Research Interests

Mircea Mustaţă

My main research interest is in algebraic geometry and in its interplay with other fields like combinatorics and commutative and noncommutative algebra. I am working on problems about singularities of algebraic varieties via jet schemes, syzygies of general sets of points on projective curves and connections with vector bundles, and toric varieties.

1. Singularities of algebraic varieties via jet schemes

The jet schemes of a complex variety X are higher order analogues of the tangent space TX, giving more refined information about the nature of the singularities of X. The mth jet scheme X_m parametrizes morphisms $\operatorname{Spec} \mathbf{C}[t]/(t^{m+1}) \longrightarrow X$. I am trying to understand how to translate the algebraic properties of these schemes (dimension, number of irreducible components) into properties of the singularities of X.

It has been understood for a long time that even if one is interested only in classifying smooth varieties, singular ones naturally enter into the picture. In the work centered around the Minimal Model Program, there are several classes of singularities which play a proeminent role: terminal, canonical, log terminal and log canonical. For example, canonical singularities are precisely the ones that appear on what are called the canonical models of varieties.

These classes of singularities are defined using a resolution of singularities for X. It turns out, however, that there are close connections with the jet schemes of X. The link is provided by the motivic integration theory, due to M. Kontsevich, V. Batyrev, and J. Denef and F. Loeser. The idea is that by embedding X in a smooth variety Y, one can encode the information about the jet schemes of X into an integral defined on the infinite dimensional space $Y_\infty = \operatorname{proj\,lim}_m Y_m$. The change of variable formula for these motivic integrals allows to compare this integral with an other integral defined on the space Y'_∞, where $Y' \longrightarrow Y$ is an embedded resolution of singularities for (Y, X).

For example, I applied this method to prove that if X is a locally complete intersection variety, then X has canonical singularities if and only if X_m is irreducible for every m. This statement had been conjectured by D. Eisenbud and E. Frenkel who applied it to deduce a generalization of Kostant's Freeness theorem in the setting of jet schemes.

This technique can be used to give a formula for the log canonical threshold. This is an invariant which appears in different approaches to singularities and it measures how far a pair (Y, X) is from being log canonical. It can be shown that if X is a subscheme of Y and Y is smooth, then the log canonical threshold of (Y, X) is given by $\dim Y - \sup_m\{(\dim X_m)/(m+1)\}$.

I hope that this connection between jet schemes and the log canonical threshold might shed new light on this invariant which in general is quite mysterious. There are several open problems about its behaviour in which this description might be useful. A better understanding of the growth pattern of the dimension of jet schemes might be relevant in this context.

2. Syzygies of general sets of points on projective curves and vector bundles

Let $X \subset \mathbf{P}^n$ be a smooth curve embedded in the projective space. There has been a lot of work in trying to understand the relation between geometric properties of the curve and the minimal free resolution of the (homogeneous coordinate ring of the) curve. The outstanding conjecture in the field is due to M. Green and says that if the curve is embedded by the canonical series, one can read off the resolution the Clifford index Cliff(X) of X, which is an invariant which measures the existence of special linear series on X.

Suppose now that $\Gamma \subset X$ is a large set of general points on X. The minimal free resolution of Γ contains the resolution of X, plus a residual part. This part is periodic (in a suitable sense) in the number of points with period deg X, an assertion which had been conjectured by S. L'vovsky. I am interested in understanding the numerical information coming from this residual part and how this is connected with geometric properties of the curve.

It is possible to give lower bounds for the ranks of the free modules which appear in the residual part. In analogy with a much studied question for general points in $\mathbf{P}^n$, one says that the curve satisfies the Minimal Resolution Conjecture (MRC) if these bounds are achieved. For example, if X is a rational quintic in $\mathbf{P}^3$, then it X satisfies MRC iff X does not lie on a smooth quadric (which is true if X is general).

More generally, if $Q = T_{\mathbf{P}^n}(-1) \mid_X$, then X satisfies MRC if and only if all the exterior powers of Q satisfy a certain generic vanishing property studied by Raynaud and which can be seen as a strong stability of Q.

For example, it has been shown by M. Popa that a curve embedded by a complete linear series of large degree does not satisfy MRC (this was used to give examples of base points of the generalized theta divisor in the moduli space of vector bundles with trivial determinant).

Recently, in joint work with G. Farkas and M. Popa, we have have proved that if X is a nonhyperelliptic curve embedded by the canonical linear series, then X satisfies MRC. This can be translated as saying that for every p, the vector bundle $\wedge^p Q$ has a theta divisor. In fact, we have shown that this divisor is the difference variety $X_{g-p-1} - X_p$. Further study of Q might be particularly interesting, as it is known from work of K. Paranjape and S. Ramanan that Green's conjecture can be reformulated as saying that $h^0(\wedge^i Q) \leq \binom{g}{i}$, for every $i \leq \mathrm{Cliff(X)}$, where g is the genus of X.

3. Toric varieties

The study of toric varieties is based on an interplay between combinatorics, algebraic geometry and commutative algebra. I am especially interested in how the description via Cox's homogeneous coordinate ring can be used to translate back and forth between geometric and algebraic properties.

The homogeneous coordinate ring is a polynomial ring graded by the class group of the variety and having a distinguished monomial ideal. It was introduced by D. Cox, based on earlier work of M. Audin, to extend the quotient construction of the projective space and Serre's correspondence to arbitrary toric varieties. It can be used to give a unified approach to vanishing results on toric varieties. In joint work with D. Eisenbud and M. Stillman we applied this description to give algorithms for computing cohomology of coherent sheaves on toric varieties.

An other application of this approach, joint with G. Smith, H. Tsai and U. Walther, is to a description of D-modules on smooth toric varieties in terms of a category of graded modules over the Weyl algebra corresponding to the homogeneous coordinate ring. Our correspondence generalizes previous results of Musson on rings of differential operators on toric varieties. A related interesting question which is open is how to give a good description for push-forward of D-modules at the level of graded modules.

A problem I find interesting is that of describing the derived category of coherent sheaves on a complete toric variety. In the case of the projective space $\mathbf{P}(V)$, a theorem of Bernstein, Gelfand and Gelfand describes this category as the derived category of graded modules over the exterior algebra $\wedge(V^*)$. This has been recently used by D. Eisenbud and F.-O. Schreyer to give a new perspective to homological methods on $\mathbf{P}(V)$. A similar description in the case of arbitrary complete toric varieties (possibly involving modules over the exterior algebra of the homogeneous coordinate ring) might have similar applications.