

Universidad Autónoma de Madrid

# Commutative Algebra for Singular Algebraic Varieties

Worksheet 1: Basic tools: localizations, blow ups

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Obergurgl, June 2012

# Localization

Let  $A$  be a ring and

$S \subset A$  a multiplicative set ( $1 \in S$  and if  $s_1, s_2 \in S$  then  $s_1 \cdot s_2 \in S$ ).

$A_S = \left\{ \frac{a}{s}, a \in A, s \in S \right\}$  where  $\frac{a}{s} = \frac{a'}{s'}$  in  $A_S \Leftrightarrow t(s'a - a's) = 0$  in  $A$ .

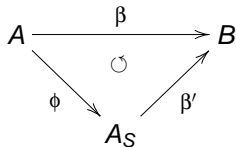
The homomorphism  $A \xrightarrow{\phi} A_S$  satisfies

$$a \longmapsto \frac{a}{1}$$

a  $\phi(S) \subset U(A_S)$ ,

b **Universal Property of the Localization**

If  $A \xrightarrow{\beta} B$  maps  $S$  in  $U(B)$  then there is a unique  $\beta'$ :

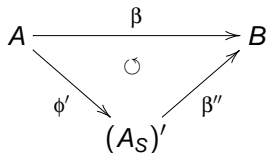


If there exists another  $A \xrightarrow{\phi'} (A_S)'$  satisfying a and b...

1.  $\phi'(S) \subset U((A_S)')$

2.  $A \xrightarrow{\phi'} (A_S)'$  satisfies the Universal Property:

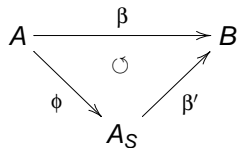
$\forall A \xrightarrow{\beta} B$  s. t.  $\beta(S) \subset U(B)$   
there exists a unique  $\beta''$  s. t.



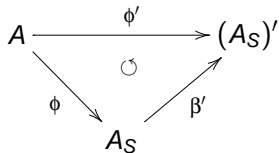
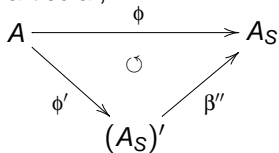
1.  $\phi(S) \subset U(A_S)$

2.  $A \xrightarrow{\phi} (A_S)$  satisfies the Universal Property:

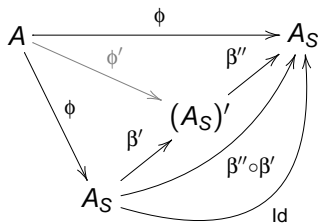
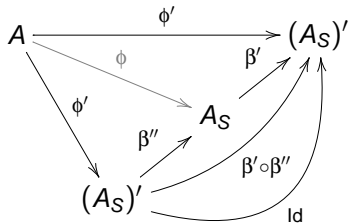
$\forall A \xrightarrow{\beta} B$  s. t.  $\beta(S) \subset U(B)$   
there exists a unique  $\beta'$  s. t.



In particular,



where  $\beta'$  and  $\beta''$  are unique. Joining both diagrams,



Universal Property of Localization  
of  $A \xrightarrow{\phi'} (A_S)'$   $\Rightarrow \text{Id} = \beta' \circ \beta''$ .

Universal Property of Localization  
of  $A \xrightarrow{\phi} (A_S)$   $\Rightarrow \text{Id} = \beta'' \circ \beta'$ .

Thus, **there is a unique isomorphism between  $A_S$  and  $(A_S)'$ .**

## Localizing twice

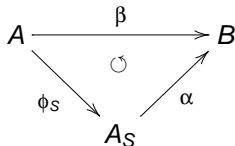
Let  $S \subset T$  be two multiplicative sets in  $A$ .  $(A_S)_T = A_T$ ?

**Aim:** Identify  $A \xrightarrow{\psi} A_S \xrightarrow{\quad} (A_S)_T$  with  $A \xrightarrow{\phi} A_T$ .

Let  $A \xrightarrow{\beta} B$  be a homomorphism s. t.  $\beta(T) \subset U(B)$ .

In particular,

- ▶  $\beta(S) \subset \beta(T) \subset U(B)$ .
- ▶ **Universal Property** of  $A \xrightarrow{\phi_S} A_S$ :  
There is a unique  $\alpha$  s. t.



## Localizing twice

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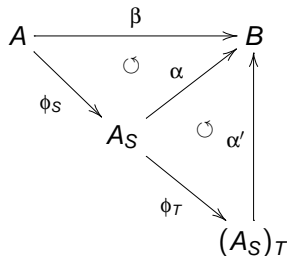
▶ **Universal Property** of  $A \xrightarrow{\phi_S} A_S$ :

There is a unique  $\alpha$  s. t.

▶  $\alpha(T) \subset U(B)$ .

▶ **Universal Property** of  $A_S \xrightarrow{\phi_T} (A_S)_T$ :

There is a unique  $\alpha'$  s. t.



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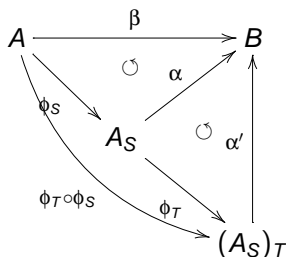
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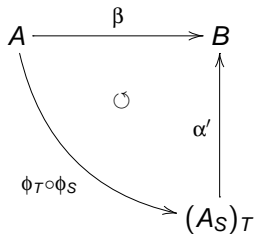
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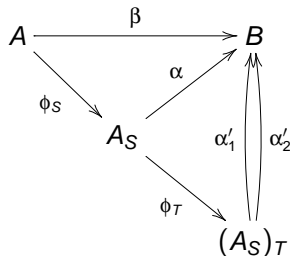


Is this  $\alpha'$  the unique homomorphism such that this diagram commutes?





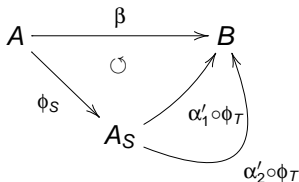
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Assume that there are two homomorphisms  $\alpha'_1$  and  $\alpha'_2$  such that

$$\beta = \alpha'_1 \circ \phi_T \circ \phi_S = \alpha'_2 \circ \phi_T \circ \phi_S$$

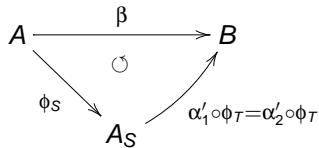
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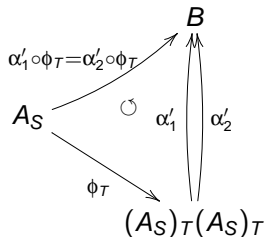
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Then there exist  $\alpha'_1 \circ \phi_T$  and  $\alpha'_2 \circ \phi_T$  making commutative this diagram.

Universal property of Localization of  $A \xrightarrow{\phi_S} A_S \Rightarrow \alpha'_1 \circ \phi_T = \alpha'_2 \circ \phi_T$ .

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