Research Interests of Terence Tao

My main research interests are in real variable harmonic analysis, in the evolution of non-linear dispersive and wave equations, and in the combinatorics of the representation theory of $U(n)$.

1. Real-variable harmonic analysis

Harmonic analysis is devoted to the quantitative estimation of various expressions and operators (linear, sub-linear, multi-linear, or non-linear) which arise in geometry, analysis and PDE. Much of my work concerns endpoint estimates of such operators, which are often just barely beyond the reach of standard techniques. Although some of these estimates are then useful in applications (e.g. to PDE), there are two other reasons to study these (often technical) endpoint estimates. The first is to perfect existing techniques, and to develop new ones, which are often of quite general application. The other is that the proof of these estimates often reveals the underlying structure of the object under study, yielding valuable insights for further applications. A typical example is the humble operation of pointwise product $(f, g) \mapsto fg$. In order to study this operation on spaces such as Sobolev spaces, it turns out to be convenient to split this product into three *para-products* $fg = \pi_1(f, g) + \pi_2(f, g) + \pi_3(f, g)$, where π_1 denotes the interaction of the high frequencies of f with the low frequencies of g, π_2 denotes the interaction of high frequencies of f with the corresponding high frequencies of g, and π_3 denotes the interaction of low frequencies of f with the high frequencies of g. The three components of the pointwise product behave quite differently under various operations; for instance, we have $\nabla(fg) \approx \pi_1(\nabla f, g) + \pi_3(f, \nabla g)$. This decomposition of the product allows one to separate various types of interactions, and is especially useful in non-linear PDE.

Currently, I am focussing on three types of operators to study. Firstly, I am interested in maximal operators of Kakeya type. A typical such operator is the *Nikodym maximal operator* $f \mapsto f^{**}$, defined by

$$f^{**}(x) := \sup_{l \ni x} \int_l f$$

where l ranges over all lines passing through x. This operator is fundamental to many continuum problems involving the incidences of lines and points in space, such as the *Kakeya needle problem* (how much area does it take to rotate a needle in the plane?). As such it is connected with many problems in arithmetic combinatorics, such as the problem of locating arithmetic progressions in a sparse set. However, very basic questions about this operator (such as its mapping properties on Sobolev spaces) remain only partially understood. Some recent progress is in [5,19,24,25,33,36,45,51].

I am also interested in oscillatory integral operator, such as the solution operator $e^{it\Delta}$ to the free Schrödinger equation. A basic question concerns L^p estimates on these operators, or more generally to understand the structure of level sets of these operators. For instance: if a wave function solving the Schrodinger equation has finite probability (L^2 norm) at time 0, what can we say about the set of points in space time where the wave function focusses (exceeds a specific threshold)? These type of estimates turn out to be fundamental to the low-regularity theory of non-linear evolution equations. A recent approach, which appears to be quite powerful, is to decompose these waves into "wave packets" which are concentrated near various lines in spacetime, and then to use the Kakeya-type operators mentioned earlier to control the number of incidences between these lines. For instance, see [1,5,6,7,10,11,14,15,17,33].

Finally, I am also interested in multi-linear operators, of which the para-products mentioned earlier are a simple example. Other examples include the bilinear Hilbert transform and the Carleson maximal operator (which appears in the summation of Fourier series). Multi-linear operators often arise as the Taylor expansion of non-linear problems; for instance, the above two operators appear in the WKB expansion of Schrödinger eigenfunctions. These operators are not localized in either space or frequency, and so one needs to analyze the entire phase plane in order to treat these operators. See [26,38,43,44]. An application (currently in preparation with C. Muscalu and C. Thiele) is that the eigenfunctions of a one-dimensional Schrödinger operator are bounded at almost every energy level, provided that the potential is in the critical space L^2.

This is of course not an exhaustive list of operators which can be studied in harmonic analysis; some results on other operators can be found in [2,3,12,30,28,30,31,35,48,50].

2. Behaviour of non-linear dispersive and wave equations

I study the Cauchy problem for various non-linear dispersive and wave equations (Korteweg-de Vries, wave maps, Yang-Mills, Maxwell-Klein-Gordon, etc.) Classically, the problems are of local well-posedness (existence, uniqueness, and continuous dependence on the data for some non-zero time), global well-posedness, persistence of regularity (do singularities develop?), scattering (does the non-linear solution eventually approach a linear solution?), or - if blowup occurs - on the nature of the blowup.

Although many of these problems are phrase for smooth initial data, it is often useful to consider the problem in much rougher spaces, such as the space of all data with finite energy. Often the solution is then constructed by an iteration argument in some Banach space. This necessitates the development of multi-linear estimates for solutions to the linear equation (which could be the linear wave or Schrödinger equation), which is basically an oscillatory integral question of the type studied earlier. Examples of such iterative techniques can be found in [6,8,9,13,22,37].

Often, these iterative techniques only give a solution for short times. To extend this local solution to a global solution, one method is to exploit energy conservation. However this approach does not work directly if the data has infinite energy (e.g. it contains shocks). Fortunately, there are methods based on frequency decomposition which allow one to treat the infinite energy case in many circumstances, see [13,22,41,46,47]. A typical result [41], joint with J. Colliander, M. Keel, G. Staffilani, and H. Takaoka, is that the Korteweg-de Vries equation with a Dirac mass as initial data has a global solution, which always has the same amount of regularity as a Dirac delta (as measured using Sobolev spaces).

For so-called critical problems, the iterative approach often fails, however there appear to be "renormalization" tricks available in certain cases to overcome this [13, 18, 52, 53]. For instance, one can show [53] that solutions to the wave map equation with smooth initial data stay smooth for all time (no singularities can develop), provided that the energy is sufficiently small.

3. Combinatorics of the representation theory and symplectic geometry of $U(n)$.

I was originally drawn to this field (which is quite different from my other two main research areas) by conversations with A. Knutson on *Weyl's eigenvalue problem*: given the eigenvalues of two Hermitian matrices, what can one say about the eigenvalues of the sum of these matrices? A complete answer was conjectured by A. Horn in 1962, and much progress was made since, culminating in the final resolution of the conjecture in [16].

This problem can be rephrased in terms of the symplectic geometry of the co-adjoint orbits of $U(n)$. By the standard procedures of geometric quantization, this problem can then be discretized into the problem of decomposing the tensor product of irreducible $U(n)$ representations. This problem has a long history in itself, and is solved by the highly combinatorial Littlewood-Richardson rule. To resolve Horn's conjecture, a key step was to recast this rule in a more symmetric and geometric form, involving an object we call a *honeycomb*.

Honeycombs have since been useful in solving a few other related problems [21,49], but they are still quite mysterious, in that we do not know exactly why honeycombs compute the answer to Weyl's problem and to the tensor product problem. (All the proofs we have are highly recursive and combinatorial, and do not shed much light on what the fundamental connection is). I believe that a satisfactory explanation of why honeycombs appear in this problem will have significant impact on both representation theory and symplectic geometry.