Rees Algebras, Elimination, and singularities of Varieties over Perfect Fields

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The Problem

Given an algebraic variety X defined over a field k,

construct a resolution of singularities

as a composition of blow-ups at regular centers,

$$X \longleftarrow X_1 \longleftarrow \dots \longleftarrow X_n.$$

Here:

- X_n is regular;
- $X \leftarrow X_n$ is birational and proper.

To address the problem we need to:

1. Identify the worst singularities of X.
2. Give a criterion to select a regular center to blow-up.
3. Compare the worst singularities of X before and after blow-up.
4. Check that the singularities after each blow-up have improved.

1. Identify the worst singularities of X.

- Use the Hilbert-Samuel function, $\mathrm{HS}(X)$:

$$X \text{ is regular } \iff \mathrm{HS}(X) \text{ is constant on } X.$$

- Use the Multiplicity, $\mathrm{Mult}(X)$,

$$X \text{ is regular } \iff \mathrm{Mult}(X) = 1 \text{ on } X.$$

Both HS(X), Mult(X):

- Are upper semi-continuous functions;
- Stratify X in locally closed strata.

Stratum corresponding to the maximum value Max HS(X), Max Mult(X):

$$\underline{\text{Max }}\text{HS}(X); \quad \underline{\text{Max }}\text{Mult}(X).$$

Both HS(X), Mult(X):

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$$\underline{\text{Max}}\,\text{HS}(X); \quad \underline{\text{Max}}\,\text{Mult}(X).$$

Notation: Use F(X) to refer to either HS(X) or Mult(X); Max F(X) for the maximum value, and set

$$\underline{\text{Max}}\,\text{F}(X) = \underline{\text{Max}}\,\text{HS}(X); \quad \text{or} \quad \underline{\text{Max}}\,\text{Mult}(X).$$

2. Give a criterion to select a regular center to blow-up.

In general, $\underline{\text{Max}}\ \mathrm{F}(X)$ will not be smooth.

Problem $\sharp 1$: Find a refinement of the function $\mathrm{F}(X)$.

This will be done using:

- Local presentations;
- Defining new invariants via elimination.

3. Compare the worst singularities of X before and after blow-up.

Consider the blow-up at a regular center $Y \subset \underline{\text{Max F}}(X)$:

$$X \leftarrow X_1$$

Problem $\sharp$2: Compare Max F(X) with Max F(X_1).

The comparison will be possible using local presentations.

FACT: Max F$(X) \geq$ Max F(X_1)

4. Check that the singularities after each blow-up have improved.

Problem $\sharp 3$:

$$\text{What if } \mathrm{F}(X) = \mathrm{F}(X_1)?$$

How do we know there is an improvement?

The improvement can be detected via other invariants that can be derived from the local presentations using techniques of elimination.

Conclusion

When trying to find a resolution of X we have to face some problems, that can be approached via:

Local presentation in regular schemes.

What are "Local presentations"...?

Hironaka's approach: Local presentations for the Hilbert-Samuel function

- There is an embedding of X in a smooth V;

Locally at $\xi \in \underline{\text{Max}}\, \text{HS}(X)$:

- There are functions $f_1, \ldots, f_s$ on V and positive integers $m_1, \ldots, m_s$, naturally attached to the $\underline{\text{Max}}\, \text{HS}(X)$ so that

$$\underline{\text{Max}}\, \text{HS}(X) = \bigcap_{1 \leq j \leq s} \{\xi \in V : \nu_\xi(f_j) \geq m_j\} \subset V. \tag{1}$$

Moreover, given any sequence of blow-ups

$$\begin{array}{ccccccc} V = V_0 & \leftarrow & V_1 & \leftarrow & \dots & \leftarrow & V_n \\ \cup & & \cup & & \dots & & \cup \\ X = X_0 & \leftarrow & X_1 & \leftarrow & \dots & \leftarrow & X_n \end{array} \tag{2}$$

with regular centers $Y_i \subset \underline{\text{Max}}\ \text{HS}(X_i)$, one has that:

$$\underline{\text{Max}}\ \text{HS}(X_i) = \cap\{\eta \in V_i : \nu_\eta(f_j^{(i)}) \geq m_j\} \subset V_i \tag{3}$$

where $f_j^{(i)}$ denotes a weighted transform of $f_j^{(i-1)}$ for $i = 1, \dots, n$.

Local presentations for the multiplicity

Work locally at $\xi \in \underline{\text{Max}}\,\text{Mult}(X)$:

- There is a finite (local) morphism onto a smooth V, $\beta : X \longrightarrow V$; with $\dim X = \dim V$.

- There are functions $f_1, \dots, f_s$ on V and positive integers $m_1, \dots, m_s$, *naturally attached* to the $\underline{\text{Max}}\,\text{Mult}(X)$ so that:

$$\beta(\underline{\text{Max}}\,\text{Mult}(X)) = \cap\{\xi \in V : \nu_\xi(f_j) \geq m_j\} \subset V.$$

Moreover, given any sequence of blow-ups with regular centers $Y_i \subset \underline{\text{Max}}\ \text{Mult}(X_i)$, there is a commutative diagram of blow-ups and finite projections:

$$\begin{array}{ccccc} X = X_0 & \longleftarrow & \cdots & \longleftarrow & X_n \\ \downarrow & & & & \downarrow \\ V = V_0 & \longleftarrow & \cdots & \longleftarrow & V_n \end{array} \tag{4}$$

and for $i = 0, \dots n$ one has that:

$$\beta_i(\underline{\text{Max}}\ \text{Mult}(X_i)) = \cap\{\eta \in V_i : \nu_\eta(f_j^{(i)}) \geq m_j\} \subset V_i$$

where $f_j^{(i)}$ denotes a weighted transform of $f_j^{(i-1)}$ for $i = 1, \dots, n$.

Conclusion

We consider two strategies to describe $\underline{\text{Max F}}(X)$:

- For the Hilbert-Samuel function: use local embeddings in smooth schemes;
- For the Multiplicity: use local finite projections onto smooth schemes.

In both cases, $\underline{\text{Max F}}(X)$ is described via weighted equations.

To handle this of information we will use Rees algebras on smooth schemes.

The role of Rees algebras

- Associate a Rees algebra to a given local presentation.
- A resolution of the Rees algebra will lead to a lowering of Max $\mathrm{F}(X)$.
- Max $\mathrm{F}(X)$ can be stratified in smooth strata via invariants defined using Rees algebras using elimination.
- When the characteristic is zero, Rees algebras can be resolved using induction in the so called codimensional type, τ.

Moreover we will see that:

- The string of invariants for resolution, defined in terms of a Rees algebra, will not depend on the local embedding, or the choice of the local finite projection.
- The codimensional type of a Rees algebra will be our tool to know how many variables can be eliminated; this will play a role in inductive arguments.
- The Canonicity Principle for Rees algebras will be our tool to globalize local invariants.

Problem

Let $\dim X = d$.

Consider upper-semi-continuous functions

$$\mathrm{F}(X) : X \to (\Lambda, \geq)$$

such that if $Y \subset \underline{\mathrm{Max}}\mathrm{F}(X)$ is smooth, then after blowing-up at Y,

$$V \leftarrow V_1$$

one has that

$$\mathrm{Max}\ \mathrm{F}(X) \geq \mathrm{Max}\ \mathrm{F}(X_1).$$

Representability in dimension d

We will say that $\mathrm{F}(X)$ is representable in dimension d if, locally, at each $\xi \in \underline{\mathrm{Max}}\mathrm{F}(X)$, there is a morphism:

$$\beta : X \to V$$

with V smooth and d-dimensional such that:

- $\underline{\mathrm{Max}}\,\mathrm{F}(X)$ is homeomorphic to $\beta(\underline{\mathrm{Max}}\,\mathrm{F}(X))$;
- There is a Rees algebra $\mathcal{G}$ on V such that

$$\mathrm{Sing}\,\mathcal{G} = \beta(\underline{\mathrm{Max}}\,\mathrm{F}(X)).$$

Moreover, any sequence of transformations

$$X \leftarrow X_1 \leftarrow \ldots \leftarrow X_n$$

at smooth centers $Y_i \subset \underline{\text{Max }}\text{F}(X_i)$, induces commutative diagrams of morphisms and blow-ups, and transforms of Rees algebras,

$$\begin{array}{ccccc}
X = X_0 & \longleftarrow & \cdots & \longleftarrow & X_n \\
\downarrow & & & & \downarrow \\
V = V_0 & \longleftarrow & \cdots & \longleftarrow & V_n \\
 & & & & \\
\mathcal{G} = \mathcal{G}_0 & & \cdots & & \mathcal{G}_n
\end{array}$$

such that

$$\beta_i(\underline{\text{Max}}\ \text{F}(X_i)) = \text{Sing}\,\mathcal{G}_i$$

for $i = 0, \ldots, n$.

Examples

- Multiplicity (Villamayor's course)
- Hilbert-Samuel (this can be shown);
- Others. . .

What if there are two different representations?