

Degenerating Geometry to Combinatorics: Research Proposal for David E Speyer

A central appeal of algebraic geometry is that its functions are polynomials which can be written down as finite expressions. This makes the theory naturally combinatorial, in that polynomials are made up of a finite number of terms which interact through discrete rules. In practice, however, the operations performed on these polynomials are usually too complex for combinatorial methods to be of use. There are a number of strategies in modern algebraic geometry which can be seen as trying to simplify the theory enough to make combinatorial methods useful. In this paper, I describe three projects I am working on and will continue pursuing that bring combinatorics into algebraic geometry, usually via degeneration. Some technical details have been omitted; please see the cited sources.

In tropical geometry, the operations of addition and multiplication are degenerated to the simpler ones of addition and minimum. This transforms the polynomial problems of algebraic geometry into piecewise linear problems of polyhedral geometry. It also has proved to illuminate hidden structures in many classical fields, such as dynamical systems [4], valuation theory [2] and compactification of algebraic varieties.

My thesis is on tropical geometry in general and as applied to several major fields of algebraic geometry: the study of linear spaces, grassmannians, curves and real algebraic geometry. In my thesis, I introduce the notion of tropical linear spaces, combinatorial analogues of classical linear spaces. There are still several remaining problems about tropical linear spaces and about the tropical grassmannian which parameterizes them, I describe the most interesting of these problems in the first section.

Another approach to bringing combinatorics into algebraic geometry is to study spaces that themselves have an underlying combinatorial structure. One of the most recent and exciting approaches of this kind is the discovery of cluster algebras. This subject can be viewed as the study of varieties which are covered with tori glued in a combinatorial manner. In the second section, I describe some progress I have made in understanding cluster algebras and how I hope to proceed in the future.

A third approach to introducing combinatorics to algebraic geometry is to degenerate the underlying ring to a more combinatorial, often in some sense “monomial”, ring. This is the approach of the theories of Gröbner and SAGBI bases. In the third section, I describe ways I have applied and

will apply these techniques to the variety of k -tuples of flags modulo GL_n -equivalence. This variety is important because it encodes the combinatorics of tensor products of GL_n representations. Much of this work is joint with Ezra Miller.

1 Tropical Linear Spaces and Grassmannians

For f a nonzero polynomial, $f = \sum_{a \in A} f_a x_1^{a_1} \cdots x_r^{a_r}$, $A \subset \mathbb{Z}^r$, denote by $\mathrm{Trop} f$ the set of points $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that the minimum of $\sum_{i=1}^r a_i x_i$ is not unique as a ranges over A .

We define an $\binom{n}{d}$ -tuple of real numbers p_I to be tropical Plücker coordinates if

$$(p_I) \in \bigcap_{S,i,j,k,l} \mathrm{Trop}(P_{Sij}P_{Skl} - P_{Sik}P_{Sjl} + P_{Sil}P_{Sjk}).$$

For p_I tropical Plücker coordinates, we define the set

$$L(p) := \bigcap_J \mathrm{Trop}\left(\sum_{j \in J} \pm P_{J \setminus j} X_j\right)$$

to be a *tropical linear space*. One way to obtain tropical Plücker coordinates is to let $P_I(t)$ be the actual Plücker coordinates of a family of linear spaces $L(t)$ and let p_I be the order of vanishing of $P_I(t)$ as $t \rightarrow 0$. In this case, we call the p_I *realizable* and $L(p)$ is the set of possible orders of vanishing of $(x_i(t)) \in L(t)$. The set of all realizable (p_I) is called the tropical Grassmannian.

In my thesis, I show that $L(p)$ is a contractible pure d -dimensional polyhedral complex and there are good notions of dual and intersection of tropical linear spaces which again produce tropical linear spaces. I provide a fairly complete description of the local geometry of tropical linear spaces. In my paper with Bernd Sturmfels [22], we classify all tropical linear spaces in the case $d = 2$ and $d = 3$, $n = 6$.

Before stating some specific problems, let me explain some of the larger goals I see for investigating tropical grassmannians. I originally introduced tropical linear spaces and the tropical grassmannian because they are the logical first steps to developing tropical notions of vector bundles, Chern classes and so forth; classically all of these objects are understood by pullback from the grassmannian. This project is still in its infancy because simply understanding the tropical grassmannian proved both more challenging and more interesting than I originally expected. I still hope to return to this

project, especially since our very complete understanding of the case $d = 2$ should allow us to develop some good notion of a tropical vector bundle of rank 2.

Secondly, tropical methods have proved very productive in computing Gromov-Witten invariants (see [17]) of toric varieties. Genus zero Gromov-Witten invariants of the Grassmannian are fairly well understood; higher genus invariants are still quite mysterious. It is possible that tropical methods might prove profitable here. Success will require a careful understanding of the tropicalizations of Schubert varieties.

There are several more specific problems concerning tropical linear spaces that still remain open. Firstly, in my thesis, I define a notion of “series-parallel” tropical linear space and provide experimental and theoretical evidence for the following conjecture:

Conjecture 1. *Let L be a tropical d -plane in n -space. Then L has at most $\binom{n-2i}{d-i} \binom{n-i-1}{i-1}$ bounded faces of dimension i . Equality holds if and only if L is a series-parallel tropical linear space.*

I will continue to attempt to prove this conjecture. One possible hope is that I have recently learned from Paul Hacking about a theory that relates the f -vectors of tropical linear spaces to the combinatorics of certain compactifications of hyperplane arrangements; perhaps this will provide topological tools that will resolve the matter.

I show in my thesis that every tropical linear space which is made from hyperplanes by the repeated use of the operations of transverse intersection and dual is series-parallel and prove the conjecture for these spaces. This leads one to hope that they might be nice in other ways. In particular, it might be possible to answer the following question, which has proved intractable for general tropical linear spaces.

Problem 2. *For a fixed combinatorial type of series-parallel linear space, determine the cone of all p_I that induce it. Show that all such p_I are realizable.*

Alexander Postnikov [19] defines a rank d matroid M on an ordered ground set to be totally positive if there is an oriented matroid with underlying matroid M for which every basis (i_1, \dots, i_d) is positively oriented when written in increasing order. Postnikov has studied the realization spaces of totally positive matroids within the Grassmannian. In [23], Lauren Williams and I define a totally positive tropical linear space analogous to the notion

of totally positive matroid. The tropical totally positive grassmannian is the parameter space of totally positive tropical linear spaces.

I think it is very likely that, by strengthening Postnikov's results, I can show that every totally positive tropical linear space is realizable. Once this is proved, there would be a large part of the tropical grassmannian which could be described fully combinatorially. As one possible application of this, the orbit of this part of the tropical grassmannian under S_n would probably be almost the entire tropical grassmannian for n small (say $n \leq 10$ or so) and might with just a little work allow us to fill in the rest. Keel and Tevelev [11] have several conjectures about the relations between the tropical grassmannian and the Chow quotient of the grassmannian by the torus, and these cases would be useful for checking and developing their theory.

2 Combinatorics of Cluster Algebras

Cluster algebras are certain commutative algebras with a specified collection of generators. We will not give a definition of a cluster algebra here, but merely a description of their properties: For every cluster algebra A there is a simplicial complex Δ of pure dimension $n - 1$ whose vertices are labeled with elements of A . For every facet $\{x_1, \dots, x_n\}$ of Δ , A is a subring of the ring of Laurent polynomials in the x_i . There is an oriented graph called the diagram associated to each cluster and one moves between clusters by a process called mutation. The variables labeling the vertices are called cluster variables and the variables labeling a given facet are called clusters.

The most intriguing combinatorial question about cluster algebras is Fomin and Zelevinsky's positivity conjecture. Let y be a cluster variable and $\mathcal{I} = \{x_1, \dots, x_n\}$ a cluster so y is a Laurent polynomial in the x_i 's. Fomin and Zelevinsky conjecture that this Laurent polynomial has positive coefficients.

Although not phrased in this language, my paper [21] proves this conjecture in many cases. In the language of cluster algebras, that paper studies a cluster algebra in which a certain subset of the cluster variables is labeled by the elements of a 3-dimensional lattice L . I show that, when $y \in L$ and $\mathcal{I} \subset L$, then y is a Laurent polynomial with positive coefficients in the variables of \mathcal{I} . In fact, all of these coefficients are 1.

Since then, Gabriel Carroll and Gregory Price [3] have proved positivity in the case of a type A_n cluster algebra. Additionally, Andr e Henriques and Joel Kamnitzer [10] have discovered the "multi-dimensional octahedron re-

currence”, a variation of the preceding cluster algebra in whose variables live on an r -dimensional lattice. This multi-dimensional octahedron recurrence contains as a special case most of the cluster algebras that have been found in combinatorics: the cluster algebras of type A_n , the cluster algebras that Postnikov uses to parameterize cells of the totally positive grassmannian in [19], the cluster algebra associated to a triple crossing diagram in [24] and others. I have mostly solved the following problem.

Problem 3. *Find a combinatorial model, and hence prove positivity, for the multidimensional octahedron recurrence.*

The case of the multi-dimensional octahedron recurrence is essentially equivalent to studying situations where the diagram of the seed associated to the cluster $\{x_1, \dots, x_n\}$ is an oriented planar graph with bipartite dual and where each mutation involves a vertex of in-degree and out-degree each at most 2. (All terminology in the last sentence is as in [8].) Moving beyond the multi-dimensional octahedron recurrence to a full proof of positivity will require removing the planarity condition and finding a way to deal with mutations of higher degree vertices.

There are two lines of attack which may help remove the planarity assumption. The first is a series of papers, [5], [6] and [9], which connect the theory of cluster algebras to Teichmüller theory. These papers implicitly study the case where the diagram is embedded on a genus g surface.

Problem 4. *Give a combinatorial formula, for an arbitrary seed but retaining the degree constraints on the mutations in terms of intersecting curves on a genus g surface in which the diagram is embedded.*

The other line of attack is based on [20], where a bijection is shown between the perfect matchings of a bipartite planar graph and certain orientations of a related graph. Applying this bijection to the combinatorial model of [21], we see that the formula for y is given in terms of certain orientations of the diagram. As [20] requires no planarity assumption, this may be a starting point for finding a combinatorial model that applies to general cluster algebras.

Problem 5. *Give a combinatorial proof of positivity by counting orientations of the diagram of \mathcal{I} subject to certain conditions.*

Solving the above problem would, in a sense, provide an exact formula for the Laurent polynomials occurring in the corresponding cluster algebras.

A less ambitious goal is to simply determine which monomials occur in this Laurent polynomial. I have begun some work with Fomin and Zelevinski towards this end; our answers definitely seem in some sense to be related to root systems.

Problem 6. *Find an exact formula for the Newton polytope of the Laurent polynomials above. Does every lattice point within this Newton polytope occur with positive coefficient in the polynomial?*

Moving beyond positivity, Lauren Williams and I [23] discovered, provably in type A and experimentally in other cases, that the totally positive part of the tropicalization of a finite type cluster algebra with respect to its cluster variables is a simplicial fan with the same combinatorics as the complex Δ . To perform these computations, we had to solve certain cases of the preceding problem, adding an additional reason to want to compute these polytopes. A similar but not precisely equivalent conjecture is made in [5]. The following problem should not be too hard.

Problem 7. *Extend this result to the other finite type cases.*

Extending the above result to non-finite type cluster algebras requires making some careful definitions of tropicalization and totally positive part. I am not sure yet which definitions are the best to use, but preliminary computations suggest that a similar result might be true in the infinite case as well. The combinatorics of Δ in the infinite type case are unknown, but it seems clear that it will in some way be related to root systems of infinite Coxeter groups.

Problem 8. *Find the right formulation of this relationship between cluster complexes and totally positive tropicalizations and prove it. A starting point might be those infinite type algebras where there are only finitely many clusters up to automorphisms of the cluster algebra; these appear to be related in some manner to affine Coxeter groups.*

As coordinate rings of Grassmannians are cluster algebras, the connection between the totally positive part of tropicalizations and the combinatorics of cluster algebras should have a fruitful interplay with the theory of positive tropical linear spaces discussed in the previous section.

3 Degeneration of the Three Flag Variety

Let G denote GL_n , B the subgroup of upper triangular matrices and N the subgroup of upper triangular matrices with 1's on the diagonal. G/B is the space of flags in \mathbb{C}^n . Set $A = G/N$, A can be thought of as the space of flags in \mathbb{C}^n with a specified volume form on each subspace. It is a standard result that $\mathcal{O}(A) = \bigoplus_{\lambda} V_{\lambda}$ as a G representation. Moreover, there is a $T := (\mathbb{C}^*)^n$ action on A such that T acts by λ on V_{λ} . Set $X = A^3/G$, which can be thought of as the space of triples of flags modulo G -equivalence. Then $\mathcal{O}(X) = \bigoplus_{\lambda, \mu, \nu} (V_{\lambda} \otimes V_{\mu} \otimes V_{\nu})^G = \bigoplus_{\lambda, \mu, \nu} \mathbb{C}^{c_{\lambda\mu\nu}}$ where $c_{\lambda\mu\nu}$ is the Littlewood-Richardson coefficient. There is a T^3 action on X which acts on the (λ, μ, ν) summand with weight (λ, μ, ν) . Knutson and Tao [13], building on work of Berenstein and Zelevinsky [1], have found a cone **HIVE** known as the hive cone with a linear map $\pi : \mathbf{HIVE} \rightarrow \mathbb{Z}^{3n}$ such that the number of lattice points in $\pi^{-1}(\lambda, \mu, \nu) \cap \mathbf{HIVE}$ is $c_{\lambda\mu\nu}$. In work currently in preparation, I answer a challenge of Allen Knutson to show:

Proposition 9. *There is a flat, T^3 -equivariant degeneration X to the toric variety associated to the (unbounded) polyhedron **HIVE**.*

This degeneration arises by using the theory of cluster algebras to display $\mathcal{O}(X)$ as a subring of a Laurent polynomial ring and then taking leading terms of the Laurent polynomials involved. This degeneration explains the result of Knutson and Tao, as the dimension of the (λ, μ, ν) weight space of $\mathcal{O}(X)$ is preserved under this degeneration.

Let $w \in S_n$ and let B_w be the subset of $A \times A$ consisting of pairs of flags (with volume forms) that are w -related. Let \bar{B}_w be the closure of B_w .

Let $c_{\lambda\mu\nu}(w)$ be the dimension of the (λ, μ, ν) weight spaces for the T^3 action on $X_w := (\bar{B}_w \times A)/G \subset X$. When w is the identity, $c_{\lambda\mu\nu}(w) = c_{\lambda\mu\nu}$. This quantity is implicitly introduced in [14] which in turn states that this concept was “formulated ... as a fallout of [18]” and further studied in [13] and [15]. In joint work with Ezra Miller, I have extended the degeneration above to a degeneration from X_w to a union of toric varieties.

Proposition 10. *There is a T^3 equivariant degeneration from X_w to a union of torus orbits in the toric variety associated to **HIVE**. The components of this union are indexed by combinatorial objects known as pipe dreams, see [12], which are essentially equivalent to the rc-graphs of [7].*

This gives a combinatorial rule for computing the $c_{\lambda\mu\nu}(w)$ which is a modification of the honeycomb rule of Knutson and Tao. Using this rule, we are fairly certain that we can give a combinatorial proof of the Strong PRV conjecture, as stated in [14], namely,

The Strong PRV Conjecture. $c_{\lambda\mu(v\lambda+vw\mu)^*}(w) = 1$, where $v \in S_n$ is such that $(v\lambda + vw\mu)^*$ is in the positive Weyl chamber. Here $(\alpha_1, \dots, \alpha_n)^* = (-\alpha_n, \dots, -\alpha_1)$.

Once this work is completed, we should hopefully be able to use these results to study the geometry of the flag manifold by relating it to the geometry of toric varieties. The most ambitious plan is to try to use it to give a combinatorial rule for computing in the cohomology ring of the flag variety. Such a rule has apparently been found by Izzet Coskun, but his results are not yet fully verified and seem to be very complex to state. Other than possibly in Coskun's work, no such combinatorial rule is known.

Set $Y_w = (A \times \bar{B}_w)/G$. The problem of computing products in $H^\bullet(G/B)$ can be shown to be equivalent to computing the cohomology class of $X_v \cap Y_w$. After degeneration, X_v and Y_w become unions of toric orbit closures in a toric variety – a setting in which the cohomology ring is far better understood. There are several challenges to this approach, of which the largest are (1) the degenerations of X_v and Y_w are often no longer transverse and (2) the relation between the cohomology of a variety and of its singular degenerations can be quite subtle. Both of these problems, however, have standard geometric tools prepared to approach them, namely the theories of excess intersection and log geometry respectively, and the latter is especially well suited for toric varieties.

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