $Q \supset P$.

There is a regular system of parameters, say $\{x_1, \ldots, x_d, \ldots x_e\}$, of A_Q, so that $PA_Q = \langle x_1, \ldots, x_d \rangle$. Moreover, there is a suitable affine neighborhood of the point Q, of the form A_f for some $f \notin Q$, so that $\{x_1, \ldots, x_d, \ldots x_e\} \subset A_f$, and $PA_f = \langle x_1, \ldots, x_d \rangle$.

We may therefore replace A by A_f and assume that the previous conditions hold at A. Now Q' is a point in a chart $U_i = Spec(A_f[\frac{x_1}{x_i}, \dots, \frac{x_d}{x_i}])$ for some index i, $1 \leq i \leq d$, mapping to the point Q via the morphism

$$A \to R_i = A\left[\frac{x_1}{x_i}, \dots, \frac{x_d}{x_i}\right]. \tag{12.1.1}$$

It remains to prove that $(R_i)_{Q'}$ is regular. The construction of R_i ensures that $P(R_i)_{Q'} = x_i(R_i)_{Q'}$, and that x_i is a non-zero divisor in that local ring. Therefore:
1) $dim((R_i)_{Q'}/x_i(R_i)_{Q'}) = dim((R_i)_{Q'}) - 1$, and
2) $(R_i)_{Q'}$ is regular if $(R_i)_{Q'}/x_i(R_i)_{Q'}$ is regular.

The assertion in 1) is a well known theorem of Krull (see [1, Corollary 11.18]). Assertion 2) is a simple consequence of 1) and the definition of regularity for local rings.

Recall here that PR_i, the ideal defining the total transform of P at R_i, and the total transform is also the ideal defined at the affine chart $Spec(R_i)$ by the inclusion of the graded ideal

$$0 \to PR \to R.$$

Moreover, one can extend this to the short exact sequence

$$0 \to PR \to R \to gr_P(A) \to 0$$

where

$$gr_P(A) = A/P \oplus P/P^2 \oplus P^2/P^3 \oplus P^3/P^4 \dots$$

Let $\mathcal{H} = Proj(gr_P(A))$, then $(R_i)_{Q'}/x_i(R_i)_{Q'} = \mathcal{O}_{\mathcal{H},\overline{Q}'}$ for a point $\overline{Q}' \in \mathcal{H}$. So it suffices to prove that $\mathcal{H}$ is a regular scheme.

By assumption, $gr_P(A) = (A/P)[X_1, \dots, X_d]$, is a polynomial ring over the regular ring A/P, where X_i denotes the class of x_i in P/P^2. One can easily check, from this description, that $\mathcal{H} = Proj(gr_P(A))$ is in fact regular. ◌

12.2. Some further properties of monoidal transformations

Consider the blow-up of a regular scheme $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ at a regular center $\mathcal{Y}$,

$$(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \xleftarrow{\pi} (\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1}).$$

Assume P is the defining ideal of $\mathcal{Y}$ in $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$. Then:
1) $\pi^{-1}(\mathcal{Y})$ is naturally identified with $\mathcal{H} = Proj(gr_P(A))$, as a closed subscheme in $(\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1})$.
2) The restriction to the open set $\mathcal{C} \setminus \mathcal{Y}$, say

$$\mathcal{C} \setminus \mathcal{Y} \xleftarrow{\pi} \mathcal{C}_1 \setminus \mathcal{H}$$

is an isomorphism.
3) $\mathcal{H} \subset \mathcal{C}_1$ is a regular hypersurface which we refer to as the *exceptional divisor*.

The geometric interpretation is that $\mathcal{Y}$ is replaced by a hypersurface, namely by $\mathcal{H} = \pi^{-1}(\mathcal{Y})$. This interpretation has also an algebraic reformulation in terms of valuations and valuation rings.

Observe that every irreducible variety $\mathcal{X}$ contains a point $x \in \mathcal{X}$, called the *generic point* of $\mathcal{X}$, such that $\overline{x} = \mathcal{X}$. Let P denote the generic point of $\mathcal{Y}$. So if $Spec(A)$ is an affine chart containing P, then $\mathcal{O}_{\mathcal{C},P} = A_P$, which is a local regular ring. The zero ideal of A is the generic point of $\mathcal{C}$, which clearly lies in $\mathcal{C} \setminus \mathcal{Y}$. So π defines an isomorphism over such point, which corresponds to the generic point of $\mathcal{C}_1$. Therefore the local rings at such points can be identified with the total quotient ring, say K, of A. This is a particular case of the so called *birational morphisms*.

On the other hand $\mathcal{H}$ is an irreducible hypersurface on the regular scheme $\mathcal{C}_1$. If Q denotes the generic point of $\mathcal{H}$, and $Spec(D)$ is an affine chart containing Q, then D_Q is a discrete valuation ring. The point is that there is an inclusion

$$A_P \subset D_Q (\subset K),$$

which is an inclusion of local rings (the maximal ideal in D_Q dominates the maximal ideal of A_P). Moreover, the valuation defined by the local regular ring A_P in the field K coincides with that defined by the discrete valuation ring $D)_Q$.

Take, for example A and $D = R_i$ as in (12.1.1). Then $Spec(R_i)$ contains Q, which is the hight one prime ideal defined by $\langle x_i \rangle$. In particular if $f \in A$, then

$$\nu_P(f) \geq n$$

if and only if $f \in \langle x_i^n \rangle$ in the regular ring R_i.

Example 12.3. Let $A = k[X, Y]$, let $P = \langle X, Y \rangle$, and let $f = X^3 - Y^4$. Consider the monomial transformation with center the origin, (i.e., with generic point $P = \langle X, Y \rangle$),

$$Spec(A) \leftarrow (\mathcal{C}_1, \mathcal{O}_{\mathcal{C}_1}).$$

In this case $\mathcal{C}_1$ can be covered by two charts: $Spec(A_1)$, and $Spec(A_2)$, where

$$A_1 = k\left[X, \frac{Y}{X}\right], \quad \text{and } A_2 = k\left[Y, \frac{X}{Y}\right].$$

In this case the generic point of $\mathcal{H}$ lies in both affine charts. It corresponds to the prime ideal $\langle X \rangle \subset A_1$, and to the prime ideal $\langle Y \rangle \subset A_2$.

The element $f \in A$ has order 3 at P. Note that

$$f = X^3 \left(1 - \left(\frac{Y}{X}\right)^3 X\right) \in A_1$$

and that

$$f = Y^3 \left(\left(\frac{X}{Y}\right)^3 - Y\right) \in A_2$$

Use finally (11.2.3) to prove that the strict transform of the ideal $\langle f \rangle$ is given by the ideal $\langle 1 - \left(\frac{Y}{X}\right)^3 X \rangle$ in A_1, and the ideal $\langle (\frac{X}{Y})^3 - Y \rangle$ in A_2.

References

[1] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.

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