Universidad Autónoma de Madrid

Commutative Algebra for Singular Algebraic Varieties

Worksheet 1: Basic tools: localizations, blow ups

O. E. Villamayor U.

Obergurgl, June 2012

Localization

Let A be a ring and

$S \subset A$ a multiplicative set ($1 \in S$ and if $s_1, s_2 \in S$ then $s_1 \cdot s_2 \in S$).

$A_S = \left\{ \frac{a}{s}, a \in A, s \in S \right\}$ where $\frac{a}{s} = \frac{a'}{s'}$ in $A_S \Leftrightarrow t(s'a - a's) = 0$ in A.

The homomorphism $\begin{array}{ccc} A & \xrightarrow{\phi} & A_S \\ a & \longmapsto & \frac{a}{1} \end{array}$ satisfies

a $\phi(S) \subset U(A_S)$,

b **Universal Property of the Localization**

If $A \xrightarrow{\beta} B$ maps S in $U(B)$ then there is a unique β':

$$\begin{tikzcd} A \arrow[rr, "\beta"] \arrow[dr, "\phi"'] & \circlearrowleft & B \\ & A_S \arrow[ur, "\beta'"'] & \end{tikzcd}$$

If there exists another $A \xrightarrow{\phi'} (A_S)'$ satisfying a and b...

1. $\phi'(S) \subset U((A_S)')$
2. $A \xrightarrow{\phi'} (A_S)'$ satisfies the Universal Property:
   $\forall\, A \xrightarrow{\beta} B$ s. t. $\beta(S) \subset U(B)$
   there exists a unique β'' s. t.

$$\begin{tikzcd} A \arrow[rr, "\beta"] \arrow[dr, "\phi'"'] & \circlearrowleft & B \\ & (A_S)' \arrow[ur, "\beta''"'] & \end{tikzcd}$$

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$$\begin{tikzcd} A \arrow[rr, "\beta"] \arrow[dr, "\phi"'] & \circlearrowleft & B \\ & A_S \arrow[ur, "\beta'"'] & \end{tikzcd}$$

In particular,

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\phi'"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & A_S \\ & (A_S)' \arrow[ur, "\beta''"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\phi'"] \arrow[dr, "\phi"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & (A_S)' \\ & A_S \arrow[ur, "\beta'"'] & \end{tikzcd}$$

where β' and β'' are unique. Joining both diagrams,

$$\begin{tikzcd} A \arrow[rr, "\phi'"] \arrow[dr, "\phi"] \arrow[dd, "\phi'"'] & & (A_S)' \\ & A_S \arrow[ur, "\beta'"] & \\ (A_S)' \arrow[ur, "\beta''"] \arrow[uurr, bend right, "\beta'\circ\beta''"'] \arrow[uurr, bend right=60, "\mathrm{Id}"'] & & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\phi'"] \arrow[dd, "\phi"'] & & A_S \\ & (A_S)' \arrow[ur, "\beta''"] & \\ A_S \arrow[ur, "\beta'"] \arrow[uurr, bend right, "\beta''\circ\beta'"'] \arrow[uurr, bend right=60, "\mathrm{Id}"'] & & \end{tikzcd}$$

Universal Property of Localization of $A \xrightarrow{\phi'} (A_S)' \Rightarrow \mathrm{Id} = \beta' \circ \beta''$.

Universal Property of Localization of $A \xrightarrow{\phi} (A_S) \Rightarrow \mathrm{Id} = \beta'' \circ \beta'$.

Thus, **there is a unique isomorphism between A_S and $(A_S)'$**.

Localizing twice

Let $S \subset T$ be two multiplicative sets in A. $(A_S)_T = A_T$**?**

Aim: Identify
$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "\psi"] & A_S \arrow[r] & (A_S)_T \end{tikzcd} \text{ with } A \xrightarrow{\phi} A_T .$$

Let $A \xrightarrow{\beta} B$ be a homomorphism s. t. $\beta(T) \subset U(B)$.
In particular,

- $\beta(S) \subset \beta(T) \subset U(B)$.
- **Universal Property** of $A \xrightarrow{\phi_S} A_S$:
  There is a unique α s. t.

$$\begin{tikzcd} A \arrow[rr, "\beta"] \arrow[dr, "\phi_S"'] & \arrow[d, phantom, "\circlearrowleft"] & B \\ & A_S \arrow[ur, "\alpha"'] & \end{tikzcd}$$

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  There is a unique α s. t.
- $\alpha(T) \subset U(B)$.
- **Universal Property** of $A_S \xrightarrow{\phi_T} (A_S)_T$:
  There is a unique α' s. t.

$$\begin{tikzcd}
A \arrow[rr, "\beta"] \arrow[dr, "\phi_S"'] & & B \\
& A_S \arrow[ur, "\alpha"] \arrow[dr, "\phi_T"'] \arrow[u, phantom, "\circlearrowleft"] \arrow[r, phantom, "\circlearrowleft"] & \\
& & (A_S)_T \arrow[uu, "\alpha'"']
\end{tikzcd}$$

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$$\begin{tikzcd}
A \arrow[rr, "\beta"] \arrow[dr, "\phi_S"] \arrow[ddrr, bend right, "\phi_T \circ \phi_S"'] & \arrow[d, phantom, "\circlearrowleft"] & B \\
& A_S \arrow[ur, "\alpha"] \arrow[dr, "\phi_T"] \arrow[r, phantom, "\circlearrowleft"] & {} \\
& & (A_S)_T \arrow[uu, "\alpha'"']
\end{tikzcd}$$

Is this α' the unique homomorphism such that this diagram commutes?

$$\begin{tikzcd} A \arrow[rr, "\beta"] \arrow[drr, bend right, "\phi_T \circ \phi_S"'] \arrow[rr, phantom, shift right=8ex, "\circlearrowleft"] & & B \\ & & (A_S)_T \arrow[u, "\alpha'"'] \end{tikzcd}$$

Is this α' the unique homomorphism such that this diagram commutes?

$$
\begin{tikzcd}
A \arrow[rr, "\beta"] \arrow[dr, "\phi_S"'] & & B \\
& A_S \arrow[ur, "\alpha"] \arrow[dr, "\phi_T"'] & \\
& & (A_S)_T \arrow[uu, shift left, "\alpha'_1"] \arrow[uu, shift right, "\alpha'_2"']
\end{tikzcd}
$$

Assume that there are two homomorphisms α'_1 and α'_2 such that

$$\beta = \alpha'_1 \circ \phi_T \circ \phi_S = \alpha'_2 \circ \phi_T \circ \phi_S$$

Is this α' the unique homomorphism such that this diagram commutes?

$$\begin{tikzcd}
A \arrow[rr, "\beta"] \arrow[dr, "\phi_S"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & B \\
& A_S \arrow[ur, "\alpha'_1 \circ \phi_T"'] \arrow[ur, bend right=60, "\alpha'_2 \circ \phi_T"'] &
\end{tikzcd}$$

Assume that there are two homomorphisms α'_1 and α'_2 such that

$$\beta = (\alpha'_1 \circ \phi_T) \circ \phi_S = (\alpha'_2 \circ \phi_T) \circ \phi_S$$

Is this α' the unique homomorphism such that this diagram commutes?

$$\begin{tikzcd} A \arrow[rr, "\beta"] \arrow[dr, "\phi_S"'] & \arrow[d, phantom, "\circlearrowleft"] & B \\ & A_S \arrow[ur, bend right, "\alpha'_1\circ\phi_T=\alpha'_2\circ\phi_T"'] & \end{tikzcd}$$

Assume that there are two homomorphisms α'_1 and α'_2 such that

$$\beta = (\alpha'_1 \circ \phi_T) \circ \phi_S = (\alpha'_2 \circ \phi_T) \circ \phi_S$$

Then there exist $\alpha'_1 \circ \phi_T$ and $\alpha'_2 \circ \phi_T$ making commutative this diagram.

Universal property of Localization of $A \xrightarrow{\phi_S} A_S \Rightarrow \alpha'_1 \circ \phi_T = \alpha'_2 \circ \phi_T$.

Is this α' the unique homomorphism such that this diagram commutes?

$$\begin{tikzcd}
 & B \\
A_S \arrow[ur, bend left, "\alpha'_1\circ\phi_T=\alpha'_2\circ\phi_T"] \arrow[dr, "\phi_T"'] \arrow[r, phantom, "\circlearrowleft"] & \\
 & (A_S)_T(A_S)_T \arrow[uu, shift left, "\alpha'_1"] \arrow[uu, shift right, "\alpha'_2"']
\end{tikzcd}$$

Assume that there are two homomorphisms α'_1 and α'_2 such that

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Universal property of Localization of $A \xrightarrow{\phi_S} A_S \Rightarrow \alpha'_1 \circ \phi_T = \alpha'_2 \circ \phi_T$.

Universal property of Localization of $A_S \xrightarrow{\phi_T} (A_S)_T \Rightarrow \alpha'_1 = \alpha'_2$.