Research Statement

Xinyi Yuan

March 7, 2008

The field of my research is number theory and arithmetic geometry. I am interested in arithmetic intersection theory, algebraic points of varieties, automorphic forms and formulae of Gross-Zagier type.

1 Birational Arithmetic Intersection Theory

Arakelov theory, also known as arithmetic intersection theory, shares many properties with the classical intersection theory. For example, the arithmetic Hilbert-Samuel formula due to Gillet-Soulé [GS] and Bismut-Vasserot counts the effective sections of arithmetcially ample line bundles. In [Yu], I proved the following birational version of the arithmetic Hilbert-Samuel formula.

Theorem 1. *If $\overline{\mathscr{L}}$ and $\overline{\mathscr{M}}$ are two ample hermitian line bundles over an arithmetic variety of dimension n, then*

$$h^0(\overline{\mathscr{L}}^{\otimes N} \otimes \overline{\mathscr{M}}^{\otimes(-N)}) \geq \frac{\hat{c}_1(\overline{\mathscr{L}})^n - n \cdot \hat{c}_1(\overline{\mathscr{L}})^{n-1}\hat{c}_1(\overline{\mathscr{M}})}{n!} N^n + o(N^n).$$

As an application, I generalized the equidistribution result of Szpiro-Ullmo-Zhang [SUZ] to algebraic dynamical systems in [Yu]. It is expected that this dynamical equidistribution will play an important role on the dynamical Bogomolov conjecture, since the equidistribution in [SUZ] was the key in the proof of the Bogomolov conjecture by Ullmo [Ul] and Zhang [Zh1].

The result in [SUZ] was deduced from the arithmetic Hilbert-Samuel formula, but the formula is not applicable in the dynamical case since negative curvature occur after perturbation. Many of the techniques in complex geometry and Arakelov theory depend heavily on positive curvature hypothesis. Theorem 1 serves as a substitute for the arithmetic Hilbert-Samuel formula, because it can deal with negative curvature quite accurately.

Controlling heights by sections with bounded norms is the main tool of Arakelov theory. It is crucial in our proof of equidistribution using Theorem 1, and it is also crucial, for example, in Vojta's proof of the Mordell conjecture in [Vo1] and Faltings's proof of Lang's conjecture in [Fa]. We expect that the study of birational arithmetic intersection theory leads to some results on Vojta's conjecture (cf. [Vo2]).

2 Modularity of Generating Functions

Let F be a number field, and V be a quadratic space over F. For any Schwartz function $\phi \in \mathscr{S}(V(\mathbb{A}_F))$, the theta function $\theta_\phi(g)$ is automorphic for $g \in \mathrm{GL}_2(\mathbb{A}_F)$ (or its metaplectic cover). If F is totally real and V is positive definite, we can actually obtain holomorphic Hilbert modular forms in this way.

In the arithmetic case, we have a similar situation. Let F be totally real, and V has signature $(n, 2)$ at one archimedean place and signature $(n+2, 0)$ at any other archimedean place. The reductive group $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSpin}(V)$ gives a Shimura variety X.

The arithmetic analogue of the theta series is the generating function

$$Z_\phi(\tau) = Z_0 + \sum_{a \in F_+} Z_a e^{2\pi i \ \mathrm{tr}(a\tau)}, \quad \tau \in \mathscr{H}^{[F:\mathbb{Q}]}.$$

Here the coefficient $Z_a \in \mathrm{Pic}(X)$ is a weighted sum of special divisors of X with "norm" a.

Theorem 2 ([YZZ1])**.** *The generating function $Z_\phi(\tau)$ is a holomorphic Hilbert modular form of parallel weight $\frac{n}{2} + 1$.*

The theorem gives the position of special divisors of X in a subtle way. It plays a central role in the proof of Gross-Zagier formula by Kudla-Rapoport-Yang [KRY]. We will use such a generating function in the proof of Gross-Zagier formula in [YZZ2] and the future project for the triple product mentioned in next section.

The previously known cases of the theorem are as follows:

1. When the coefficients are considered as cohomology classes, the theorem is a result of Kudla-Millson. See [KM1], [KM2], [KM3].

2. In the case of modular curves, the theorem is just the Gross-Kohnen-Zagier theorem in [GKZ]. It is related to their famous result that all Heegner points are colinear in the Mordell-Weil group of an elliptic curve.

3. In the case of Hilbert modular surfaces, it is proved by Hirzebruch-Zagier [HZ].

4. In the case that $F = \mathbb{Q}$, the general result is proved by Borcherds [Bo].

Their proofs don't work in our case. Borcherds uses some holomorphic modular forms of SL_2 with poles at cusps, which do not exist if $F \neq \mathbb{Q}$. Our idea is to take a positive definite quadratic space V' and use the product formula $Z_{\phi\otimes\phi'} = \theta_{\phi'} Z_\phi$ to reduce the problem to a high dimensional Shimura variety X'. On X', divisor classes are the same as cohomology classes, and thus the result of Kudla-Millson is applicable.

3 Gross-Zagier Formula

Gross and Zagier proved their original formula on certain modular curves in [GZ], and S. Zhang extended it to quite general cases over totally real fields in [Zh2], [Zh3], [Zh4]. There are many difficulties in the computation of automorphic forms and arithmetic intersection on Shimura curves, so these proofs only work under a lot of unramified conditions.

In [YZZ2], another joint work with Shou-wu Zhang and Wei Zhang, we prove the Gross-Zagier formula in the most general setting, as expected by Gross [Gr]. More precisely, we compute the central derivative of the Rankin-Selberg L-function $L(s, \pi, \chi)$ with odd sign in the following case:

- $\pi = \otimes_v \pi_v$ is cuspidal automorphic representation of GL_2 over a totally real number field F, discrete of weight 2 at all archimedean places;
- $\chi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ is a character of finite order for a totally imaginary quadratic extension E over F. Assume that $\chi|_{\mathbb{A}^\times}$ is reciprocal to the central character of π.

This general formula automatically simplifies many important results on Heegner points depending on the original formulae of Gross-Zagier and S. Zhang. For example, we can simply sharpen the indefinite case of Mazur's conjecture proved in the recent work of Cornut-Vatsal [CV].

Our new treatment simplifies the previous proofs in many respects. It follows the framework of the central value formula by Waldspurger [Wa], uses the generating function introduced above as a substitute of the theta function, and employs a more representation-theoretic point of view. The identity at unramified places follows from the usual Siegel-Weil formula, and computation of the identity at bad places is avoided by an argument using the result of local linear functionals by Tunnell [Tu] and Saito [Sa].

Along the line above, we plan to treat the triple product problem proposed by Gross-Kudla [GK]. We are going to prove a derivative formula parallel to the central value formula of Ichino [Ic]. This is another important example of Beilinson-Bloch conjecture and Tate conjecture. Moreover, by recent work of S. Zhang [Zh5], the height pairing of the modified diagonal cycle is closely related to the self-intersection of the arithmetic canonical class on the corresponding Shimura curve. Hence, the triple product formula relates the triple L-function to the Bogomolov conjecture, the effective Mordell conjecture and the abc conjecture significantly.

References

[Bo] R. Borcherds: The Gross-Kohnen-Zagier theorem in higher dimensions. Duke Math. J. 97 (1999), no. 2, 219–233.

[CV] C. Cornut; V. Vatsal: Nontriviality of Rankin-Selberg L-functions and CM points, to appear in Proc. LMS Durham Symposium: L-functions and Galois representations (2004).

[Fa] G. Faltings: Diophantine approximation on abelian varieties. Ann. of Math. (2) 133 (1991), no. 3, 549–576.

[GK] B. Gross; S. Kudla: Heights and the central critical values of triple product L-functions. Compositio Math. 81 (1992), no. 2, 143–209.

[GS] H. Gillet, C. Soulé, An arithmetic Riemann-Roch theorem, Inventiones Math., 110 (1992), 473–543.

[GKZ] B. Gross; W. Kohnen; D. Zagier: Heegner points and derivatives of L-series. II. Math. Ann. 278 (1987), no. 1-4, 497–562.

[Gr] B. Gross: Heegner points and representation theory. Heegner points and Rankin L-series, 37–65, Math. Sci. Res. Inst. Publ., 49, Cambridge Univ. Press, Cambridge, 2004.

[GZ] B. Gross; D. Zagier: Heegner points and derivatives of L-series. Invent. Math. 84 (1986), no. 2, 225–320.

[HZ] F. Hirzebruch; D. Zagier: Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. Invent. Math. 36 (1976), 57–113.

[Ic] A. Ichino: Trilinear forms and the central values of triple product L-functions. Preprint.

[KM1] S. Kudla; J. Millson: The theta correspondence and harmonic forms. I. Math. Ann. 274 (1986), no. 3, 353–378.

[KM2] S. Kudla; J. Millson: The theta correspondence and harmonic forms. II. Math. Ann. 277 (1987), no. 2, 267–314.

[KM3] S. Kudla; J. Millson: Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables. Inst. Hautes Études Sci. Publ. Math. No. 71 (1990), 121–172.

[KRY] S. Kudla; M. Rapoport; T. Yang: Modular forms and special cycles on Shimura curves. Annals of Mathematics Studies, 161. Princeton University Press, Princeton, NJ, 2006.

[Sa] H. Saito: On Tunnell's formula for characters of GL(2). Compositio Math. 85 (1993), no. 1, 99–108.

[SUZ] L. Szpiro; E. Ullmo; S. Zhang: Équidistribution des petits points. Invent. Math. 127 (1997) 337–348.

[Tu] J. Tunnell: Local ϵ-factors and characters of GL(2). Amer. J. Math. 105 (1983), no. 6, 1277–1307.

[Ul] E. Ullmo: Positivité et discrétion des points algébriques des courbes. Ann. of Math. (2), 147 (1998), 167–179.

[Vo1] P. Vojta: Siegel's theorem in the compact case. Ann. of Math. (2) 133 (1991), no. 3, 509–548.

[Vo2] P. Vojta: Diophantine approximations and value distribution theory. Lecture Notes in Mathematics, 1239. Springer-Verlag, Berlin, 1987.

[Wa] J. Waldspurger: Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie. Compositio Math. 54 (1985), no. 2, 173–242.

[Yu] X. Yuan: Big Line Bundles over Arithmetic Varieties, to appear in Invent. Math.

[YZZ1] X. Yuan; Shou-wu Zhang; Wei Zhang: The Gross-Kohnen-Zagier Theorem over Totally Real Fields. Submitted.

[YZZ2] X. Yuan; Shou-wu Zhang; Wei Zhang: A General Gross-Zagier Formula. In Preparation.

[Zh1] S. Zhang: Equidistribution of small points on abelian varieties. Ann. of Math. 147 (1) (1998) 159–165.

[Zh2] S. Zhang: Heights of Heegner points on Shimura curves. Ann. of Math. (2) 153 (2001), no. 1, 27–147.

[Zh3] S. Zhang: Gross-Zagier formula for GL_2. Asian J. Math. 5 (2001), no. 2, 183–290.

[Zh4] S. Zhang: Gross-Zagier formula for GL(2). II. Heegner points and Rankin L-series, 191–214, Math. Sci. Res. Inst. Publ., 49, Cambridge Univ. Press, Cambridge, 2004.

[Zh5] S. Zhang: Gross-Schoen cycles, dualising sheaves, and triple product L-series. Preprint.

Address: *Department of Mathematics, Columbia University, New York, NY 10027.*
Email: *yxy@math.columbia.edu.*