Research prospectus

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July 19, 2004

My research interests lie in the area of the moduli of Riemann surfaces and its interplay with other fields of mathematics, especially hyperbolic geometry, algebraic geometry, symplectic geometry and ergodic theory.

Let X be a complete hyperbolic Riemann surface of genus g with n punctures. My work has been motivated by the problem of estimating $s_X(L)$, the number of primitive *simple* closed geodesics of hyperbolic length less than L on X. To explore this problem, we have followed two approaches: the first using symplectic geometry of moduli spaces of curves, and the second using ergodic theory of the earthquake flow. Both methods provide new results and insights about the moduli space $\mathcal{M}_{g,n}(\ell_1,\dots,\ell_n)$ of Riemann surfaces with geodesic boundary components, the bundle of holomorphic quadratic differentials over $\mathcal{M}_{g,n}$ and the space of measured laminations.

Volumes of moduli spaces of curves. Let $S_{g,n}$ be a compact, connected, oriented surface of genus g with n boundary components $\{\beta_i\}_{i=1}^n$ with $\chi(S_{g,n})<0$. The mapping class group $\mathrm{Mod}_{g,n}$ of $S_{g,n}$ acts on the Teichmüller space $\mathcal{T}_{g,n}$ of complete hyperbolic Riemann surfaces marked by $S_{g,n}$. The quotient space

$$\mathcal{M}_{g,n}=\mathcal{T}_{g,n}/\,\mathrm{Mod}_{g,n}$$

is the moduli space of hyperbolic Riemann surfaces of genus g with n cusps. The space $\mathcal{T}_{g,n}$ is a finite-dimensional complex manifold equipped with the Weil-Petersson Kähler metric. The Weil-Petersson volume of the moduli space $\mathcal{M}_{g,n}$ is a finite number and its value as a function of g and n arises naturally in different contexts. We find it fruitful to consider more generally the moduli space of bordered Riemann surfaces with fixed geodesic boundary lengths. We approach the study of the volumes of these moduli spaces via the length functions of simple closed geodesics on a hyperbolic surface. We establish:

Theorem 1 *The volume $V_{g,n}(\ell_1,\dots,\ell_n)=\mathrm{Vol}(\mathcal{M}_{g,n}(\ell_1,\dots,\ell_n))$ is a polynomial in $\ell_1,\dots,\ell_n$, namely:*

$$V_{g,n}(\ell)=\sum_{|\alpha|\le 3g-3+n} C_\alpha\cdot \ell^{\,2\alpha},$$

where $C_\alpha>0$ lies in $\pi^{6g-6+2n-2|\alpha|}\cdot\mathbb{Q}$.

Here the exponent $\alpha = (\alpha_1, \ldots, \alpha_n)$ ranges over elements in $(\mathbb{Z}_+)^n$, $\ell^\alpha = \ell_1^{\alpha_1} \cdots \ell_n^{\alpha_n}$, and $|\alpha| = \sum \alpha_i$. We also give an explicit recursive method for calculating these polynomials. The constant term of the polynomial $V_{g,n}(\ell)$ is the volume of $\mathcal{M}_{g,n}$, the traditional moduli space of closed surfaces of genus g with n marked points.
Example: Using our recursive method, we get:

$$V_{1,1}(\ell_1) = \ell_1^2/24 + \pi^2/6,$$

$$V_{1,2}(\ell_1, \ell_2) = (4\pi^2 + \ell_1^2 + \ell_2^2)(12\pi^2 + \ell_1^2 + \ell_2^2)/384,$$

and

$$V_{0,4}(\ell_1, \ldots, \ell_4) = (4\pi^2 + \ell_1^2 + \ell_2^2 + \ell_3^2 + \ell_4^2)/4.$$

Our point of departure for calculating these volume polynomials is the following result of G. McShane:

Theorem 2 (McShane) *Let X be a hyperbolic once-punctured torus. Then we have*

$$\sum_\gamma (1 + e^{\ell_\gamma(X)})^{-1} = \frac{1}{2}, \tag{0.1}$$

where the sum is over all simple closed geodesics γ on X.

One challenge is to find a McShane-type formula for closed surfaces. The discussion above motivates the study of the *simple* length spectrum of a hyperbolic surface. It is interesting to know if there is a bound on the multiplicities (the number of simple closed geodesics of the same length) depending only on g and n. The special case of this question for $g = n = 1$ is related to the uniqueness conjecture for Markoff triples.
The Kontsevich-Witten formula. By applying the method of symplectic reduction, we obtain a formula for the volume polynomial $V_{g,n}(\ell)$ in terms of the intersection numbers of tautological line bundles over $\overline{\mathcal{M}}_{g,n}$.

Theorem 3 *The coefficients of the volume polynomial* $\mathrm{Vol}(\mathcal{M}_{g,n}(\ell_1, \ldots, \ell_n)) = \sum C_\alpha \ell^\alpha$ *are given by*

$$C_\alpha = \frac{2^{|\alpha|}}{\alpha!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \cdot w^{3g-3-|\alpha|},$$

where ψ_i is the first Chern class of the ith tautological line bundle, w is the Weil-Petersson symplectic form, $\alpha! = \prod \alpha_i!$ and $|\alpha| = \sum \alpha_i$.

Thus our algorithm for calculating volumes leads to a recursive formula for these intersection numbers and gives a new proof of the Witten-Kontsevich formula for the intersection numbers of tautological classes on $\overline{\mathcal{M}}_{g,n}$.

It would be interesting to understand the intersection numbers involving the Chern classes of the Hodge bundle using similar methods.
Our proof of the Witten-Kontsevich formula suggests some similarities between

$\mathcal{M}_{g,n}$ and the variety $\mathrm{Hom}(\pi_1(S), G)/G$ of representations of the fundamental group of the surface S in a compact Lie group G, up to conjugacy.

Growth of the number of simple closed geodesics. For $X \in \mathcal{M}_{g,n}$, let $c_X(L)$ be the number of primitive closed geodesics on X of length $\leq L$. By work of Delsart, Huber, Selberg and Margulis, we have

$$c_X(L) \sim e^L/L$$

as $L \to \infty$. However, very few closed geodesics are *simple* and it is hard to discern them in $\pi_1(S_{g,n})$.

Let $\mathcal{ML}_{g,n}$ be the space of *measured laminations* on $S_{g,n}$. There is a one-to-one correspondence between the integral measured laminations, $\mathcal{ML}_{g,n}(\mathbb{Z})$, and unions of disjoint essential simple closed curves on $S_{g,n}$, up to isotopy. There is a natural symplectic form on $\mathcal{ML}_{g,n}$ preserved by the action of $\mathrm{Mod}_{g,n}$.

For any $X \in \mathcal{T}_{g,n}$ and $\lambda \in \mathcal{ML}_{g,n}$, let $\ell_\lambda(X)$ denote the hyperbolic length of λ on X.

To understand the growth of $s_X(L)$, it proves fruitful to fix a simple closed curve $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$ and consider more generally the counting function

$$s_X(L,\gamma) = \#\{\alpha \in \mathrm{Mod}_{g,n} \cdot \gamma \mid \ell_\alpha(X) \leq L\}.$$

There are only finitely many isotopy classes of simple closed curves on $S_{g,n}$ up to the action of the mapping class group. Therefore, summing $s_X(L,\gamma)$ over representatives of these orbits gives $s_X(L)$, and the asymptotics of the $s_X(L,\gamma)$'s determines the asymptotics of $s_X(L)$. We show :

Theorem 4 *For any* $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$*, we have*

$$\lim_{L\to\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = n_\gamma(X),$$

where $n_\gamma(X)$ *is a smooth proper function of* $X \in \mathcal{M}_{g,n}$*.*

In the case of $\mathcal{M}_{1,1}$, this result was previously obtained by McShane and Rivin. The upper and lower estimates for $S_X(L)$ when $X \in \mathcal{M}_{g,n}$ were obtained by M. Rees and I. Rivin.

Frequencies of different types of simple closed curves. We now discuss more precisely how $n_\gamma(X)$, the constant in the growth rate of $s_X(L,\gamma)$, depends on X and on the simple closed curve γ.

Let B_X be the unit ball in the space of measured geodesic laminations with respect to the length function at X:

$$B_X = \{\lambda \mid \ell_\lambda(X) \leq 1\} \subset \mathcal{ML}_{g,n}.$$

We show that B_X is convex with respect to the piecewise linear structure of $\mathcal{ML}_{g,n}$. Let $B(X) = \mathrm{Vol}(B_X)$ with respect to the Thurston volume form on $\mathcal{ML}_{g,n}$. We show that

$$b_{g,n} = \int_{\mathcal{M}_{g,n}} B(X)\, dX$$

is a finite number in $\pi^{6g-6+2n} \cdot \mathbb{Q}$ which can be calculated in terms of the leading coefficients of the volume polynomials.

We show that the contributions of X and γ to $n_\gamma(X)$ separate as follows:

Theorem 5 *For any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$, there exists a rational number c_γ such that we have:*

$$n_\gamma(X) = \frac{c_\gamma \cdot B(X)}{b_{g,n}}.$$

It follows that the relative frequencies of different types of simple closed curves on X are universal rational numbers.

Corollary 6 *For $X \in \mathcal{M}_{g,n}$ and $\gamma_1, \gamma_2 \in \mathcal{ML}_{g,n}(\mathbb{Z})$, we have*

$$\lim_{L \to \infty} \frac{s_X(L, \gamma_1)}{s_X(L, \gamma_2)} = \frac{c_{\gamma_1}}{c_{\gamma_2}} \in \mathbb{Q}.$$

The limit is a positive rational number independent of X.

Remark. The exact same result holds when the surface X has variable negative curvature.

Example: Let γ_i be a simple closed curve on $S_{g,0}$ such that $S_{g,0} - \gamma_i \cong S_{i,1} \cup S_{g-i,1}$. Then we have

$$\frac{s_X(L, \gamma_i)}{s_X(L, \gamma_j)} \to \frac{\binom{g}{i}}{\binom{g}{j}}$$

as $L \to \infty$.

We can also calculate c_γ recursively using our recursive formula for $V_{g,n}(\ell)$. In fact, we can write the number c_γ in terms of the intersection numbers of tautological line bundles over the moduli space of Riemann surfaces of type $S_{g,n} - \gamma$.

Similar counting problems for the number of saddle connections for a *generic* Abelian differential have been studies by Masur, Eskin and Zorich. Constants in the quadratic asymptotics and frequencies of different types of saddle connections are related to volumes of moduli spaces of holomorphic Abelian differentials.

From $\mathcal{ML}_{g,n}$ to holomorphic quadratic differentials. We study the relationship between the earthquake flow on $\mathcal{PM}_{g,n}$ the bundle of geodesic measured laminations and the Teichmuller horocycle flow on $\mathcal{QM}_{g,n}$ the bundle of holomorphic quadratic differentials. Our main result is:

Theorem 7 *The earthquake flow and the Teichmüller horocycle flow are measurably isomorphic.*

It is known that the Teichmüller horocycle flow is ergodic with respect to the Lebesgue measure class. Therefore, we have:

Corollary 8 *The earthquake flow on $\mathcal{P}^1\mathcal{M}_{g,n}$ is ergodic with respect to the Lebesgue measure class.*

We also study the invariant measure for the earthquake flow on $\mathcal{P}^1\mathcal{M}_{g,n}$.

Theorem 9 *There exists a finite invariant measure $\nu_{g,n}$ for the earthquake flow on $\mathcal{P}^1\mathcal{M}_{g,n}$. This measure projects to the volume form given by $B(X)\cdot\mu_{wp}$ on $\mathcal{M}_{g,n}$.*

Then by using the relation between $\mathcal{P}^1\mathcal{M}_{g,n}$ and $\mathcal{Q}^1\mathcal{M}_{g,n}$, we obtain:

Theorem 10 *We have:*

$$\mathrm{Vol}(\mathcal{Q}^1\mathcal{M}_{g,n}) = \int\limits_{\mathcal{M}_{g,n}} B(X)\, dX.$$

The value of $\mathrm{Vol}(\mathcal{Q}^1\mathcal{M}_{g,n})$ arises in several problems related to billiards and dynamics of interval exchange maps. Volumes of different strata of moduli spaces of holomorphic Abelian differentials have been calculated by A. Eskin and A. Okounkov.

Equidistribution results and counting simple closed geodesics. The growth of the number of simple closed geodesics, $s_X(L)$, can be also investigated via the dynamics of Thurston's earthquake flow on moduli space. Here we present our results on equidistribution of *horospheres*.

Our equidistribution results suggest more analogies between the earthquake flow and unipotent flows on homogeneous spaces. The latter are now rather well-understood by work of Ratner, Margulis and Dani.

For any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$ and $L > 0$, we define an ergodic, earthquake-flow invariant probability measure $\nu_\gamma(L)$ on $\mathcal{PM}_{g,n}$ supported on the set

$$H_\gamma(L) = \{(X,\lambda) \mid \ell_\gamma(X) = L,\ \ell_\lambda(X) = 1,\ i(\gamma,\lambda) = 0\}/\operatorname{Mod}_{g,n} \subset \mathcal{P}^1\mathcal{M}_{g,n}.$$

The preimage of $H_\gamma(L)$ in $\mathcal{PT}_{g,n}$ lies above the set

$$\{X \mid \ell_\gamma(X) = L\ \} \subset \mathcal{T}_{g,n}$$

which is analogous to a *horosphere*. We prove the following equidistribution result:

Theorem 11 *As L tends to infinity, the horosphere meaure $v_\gamma(L)$ become equidistributed with respect to the unique invariant probability measure in the Lebesgue measure class; that is,*

$$\nu_\gamma(L) \to \nu_{g,n}/b_{g,n}$$

as $L \to \infty$.

In particular, if γ is a maximal system of simple closed curves then by using the fact that $v_X(\mathcal{PML}_{g,n}) = B(X)$, we show that the images of the level sets

$$\{X \in \mathcal{T}_{g,n} \mid \ell_\gamma(X) = L\}$$

become equidistributed with respect to the measure $B(X)\cdot\mu_{wp}$ in $\mathcal{M}_{g,n}$ as $L \to \infty$.

It is an interesting open problem to know if the earthquake flow and horocycle flow are actually topologically equivalent. Also, we do not know if the earthquake flow can be extended to a $PSL(2,\mathbb{R})$ action on $\mathcal{PM}_{g,n}$. Generalizing our methods also would yield information about the volume of $\mathcal{Q}^1_{g,n}(a_1,\ldots,a_n)$.

We would like to classify the ergodic measures for the earthquake flow and prove results analogous to Ratner's rigidity theorems for unipotent flows on homogeneous spaces. We speculate that any ergodic earthquake flow invariant measure is "geometric". Understanding the closure of orbits of the earthquake flow in $\mathcal{P}^1\mathcal{M}_{g,n}$ would shed light to the classification of ergodic measures for $PSL(2,\mathbb{R})$ action on the moduli space of holomorphic quadratic differentials. The problem of the classification of all ergodic measures for the earthquake or horocycle flow would shed light on the asymptotic behaviour of the number of saddle connections of *all* 1-forms (not just generic ones). A more approachable problem is the classification of the ergodic measures of the action of the mapping class group on $\mathcal{ML}_{g,n}$.