Research Statement

I am a low dimensional topologist. I work with knots and links inside three dimensional manifolds.

A knot K is an embedding of the circle S^1 inside a three-manifold Y. We throughout work in either the smooth category or the piecewise linear category, and depending on that, knots are either smooth embeddings or piecewise linear embeddings of S^1 inside Y. Two knots K_1 and K_2 are said to be equivalent if there is an isotopy (smooth or piecewise linear) that takes K_1 to K_2. A central question in knot theory is the following. Given two knots K_1 and K_2, when are they equivalent? A very natural way to approach this problem is by constructing powerful knot invariants. Given a knot, we associate to it an object of some category, such that if two knots are equivalent, then the corresponding objects are isomorphic in that category.

In 2001 [14] [13], Zoltán Szabó and Peter Ozsváth came up with some extremely powerful invariants for 3-manifolds, called Heegaard Floer homologies $HF(Y)$. There are four flavors, denoted by $\widehat{HF}(Y)$, $HF^-(Y)$, $HF^+(Y)$ and $HF^\infty(Y)$. To a particular presentation of a closed oriented 3-manifold Y, they associate certain chain complexes $\widehat{CF}(Y)$, $CF^-(Y)$, $CF^+(Y)$ and $CF^\infty(Y)$, such that the homologies of these chain complexes are independent of the presentation, and hence are invariants. Soon afterwards, they [12] and independently Jacob Rasmussen [17] refined these invariants to knot Floer homologies $HFK(K, Y)$, invariants of a knot K in a 3-manifold Y, which were eventually extended to include the case of links [16]. Even restricted to the case of knots in S^3, the knot Floer homology is a very powerful invariant, and due to its geometric definition, it has helped in the understanding of various properties of knots. For example, just by looking at the hat version $\widehat{HFK}(K, S^3)$ of the knot Floer homology for a knot K in S^3, one can recover its symmetric Alexander polynomial [12], one can calculate its genus [11] and one can determine whether or not K is fibered [10]. However this extremely beautiful theory of Heegaard Floer homology suffered one single flaw. The boundary map in all the chain complexes were defined by counting the number of points in certain moduli spaces, and the very geometric nature of such a definition which allowed people to prove many powerful results, also prevented a combinatorial understanding of the theory. There were a few specific cases, where a few specific tricks worked for computing the invariants, but there was no algorithm to compute the invariants in general.

1. Grid diagrams

In 2006, along with Ciprian Manolescu and Peter Ozsváth, I discovered an algorithm to compute all versions of the knot Floer homology and the link Floer homology for knots and links in S^3 [8]. The idea was to represent a knot or a link by a grid diagram, and then regard the grid diagram as a multipointed genus 1 Heegaard diagram for the knot or the link. A Heegaard diagram obtained by such a method turned out to be *nice* as defined in [19], and hence the count of holomorphic disks was combinatorial given by counting certain rectangles on the torus. Thus an $n \times n$ grid diagram Γ of a knot K allows us to define a chain complex $(C(\Gamma, \partial))$ combinatorially such that the following holds with coefficients in $\mathbb{Z}/2\mathbb{Z}$,

Theorem 1.1. [8] *The homology* $H_*(C(\Gamma, \partial))$ *is isomorphic to* $\widehat{HFK}(K, S^3) \otimes^{n-1} (\mathbb{Z}/2\mathbb{Z})^2$.

Further Research. Indeed the whole analysis can easily be extended for links, and similar statements hold for all versions of knot Floer and link Floer homologies. Later Ciprian Manolescu, Peter Ozsváth, Zoltán Szabó and Dylan Thurston gave a combinatorial proof that $H_*(C(\Gamma, \partial))$ is a knot invariant [9], thus giving a combinatorial proof of the invariance of knot Floer homology. However knot Floer homology detects many geometric structures of knots, like the genus or fiberedness, and even after having a complete combinatorial description of knot Floer homology in terms of grid diagrams, there has not been any combinatorial proof of these facts. I would like to investigate this in the future, maybe by using minimal genus Seifert surfaces adapted to a knot drawn on a grid as in [1].

2. Nice Heegaard diagrams

Very soon after the first paper, along with Jiajun Wang, I produced an algorithm to compute $\widehat{HF}(Y)$, the hat version of Heegaard Floer homology of a closed 3-manifold Y, and $\widehat{HFK}(K,Y)$, the hat version of the knot Floer homology for a knot K in Y. The idea was again to represent either the 3-manifold Y or the knot K inside Y by a *nice* pointed Heegaard diagram. The only non-combinatorial part of the Heegaard Floer theory is the count of certain pseudo-holomorphic disks. However in *nice* Heegaard diagrams that count is combinatorial, and thus the following two theorems provide an algorithm to compute the hat version of the Heegaard Floer homology.

Theorem 2.1. [19] *Given a nice Heegaard diagram for Y, $\widehat{HF}(Y)$ can be computed combinatorially. Similarly given a nice Heegaard diagram for a knot $K \subset Y$, $\widehat{HFK}(K,Y)$ can be computed combinatorially.*

Theorem 2.2. [19] *There is an algorithm to convert any pointed Heegaard diagram to a nice pointed Heegaard diagram by handleslides and isotopies.*

3. Maslov index of triangles

In [15], Peter Ozsváth and Zoltán Szabó introduced a smooth 4-manifold invariant with origins in Heegaard Floer homology, but with similar properties to that of Seiberg Witten invariants. The Heegaard Floer homology invariants for 3-manifolds and knots inside 3-manifolds involved counting the number of pseudo-holomorphic maps from a disk with two marked points on its boundary to certain spaces satisfying certain boundary conditions. To be able to count, they look at only those homotopy classes of disks where the expected dimension of the relevant moduli space is zero. This expected dimension is called the Maslov index, and for maps from the disk with two marked points on its boundary, the Maslov index is given by Lipshitz' formula [6]. This is the very first step for making the Heegaard Floer homology combinatorial, and in fact both the papers [8] and [19] rely heavily on the formula. However for 4-manifolds, the definition of the Heegaard Floer invariants required the count of the number of pseudo-holomorphic maps from a disk with three marked points in its boundary to some space satisfying some boundary conditions. In an attempt to understand the hat version of the 4-manifold invariants, I proved the following.

Theorem 3.1. [18] *The Maslov index of a map ϕ from a disk with three marked points on its boundary is given by $\mu(\phi) = e(\phi) + \mu_x(\phi) + \mu_y(\phi) + b(\phi).a(\phi)$, where e is the Euler measure, μ_x and μ_y are point measures, and $b(\phi)$ and $a(\phi)$ are homotopy classes of curves obtained as the image of ϕ restriced to certain parts of its boundary.*

Using this, and generalising the definition of *niceness* from [19], I could prove

Theorem 3.2. [18] *If a 4-dimensional cobordism W between Y and Y' is represented by a nice triple Heegaard diagram, then the induced map $\widehat{F}_W : \widehat{HF}(Y) \to \widehat{HF}(Y')$ can be computed combinatorially.*

Further Research. However to actually have an algorithm to compute this map $\widehat{F}_W$, one should be able to represent any cobordism by a *nice* diagram. This was achieved by Robert Lipshitz, Ciprian Manolescu and Jiajun Wang in [7], where they also prove certain versions of the above two theorems. However since there is no combinatorial proof of the naturality of $\widehat{HF}(Y)$, only the rank of $\widehat{F}_W$ can be computed. One way to address this problem would be to connect any two *nice* Heegaard diagrams representing the same 3-manifold by some *nice* isotopies and handleslides. This problem seems to be closely related to that of converting any pointed triple Heegaard diagram to a nice one by isotopies, handleslides and stabilisations. It is still open whether or not there is such an algorithm, and I plan to work on this problem in the future.

4. Distinguishing Seifert surfaces

In the meanwhile, I got interested in sutured manifolds, introduced by David Gabai in [2]. András Juhász has constructed a version of Floer homology for sutured manifolds (M, γ), called sutured Floer homology and denoted by $SFH(M, \gamma)$ [4]. However there is no algorithm to construct a Heegaard diagram representing a general sutured manifold. In [3], along with Matt Hedden and András Juhász, I provide an algorithm to produce Heegaard diagrams representing the sutured manifold obtained as the complement of a Seifert surface. This allows us to compute the sutured Floer homology of the Seifert surface complement, which for minimal genus surfaces, is simply the top term of the hat version of the knot Floer homology. However, the computation allows us to identify the relative $Spin^C$ gradings of the different generators of the sutured Floer homology, which is some extra information. Using this, we distinguish two minimal genus Seifert surfaces of the knot 8_3.

Theorem 4.1. [3] *The knot 8_3 has two minimal genus Seifert surfaces such that no isotopy of S^3 takes one surface to another.*

Further Research. The notion of equivalence that we use is that of weak equivalence. Two Seifert surfaces for the same knot are equivalent if there is an isotopy of S^3 that takes one surface to another. However there is also a notion of strong equivalence, where we in addition require the knot to remain fixed during the isotopy. The minimal genus Seifert surfaces for all prime knots upto 10 crossings has been classified [5] upto strong equivalence. In the future, we intend to investigate this list and classify all minimal genus Seifert surfaces upto weak equivalence.

5. Knot Floer homotopy

The project that I am currently working on is an attempt to generalise some of these Heegaard Floer homology invariants to stable homotopy invariants. In other words, given a knot K in S^3, I want to construct a CW complex X_K in a natural way, such that the stable homotopy type of the CW complex is a knot invariant and $H_*(X_K) = HFK^-(K, S^3)$. However the right hand side is related to the homology of a chain complex $C^-(\Gamma, \partial^-)$ defined in terms of a grid diagram Γ for the knot K [8]. We impose the additional restriction that X_K should have exactly one cell for each generator of this chain complex. Using the fact that certain posets are shellable and thin, and hence their order complexes are balls, I can prove

Theorem. *Given a knot K drawn on grid Γ, there is a CW complex $X_{K,i}$ whose stable homotopy type depends only on K and i, such that it has exactly one cell for each generator of $C^-(\Gamma, \partial^-)$ with Alexander grading i, and its homology is $H_*(C^-(\Gamma, \partial^-))$ in Alexander grading i.*

Further Research. There is a similar statement for $\widehat{HFK}(K, S^3) \otimes^{n-1} \mathbb{Z}^2$. Since this gives us a space whose stable homotopy type is a knot invariant, hence its stable homotopy groups are also knot invariants. Steenrod operations are stable cohomology operations that increase the grading, and hence they will act on knot Floer homology by decreasing the Maslov grading. I would like to investigate the action of Sq^k in the future to check if they are determined by some positive Maslov index k domains.

There is also a natural question as to whether this invariant computes anything new. I would like to find examples of knots K_1 and K_2 such that their knot Floer homologies are same but the stable homotopy types of X_{K_1} and X_{K_2} are different.

In a related project, using grid diagrams, I would like to convert knot Floer homology into a $(3+1)$ TQFT. For this I would like to construct maps on $HFK(K, S^3)$ corresponding to properly embedded surfaces in $S^3 \times I$. If such maps exist, and are invariants for the surfaces in $S^3 \times I$, then I would like to view these maps on homology as maps induced from a maps on stable homotopy. Such invariants might later lead to an invariant for smoothly embedded surfaces in 4-manifolds.

References

1. Ivan Dynnikov, *Arc-presentation of links. Monotone simplification*, arXiv:math/0208153v2.
2. David Gabai, *Foliations and the topology of* 3*-manifolds*, Journal of Differential Geometry **18** (1983), 443–503.
3. Matt Hedden, András Juhász, and Sucharit Sarkar, *On sutured Floer homology and the equivalence of Seifert surfaces*, arXiv:0811.0178v1.
4. András Juhász, *Floer homology and surface decompositions*, Geometry and Topology **12** (2008), no. 1, 299–350.
5. Osamu Kakimizu, *Classification of the incompressible spanning surfaces of prime knots of* 10 *or less crossings*, Hiroshima Math Journal **35** (2005), no. 1, 47–92.
6. Robert Lipshitz, *A cylindrical reformulation of Heegaard Floer homology*, **10** (2006), 955–1096.
7. Robert Lipshitz, Ciprian Manolescu, and Jiajun Wang, *Combinatorial cobordism maps in hat Heegaard Floer theory*, Duke Mathematical Journal, to appear.
8. Ciprian Manolescu, Peter Ozsváth, and Sucharit Sarkar, *A combinatorial description of knot Floer homology*, Annals of Mathematics, to appear.
9. Ciprian Manolescu, Peter Ozsváth, Zoltán Szabó, and Dylan Thurston, *On combinatorial link Floer homology*, Geometry and Topology **11** (2007), 2339–2412.
10. Yi Ni, *Knot Floer homology detects fibred knots*, Inventiones Mathematicae **170** (2007), no. 3, 577–608.
11. Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and genus bounds*, Geometry and Topology **8** (2004), 311–334.
12. ______, *Holomorphic disks and knot invariants*, Advances in Mathematics **186** (2004), no. 1, 58–116.
13. ______, *Holomorphic disks and three-manifold invariants: properties and applications*, Annals of Mathematics **159** (2004), no. 3, 1159–1245.
14. ______, *Holomorphic disks and topological invariants for closed three-manifolds*, Annals of Mathematics **159** (2004), no. 3, 1027–1158.
15. ______, *Holomorphic triangles and invariants for smooth four-manifolds*, Advances in Mathematics **202** (2006), no. 2, 326–400.
16. ______, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebraic and Geometric Topology **8** (2008), 615–692.
17. Jacob Rasmussen, *Floer homology and knot complements*, Ph.D. thesis, Harvard University, 2003.
18. Sucharit Sarkar, *Maslov index of holomorphic triangles*, arXiv:math/0609673v2.
19. Sucharit Sarkar and Jiajun Wang, *An algorithm for computing some Heegaard Floer homologies*, Annals of Mathematics, to appear.