RESEARCH STATEMENT

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My main mathematical interest is the interaction between geometry and analysis, more specifically partial differential equations (PDEs), especially in the areas in which a geometrical approach can bring a clear understanding to the analytical problem. Such connections show up in as diverse areas as N-body scattering in quantum mechanics, the study of symmetric spaces and wave propagation in domains with corners.

There is a familiar relationship between classical mechanics (which is 'geometric') and quantum mechanics (which is described by PDEs, hence 'analytic'). In the description of light these two approaches are geometric optics and the study of the wave equation (a PDE) respectively, while in mechanics they correspond to Hamiltonian dynamics, where particles are points in space moving along their trajectories, and the study of the Schrödinger equation, where particles are represented by wave functions. In either case, while solutions of the PDEs do not follow the geometric rules (as shown, for example, by diffraction), their 'singularities' do. For light, this means that the propagation of sharp signals, which are the signals used to carry information, is *precisely* described by geometric optics, as shown by Hörmander, Melrose, Sjöstrand, Taylor and Lebeau in various settings. For quantum particles this means that the evolution of wave functions 'at infinity' is described by the classical picture.

My most recent contribution in these areas has been the proof of the propagation of smooth (C^∞) and Sobolev singularities for solutions of the wave equation (with Dirichlet ot Neumann boundary conditions) on manifolds with corners M equipped with a smooth Riemannian metric g. This result is a C^∞ counterpart of Lebeau's results for the propagation of analytic singularities on real analytic manifolds with appropriately stratified boundary, and it extends results of Melrose, Sjöstrand and Taylor for manifolds with a smooth boundary (and no corners). The methods rely on microlocal (or phase space) positive commutator estimates, thus providing a new proof for the propagation of singularities at so-called hyperbolic points (corresponding to rays hitting a boundary face non-tangentially) even if M has a smooth boundary. They have much in common with the methods employed in the study of N-body quantum Hamiltonians, which had been the central part of my research. There are natural extensions that I expect to work on with Richard Melrose and Jared Wunsch, such as more complicated geometric singularities, allowing for domains with corners whose boundary has an appropriate fibred structure, and a metric corresponding to this structure. An example of this setting is the blow-up of corners of a manifold with corners equipped with a smooth metric, in which the preceeding result already applies, but it is much more general. Its immediate relevance is that natural boundary conditions for the Laplacian on forms necessitate a resolution of the geometry at the corners, even if the metric itself is smooth.

During the previous six years I carried out a program in which I analyzed the microlocal structure of N-body Hamiltonians H in detail. Thus, I described propagation of 'singularities', here meaning lack of decay at infinity, for generalized eigenfunctions of H and analyzed its consequences, extending and generalizing geometric two-body type results of Melrose, Zworski and myself. More explicitly, these results give the precise form of the asymptotics of the wave function of the particles at infinity, which in turn has immediate implications for the structure of the scattering matrices, objects describing the outcome of scattering experiments. In this new setting, even at infinity, both classical and quantum effects are present (the latter arising when some subsystems have bound states), giving rise to phenomena which have no analogues for wave propagation.

Such a precise microlocal understanding of N-body scattering has numerous applications. Two examples are the analysis of the spectral shift function, which is an analogue of the counting function for the Laplacian in bounded domains (jointly with Xue Ping Wang), and inverse scattering, i.e. the determination of interactions from the scattering matrices (jointly with Gunther Uhlmann). Many interesting questions remain, such as the study of natural analogues of the spectral shift function for (locally) symmetric spaces, see the discussion below.

There are some points in phase space at which the theorem on the propagtion of singularities gives no new information; these are called radial points. While such points are absent for operators such as the wave operator, they play a central role in scattering theory – in fact, in the N-body setting these correspond to the 'propagation set' of Sigal and Soffer. Some basic questions about the behavior of PDEs at such points were answered by Guillemin and Schaeffer in the 70s under generic 'non-resonant' assumptions. A more precise understanding would be ultimately of significant help in N-body scattering, but applicability in that setting may be quite some time away. Andrew Hassell, Richard Melrose and I have obtained a detailed description of solutions of PDEs at radial points under a non-degeneracy hypothesis for the linearization of the classical mechanical system, but without resonance assumptions, both in the traditional setting of Guillemin-Schaeffer, and also in the closely related setting of scattering by a class of bounded potentials that do not decay at infinity (symbolic potentials of order 0). This work parallels recent work of Herbst and Skibsted, but gives more precise results. The key tool is positive commutator estimates using a module of pseudo-differential operators, which in turn captures the precise phase space structure of solutions of the PDE. (In the simplest setting, this would be a Lagrangian structure.)

Perhaps surprisingly, the structure of flats in higher rank symmetric spaces is very similar to the structure of the configuration space in N-body scattering, with the walls of the Weyl chambers playing the role of collision planes – now eigenvalues, rather than particles, collide! In a joint project with Rafe Mazzeo we have used the very detailed understanding of three-body problems to obtain results such as a uniform description of Green's function up to the walls on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$, giving a new approach to (slightly improved versions of) recent results of Anker, Guivarch, Ji and Taylor. Thus, our method contrasts with the classical methods of Harish-Chandra, Trombi and Varadarajan, which have been the main available tools in the field, and which rely on convergent perturbation series expansions for the spherical functions. Perhaps more interestingly, we have also shown that the resolvent of the Laplacian on symmetric spaces of non-compact type, of arbitrary rank, such as $\mathrm{SL}(N,\mathbb{R})/\mathrm{SO}(N,\mathbb{R})$, continues analytically across the continuous spectrum as an operator between appropriate spaces. Our method relies on complex scaling, which has played a central role in Euclidean scattering, and gives the analytic continuation rather directly. This shows the close connection between quantum scattering and very algebraic objects, namely symmetric spaces.

It is expected that these methods can also be adapted to locally symmetric spaces; this is the subject of ongoing research with Rafe Mazzeo.