Commutative Algebra for Singular Algebraic Varieties

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Some facts about blow-ups. Stability of local presentations

Blow-ups: the basics

Let A be a domain with quotient field K. Consider the blow-up at an ideal $I = \langle f_1, \dots, f_r \rangle$. Note that each chart

$$A_i = A\left[\frac{f_1}{f_i}, \dots, \frac{f_r}{f_i}\right]$$

is included in K, and for each prime $\mathfrak{p} \subset A_i$, the local ring $(A_i)_{\mathfrak{p}}$ is included in K. In addition, if $\frac{f_j}{f_i} \notin \mathfrak{p}$, there is a natural identification of $\mathfrak{p}$ with a prime, say $\mathfrak{q} \subset A_j$, inducing the same local ring in K, namely $(A_i)_{\mathfrak{p}} = (A_j)_{\mathfrak{q}} (\subset K)$. If we denote the blow up by $Spec(A) \leftarrow X$, then $\mathfrak{p}$ defines a point, say $x \in X$, and $\mathcal{O}_{X,x} = (A_i)_{\mathfrak{p}} = (A_j)_{\mathfrak{q}}$ as subrings of the quotient field K.

This occurs, for example, when we consider the ideal zero in A_i and the ideal zero in A_j, both localizations define the field K. Moreover, these define a point, say $x_0 \in X$, called the generic point of X. We say that X is a reduced irreducible scheme with quotient field K.

Let $Y \leftarrow X$ be a morphism of schemes, if $x \in X$ maps to $y \in Y$, then there is an homomorphism of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. When A is a domain and $\mathrm{Spec}(A) \leftarrow X$ is the blow up, the generic point $x_0 \in X$ maps to the generic point of $\mathrm{Spec}(A)$, and the local homomorphism is $K = K$. This is an example of a birational morphism. Moreover, given a point $x \in X$, with image $y \in \mathrm{Spec}(A)$, the local homomorphism is given by an inclusion of local rings in the quotient field K.

Blow ups at regular (irreducible) varieties and the stability of local presentations

There is a naturally defined morphism in the class of algebraic varieties, and more generally in the class of schemes, called the blow up. Of particular interest, at least in our discussion, is the blow up of a closed and regular subvariety. Let Y be a regular variety included in X and let

$$X \xleftarrow{\pi_Y} X'$$

be the blow-up at Y; namely the blow-up of X at the ideal I_Y defining Y. This is a projective morphism, and we state here some basic properties.

• $\pi_Y^{-1}(Y)$ is a hypersurface in X' defined by the invertible ideal $I_Y \mathcal{O}_{X'}$.

• If $X \subset Z$ is a closed inclusion of algebraic varieties of schemes, the previous inclusion $Y \subset X$ induces an inclusion $Y \subset Z$ and a blow up

$$Z \xleftarrow{\pi_Y} Z'.$$

Then there is a closed inclusion $X' \subset Z'$, and $X \xleftarrow{\pi_Y} X'$ is obtained by restriction.

• If Z is regular and Y is regular, then Z' is regular, and moreover:

a) $\pi^{-1}(Y) = H$ is a regular hypersurface in Z'.

b) If Y is irreducible with generic point y, then the smooth hypersurface $\pi_Y^{-1}(Y) = H$ is irreducible. Let h be the generic point of H, then $\mathcal{O}_{Z',h}$ is a valuation ring, which dominates $\mathcal{O}_{Z,y}$ at the maximal ideal. Moreover, the order at the regular local ring $\mathcal{O}_{Z,y}$ is the valuation at $\mathcal{O}_{Z',h}$.

To be precise, here $Z \stackrel{\pi_Y}{\longleftarrow} Z'$ is a birational morphism of schemes with the same quotient field, say K. Therefore, if $x' \in Z'$ maps to $x \in Z$, we get an inclusion of local (and regular!) rings, say

$$\mathcal{O}_{Z,x} \subset \mathcal{O}_{Z',x'} \subset K,$$

and K is the quotient field of both rings.

Given a local regular ring (R, m) we define the order as function,

$$\nu_m : R \setminus \{0\} \to \mathbb{N}$$

where $\nu_M(f) = n$ if $f \in m^n \setminus m^{n+1}$. Given non-zero elements $f, g \in R$

$$\nu_m(fg) = \nu_M(f) + \nu_M(g)$$

$$\nu_m(f+g) \geq \min\{\nu_m(f), \nu_m(g)\}.$$

As $\mathcal{O}_{Z,x} \subset \mathcal{O}_{Z',x'} \subset K$, it is natural to ask, how does the order of an element $f \in \mathcal{O}_{Z,x}$ relate to the order of the same element in $\mathcal{O}_{Z',x'}$. In general this relation is not simple to specify.

The claim in b) is that the point $h \in Z'$ maps to $y \in Z$, and moreover the inclusion of local regular rings $\mathcal{O}_{Z,y} \subset \mathcal{O}_{Z',h} \subset K$, has the property that for any $f \in \mathcal{O}_{Z,x}$, the order in this local ring is the same as the order in $\mathcal{O}_{Z',h}$.

Corollary. *Let J be an ideal in Z, and let b denote the order of J at $\mathcal{O}_{Z,y}$ (at the generic point of the center Y. Then, there is a factorization of the form*

$$J\mathcal{O}_{Z'} = I(H)^b J'$$

and b is the biggest exponent with this property.

Proof. Fix a point $x \in H$. As Z' is regular, the local ring $(R, m) = \mathcal{O}_{Z',x}$ is regular, and one can choose a regular system of parameters, say $\{x_1, \ldots, x_d\}$, so that $\mathfrak{p}_1 = \langle x_1 \rangle$ is the ideal defining the hypersurface $I(H)$ locally at R (see a)). A local regular ring is a unique factorization domain, so given an element $f \in R$, one can find an expression

$$\langle f \rangle = \mathfrak{p}_1^r \langle f' \rangle$$

so that $f' \notin \mathfrak{p}_1$, where $r = \nu_{\mathfrak{p}_1}(f)$, is the order of f at the localization of R at $\mathfrak{p}_1$ (at $R_{\mathfrak{p}_1}$).

Note that $R_{\mathfrak{p}_1} = \mathcal{O}_{Z',h}$. The statement in b) says that there is an inclusion of local rings

$$\mathcal{O}_{Z,y} \subset \mathcal{O}_{Z',h} (\subset K),$$

and if $f \in \mathcal{O}_{Z,y}$ then the order of f in this regular ring is also the order of f in $\mathcal{O}_{Z',h}$. In particular, if J is an ideal in Z, with order b at $\mathcal{O}_{Z,y}$, then

$$J\mathcal{O}_{Z',x} = I(H)^b J'$$

so that J' is not contained in $\mathfrak{p}_1$. This proves the claim. ⟲

The order at a regular local ring and valuations

Let k be a field, and consider the polynomial ring in two variables, $k[x_1, x_2]$. Let $M = \langle x_1, x_2 \rangle$. Then $k[x_1, x_2]_M$ is a local regular ring, and the order at M defines a function $\nu_M : k[x_1, x_2] \setminus \{0\} \to \mathbb{N}$. where $\nu_M(f) = n$ if $f \in M^n \setminus M^{n+1}$, which resembles the property of a valuation ring. With the notion of blow up this analogy becomes very explicit. Let

$$\mathbb{A}^2 = Spec(k[x_1, x_2]) \stackrel{\pi}{\leftarrow} X$$

be the blow up at M. Then X is covered by two affine charts,

$$k[x_1, x_2] \longrightarrow k[x_1, x_2]\left[\frac{x_1}{x_2}\right] = k\left[x_2, \frac{x_1}{x_2}\right]$$

and

$$k[x_1, x_2] \longrightarrow k[x_1, x_2]\left[\tfrac{x_2}{x_1}\right] = k\left[x_1, \tfrac{x_2}{x_1}\right].$$

The two charts are patched via the identification

$$k\left[x_2, \frac{x_1}{x_2}\right]\left[\frac{x_2}{x_1}\right] = k\left[x_1, \frac{x_2}{x_1}\right]\left[\frac{x_1}{x_2}\right],$$

and π is constructed by patching the affine morphism, which grows from

$$\begin{array}{ccccc} & & k\left[x_2, \tfrac{x_1}{x_2}\right] & & \\ & \nearrow & & \searrow & \\ k[x_1, x_2] & & & & k\left[x_2, \tfrac{x_1}{x_2}\right]\left[\tfrac{x_2}{x_1}\right] = k\left[x_1, \tfrac{x_2}{x_1}\right]\left[\tfrac{x_1}{x_2}\right] \\ & \searrow & & \nearrow & \\ & & k\left[x_1, \tfrac{x_2}{x_1}\right] & & \end{array}$$

Let $f = x_1^3 - x_2^9 \in k[x_1, x_2]$. Observe that $\nu_M(f) = 3$. Note that

$$fk\left[x_2, \frac{x_1}{x_2}\right] = x_2^3\left(\left(\frac{x_1}{x_2}\right)^3 - x_2^6\right)$$

and that f has order 3 at the discrete valuation ring obtained when localizing the regular ring $k\left[x_2, \frac{x_1}{x_2}\right]$ at the height one prime ideal $\langle x_2\rangle$. The point is that there is an inclusion of local rings

$$k[x_1, x_2]_M \subset k\left[x_2, \frac{x_1}{x_2}\right]_{\langle x_2\rangle}$$

and the order of an element $f \in k[x_1, x_2]_M$ is the valuation of f at the valuation ring $k\left[x_2, \frac{x_1}{x_2}\right]_{\langle x_2\rangle}$.

In a similar manner one shows that

$$fk\left[x_1, \frac{x_2}{x_1}\right] = x_1^3\left(1 + \left(\frac{x_1}{x_2}\right)^6\right).$$

Note here that $k\left[x_2, \frac{x_1}{x_2}\right]_{\langle x_2\rangle} = k\left[x_1, \frac{x_2}{x_1}\right]_{\langle x_1\rangle}$ as subrings in the quotient field of $k[x_1, x_2]$.

Stability of local presentations

Let f be a function on a smooth scheme Z, and let m be an integer. Consider a closed set given by a expression of the form

$$F = \{x \in Z, \nu_x(f) \geq m\} \subset Z.$$

Let

$$Z \xleftarrow{\pi_Y} Z'.$$

be the blow-up at a smooth center $Y \subset F$.

- The function defined by f over Z' vanishes along H, and, locally

$$f = I(H)^m f_1$$

for some function f_1 on Z'. Moreover, the set

$$F' = \{x \in Z', \nu_x(f') \geq m\}(\subset Z')$$

is well defined, independently of the local choice of f_1.

- More generally, given functions $f_1, \dots, f_s$ on Z and positive integers $m_1, \dots, m_s$, define

$$F = \cap_{1 \leq i \leq s} \{x \in Z, \nu_x(f_j) \geq m_j\} \subset Z$$

and note that it is a closed set.

Let

$$Z \xleftarrow{\pi_Y} Z'.$$

be the blow-up at a smooth center $Y \subset F$. Note that there is a natural lifting to previous expression to one of the form

$$\cap_{1 \leq i \leq s} \{x \in Z', \nu_x(f'_j) \geq m_j\} \subset Z'.$$

which is closed in Z'.