Universidad Autónoma de Madrid

Commutative Algebra for Singular Algebraic Varieties

Thursday

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$(S,\mathfrak{n})$ is local ring of dim d

Definition

$(S,\mathfrak{n})$ is regular iff $\mathfrak{n}$ is spanned by a system of parameters.

Properties of a regular local ring $(S,\mathfrak{n})$

- S is a domain,
- $\nu : S \setminus \{0\} \to \mathbb{N}$ where if $\nu(f) = r \Leftrightarrow f \in \mathfrak{n}^r \wedge f \notin \mathfrak{n}^{r+1}$
  - $\nu(fg) = \nu(f) + \nu(g)$
  - $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$

Completions and Local rings

$(R, \mathfrak{m})$ local ring

$$R/\mathfrak{m}^n \longleftarrow R/\mathfrak{m}^{n+1} \longleftarrow R/\mathfrak{m}^{n+2}$$

Definition

$(\hat{R}, \hat{\mathfrak{m}})$ is a subset of

$$\prod_{k \geq 0} R/\mathfrak{m}^k.$$

$x \in \prod_{k \geq 0} R/\mathfrak{m}^k \Leftrightarrow x = (x_0, x_1, \ldots, x_n, \ldots)$ where $x_k \in R/\mathfrak{m}^k$.

$x = (x_0, x_1, \ldots, x_n, \ldots) \in \hat{R}$ when $x_{i-1} \longleftarrow\!\dashv x_i$ via $R/\mathfrak{m}^{i-1} \longleftarrow R/\mathfrak{m}^i$

$$
\begin{array}{ccc}
 & a \in R & \\
\swarrow & & \searrow \\
R/\mathfrak{m}^{i-1} & \longleftarrow & R/\mathfrak{m}^{i} \\
a_{i-1} & \longleftarrow\!\!\!\shortmid & a_i
\end{array}
$$

$$0 \longrightarrow R \longrightarrow \hat{R}$$

Example

$R = k[t]_{\langle t \rangle}$

$$
\begin{array}{ccc}
 & k[t]_{\langle t \rangle} & \\
\swarrow & & \searrow \\
k[t]/\langle t \rangle^{i-1} & \longleftarrow & k[t]/\langle t \rangle^{i}
\end{array}
$$

$$R = k[t]_{\langle t \rangle} \longrightarrow \hat{R} = k[[t]]$$

Facts

- $(S, \mathfrak{n}, k)$ regular local ring with residue field k.
  If S contains a field, and if $\{x_1, \dots, x_d\}$ is a r. s. p., then

$$\hat{S} = k[[x_1, \dots, x_d]]$$

- $(R, \mathfrak{m})$ and $I \subset \mathfrak{m}$:

$$(\widehat{R/I}) = \hat{R}/I\hat{R}.$$

$\mathcal{O}_{X,x}$ $\qquad$ $\mathcal{P} = \{x_1, \ldots, x_d\}$ s. of p.

$$\begin{array}{ccc} R = k[x_1, \ldots, x_d]_{\langle x_1, \ldots, x_d \rangle} & \longrightarrow & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \hat{R} = k[[x_1, \ldots, x_d]] & \xhookrightarrow{\text{finite}} & \hat{\mathcal{O}}_{X,x} \end{array}$$

$\mathcal{O}_{X,x}$ $\qquad\qquad \mathcal{P} = \{x_1, \ldots, x_d\}$ s. of p.

$$
\begin{array}{ccc}
R = k[x_1, \ldots, x_d]_{\langle x_1, \ldots, x_d \rangle} & \longrightarrow & \mathcal{O}_{X,x} \\
\big\downarrow & & \big\downarrow \\
\hat{R} = k[[x_1, \ldots, x_d]] & \xhookrightarrow{\text{finite}} & \hat{\mathcal{O}}_{X,x} \\
\big\downarrow & & \big\downarrow \\
K = k((x_1, \ldots, x_d)) & \xhookrightarrow{\text{finite}} & K \otimes_{\hat{R}} \hat{\mathcal{O}}_{X,x}
\end{array}
$$

$\mathcal{O}_{X,x}$ $\qquad$ $\mathcal{P} = \{x_1, \ldots, x_d\}$ s. of p.

$$
\begin{array}{ccc}
R = k[x_1, \ldots, x_d]_{\langle x_1, \ldots, x_d \rangle} & \longrightarrow & \mathcal{O}_{X,x} \\
\big\downarrow & & \big\downarrow \\
\hat{R} = k[[x_1, \ldots, x_d]] & \xhookrightarrow{\text{finite}} & \hat{\mathcal{O}}_{X,x} \\
\big\downarrow & & \big\downarrow \\
K = k((x_1, \ldots, x_d)) & \xhookrightarrow{\text{finite}} & K \otimes_{\hat{R}} \hat{\mathcal{O}}_{X,x}
\end{array}
$$

$$n_{\mathcal{P}} = \dim_K (K \otimes_{\hat{R}} \hat{\mathcal{O}}_{X,x})$$

$$\mathcal{O}_{X,x} = \left(k[x,y]/\langle x^2 - y^3 \rangle\right)_{\langle x,y \rangle} = B \qquad \mathcal{P} = \{x\} \text{ s. p.}$$

$$\begin{array}{ccc}
R = k[x]_{\langle x \rangle} & \longrightarrow & B \\
\big\downarrow & & \big\downarrow \\
\hat{R} = k[[x]] & \xhookrightarrow{\text{finite}} & \hat{B} \\
\big\downarrow & & \big\downarrow \\
K = k((x)) & \xhookrightarrow{\text{finite}} & K \otimes_{\hat{R}} \hat{B}
\end{array}$$

$$\mathcal{O}_{X,x} = \big(k[x,y]/\langle x^2 - y^3\rangle\big)_{\langle x,y\rangle} = B \qquad \mathcal{P} = \{x\} \text{ s. p.}$$

$$\begin{array}{ccc}
R = k[x]_{\langle x\rangle} & \longrightarrow & B \\
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\big\downarrow & & \big\downarrow \\
K = k((x)) & \xhookrightarrow{\text{finite}} & K \otimes_{\hat{R}} \hat{B}
\end{array}$$

$$n_{\mathcal{P}} = \dim_K(K \otimes_{\hat{R}} \hat{B}) = 3$$

$$\mathcal{O}_{X,x} = \left(k[x,y]/\langle x^2 - y^3\rangle\right)_{\langle x,y\rangle} = B \qquad\qquad \mathcal{P} = \{y\} \text{ s. p.}$$

$$\begin{array}{ccc}
R = k[y]_{\langle y\rangle} & \longrightarrow & B \\
\big\downarrow & & \big\downarrow \\
\hat{R} = k[[y]] & \xhookrightarrow{\text{finite}} & \hat{B} \\
\big\downarrow & & \big\downarrow \\
K = k((y)) & \xhookrightarrow{\text{finite}} & K \otimes_{\hat{R}} \hat{B}
\end{array}$$

$$\mathcal{O}_{X,x} = \left(k[x,y]/\langle x^2 - y^3\rangle\right)_{\langle x,y\rangle} = B \qquad \mathcal{P} = \{y\} \text{ s. p.}$$

$$\begin{array}{ccc}
R = k[y]_{\langle y\rangle} & \longrightarrow & B \\
\big\downarrow & & \big\downarrow \\
\hat{R} = k[[y]] & \xrightarrow{\text{finite}} & \hat{B} \\
\big\downarrow & & \big\downarrow \\
K = k((y)) & \xrightarrow{\text{finite}} & K \otimes_{\hat{R}} \hat{B}
\end{array}$$

$$n_{\mathcal{P}} = \dim_K(K \otimes_{\hat{R}} \hat{B}) = 2$$

$\mathcal{P} = \{x_1, \ldots, x_d\}$ s. p. $\mathcal{O}_{X,x}$

$$\begin{array}{ccc} \hat{R} = k[[x_1, \ldots, x_d]] & \xhookrightarrow{\text{finite}} & \hat{\mathcal{O}}_{X,x} \\ \big\downarrow & & \big\downarrow \\ K = k((x_1, \ldots, x_d)) & \xhookrightarrow{\text{finite}} & K \otimes_{\hat{R}} \hat{\mathcal{O}}_{X,x} \end{array}$$

$$n_{\mathcal{P}} = \dim_K(K \otimes_{\hat{R}} \hat{\mathcal{O}}_{X,x})$$

“Definition”

multiplicity of $\mathcal{O}_{X,x}$ $=$ smallest possible $n_{\mathcal{P}}$, $\mathcal{P} = \{x_1, \ldots, x_d\}$ is a s. p.

Question 1: How do we decide when, given a s. p.
$\mathcal{P} = \{x_1, \dots, x_d\} \in \mathcal{O}_{X,x}$,
$n_{\mathcal{P}} = \dim_K(K \otimes_S \hat{\mathcal{O}}_{X,x})$ is the multiplicity of the local ring?

Question 2: Is it possible to define the multiplicity of $\mathcal{O}_{X,x}$ without passing to the completion?

$$\begin{array}{ccc} k[x_1, \dots, x_d] & \xhookrightarrow{\text{finite}} & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ K = k(x_1, \dots, x_d) & \xhookrightarrow{\text{finite}} & K \otimes_R \mathcal{O}_{X,x} \end{array}$$

$$n_{\mathcal{P}} = dim_K(K \otimes_R \mathcal{O}_{X,x})$$