Research description

The general area of my interests is partial differential equations, more specifically nonlinear wave and Schrödinger equations.

One of the most fascinating aspects of the subject of partial differential equations is its connection and interaction with the other branches of mathematics such as geometry, analysis, and physics.

A large part of my research focuses on the study of the nonlinear evolution equations originating in the problems of mathematical physics, most importantly the Einstein equations.

Geometry, analysis, and physics naturally come together in the study of the Einstein equations. The Einstein equations describe a 4-dimensional Lorentzian manifold which is Ricci flat (in vacuum) or, in general, whose gravitational tensor, composed from the Ricci and scalar curvatures, is set to be equal to an energy-momentum tensor of the matter fields (scalar, electromagnetic, fluids).

The Einstein equations have several unique features. They are overdetermined, which leads to the constraint equations on the initial data (given by a 3-dimensional riemannian hypersurface and its second fundamental form). The concept of evolution is also different since there is no a priori given time function. In addition, there is a constant interplay between the properties of the metric, obtained as a solution of the PDE: the Einstein equations, and the generated geometry, expressed in terms of its causal structure.

This relation can be illuminated through the study of the rough solutions of the Einstein equation. In this approach, one chooses a set of the initial data, a Riemannian metric and a second fundamental form from a Sobolev space of low regularity, and shows that it can be extended, at least locally, as a solution of the Einstein equations of the same regularity. This question is of the particular interest to me and is a subject of my joint work with Sergiu Klainerman. We have recently been able to extend the local well-posedness theory of the Einstein equations to the space of metrics whose curvature is slightly above L^2 in the scale of Sobolev spaces [12].

While the classical solutions detect only the Lorentzian nature of the metric, the rough solutions require and provide a much more refined information on its casual structure [12,15,16,17].

From the PDE point of view the Einstein equations become well defined evolution equations only after a choice of the appropriate gauge. In particular, in wave coordinates the Einstein equations can be written as a quasilinear hyperbolic system, which can be thought of as a wave equation for a metric evolving on the background prescribed by the same unknown metric.

To attack the problem of local existence and uniqueness of low regularity solutions for such PDE requires development of new analytic tools [9,11,12]. The standard Fourier based methods, so effective in the study of the wave equation on a fixed Minkowski background, fail in this setting.

The mathematical theory of the Einstein equations is still in the process of development. There is a tremendous wealth of open problems. Among them is the L^2 conjecture asserting the existence of the Einstein metrics with curvature in L^2, strong stability of Minkowski space, stability of Kerr solutions.

I am also interested in understanding the behavior of solutions of the Yang-Mills, Maxwell-Klein-Gordon, and Wave Maps equations. These equations describe nonlinear field theories set on a fixed Lorentzian background (most popularly the Minkowski space). They can be loosely described by the systems of semilinear hyperbolic equations of the type

$$\partial_t^2 \phi - \triangle \phi = Q(\phi, \partial\phi)$$

The nonlinearity Q has a special structure which often becomes apparent only after the choice of the appropriate gauge. It is precisely the structure of the nonlinear term that controls the answers to the questions of global existence and long time behavior of the solutions, as well the issue of the optimal regularity of local solutions.

While in general, with exception of several specific and very important dimensions where one is aided by the presence of strong conserved quantities, solutions corresponding to the large initial data are expected to blow up in finite time, the small data solutions exist globally in time. However, it is important to understand the question of existence and uniqueness of small data solutions which live precisely in the space of functions critical with respect to the natural scaling symmetry associated with the equation. There has been a considerable progress on this problem in the case of the Wave Maps equation (for some results see [8] and, certainly, the work of T. Tao and others) but it remains completely open for the Yang-Mills and Maxwell-Klein-Gordon equations. The study of the small data critical regularity should also potentially lead to better understanding of the blowup of the large solutions (or their global existence as hoped for the Wave Maps in dimension $n = 2$), where the questions range from describing the nature of the blowup to the stability of the blowup solutions.

Another aspect of my research is concerned with the study of the dispersive equations, more specifically linear and nonlinear Schrödinger equations

$$i\partial_t\psi + \triangle\psi + V(t,x)\psi(\text{ or } F(|\psi|^2)\psi) = 0,$$

which arise naturally in quantum mechanics and nonlinear optics.

One of the important questions in the study of the nonlinear Schrödinger equation is the question of the final state of the evolution. The first step on this road is the asymptotic stability of special solutions, so called solitons, and their combinations, multi-soliton states.

The investigation of stability leads to the splitting of the dynamics into the finite dimensional unstable part corresponding to the manifold of special solutions and the infinite-dimensional part, which is needed to disperse. Interestingly enough, the interaction between the above two dynamics is described, in the first approximation, by the linear Schrödinger equation with a time-dependent potential $V(t,x)$. The dispersion needed for the infinite dimensional part of the dynamics is related to time decay properties of the solutions of the linear equation.

In turn, already the decay or dispersive properties of the solutions of the Schrödinger equation with a time-dependent potential is to a large extent an untapped area. However, we are able to obtain the desired decay results in some important cases [14,18].

I am also interested in understanding refined properties of solutions of the spatially periodic Schrödinger equation, for which the dispersion manifests itself in a much more subtle way [4,5]. In particular, solutions exhibit fractal behavior [7]. This behavior is connected with the diophantine properties of the real numbers.

The diophantine properties of more complicated objects are themselves focus of interest. One example of this is the behavior of the words generated by two elements of the group $SO(3)$. The Gamburd-Jacobson-Sarnak conjecture asserts that the analogue of the Khintchine's Theorem for real numbers holds true in this setting. Although it is till open, the first step in understanding this problem was taken in [10], where a weaker version of the conjecture was proved.