RESEARCH DESCRIPTION

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My main area of interest is algebraic geometry and its connections with other fields such as symplectic geometry, mathematical physics, and combinatorics. More specifically, I have been working in the area of Gromov-Witten theory – the study of holomorphic maps from Riemann surfaces to an algebraic variety. This subject includes many classical questions of enumerative algebraic geometry, such as Kontsevich's calculation of the number of degree d rational curves in the plane through $3d-1$ general points. However, recent interaction with mathematical physics has also motivated a network of conjectural relations to gauge and orbifold theories. In my future research, I hope to pursue a better understanding of these enumerative questions and the related conjectures.

1. Gromov-Witten theory of threefolds

Enumerative geometry concerns counting the number of curves of given degree through a fixed collection of cycles. Given a three-dimensional, smooth, projective variety X, there are two natural approaches to studying these questions. In the approach of Gromov-Witten theory, one considers the moduli space of maps from curves of genus g to X. Incidence conditions on X describe cycles on this moduli space and, by studying their intersection theory (suitably defined), one obtains Gromov-Witten invariants approximating the desired enumerative data. In another approach, known as Donaldson-Thomas theory [1, 17], curves are described by their defining equations rather than by their parametrization. From this perspective, the natural geometric space is the Hilbert scheme of curves on X and one can again define invariants by a suitable intersection of cycles.

While there is a heuristic similarity between these approaches, there is no direct geometric relationship between them, as the actual moduli spaces are completely different. Nevertheless, in joint work [7, 8] with Nekrasov, Okounkov, and Pandharipande, we have conjectured the following.

Conjecture 1.1. *Given a smooth projective threefold X, there is a precise equivalence between the Gromov-Witten theory of X in all genus and the Donaldson-Thomas theory of X.*

As the correspondence is rather complicated to state concretely, the reader can consult the references for a detailed statement. This conjecture can be viewed as an example of the equivalence between string theory and gauge theory predicted by physicists. From a mathematical point of view, it generalizes integrality conjectures regarding Gromov-Witten invariants as well as Hodge integral identities on the moduli space of curves. Recently, in joint work with Oblomkov, Okounkov, and Pandharipande [9], we have proven the following.

Theorem 1.2. *The Gromov-Witten/Donaldson-Thomas correspondence holds for toric threefolds.*

This is the strongest known confirmation of the conjecture and it extends work of many other mathematicians on Hodge integrals and symmetric functions [15, 3, 4]

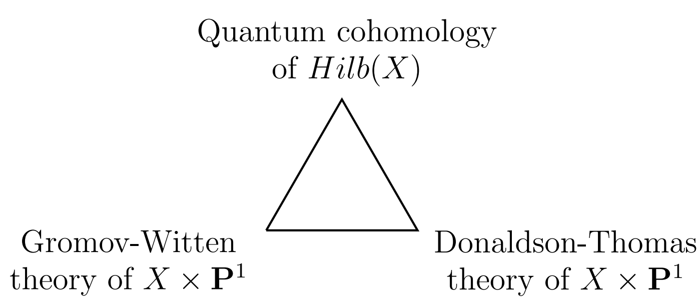
While this is good evidence for the conjecture, there are still many open directions that I would like to pursue. First, our solution for toric varieties suggests an algorithm for explicitly computing these invariants in terms of symmetric functions. A precise expression would be very useful for the related enumerative questions and should also be of independent interest in the theory of symmetric functions. Second, there are many universal structures on the Gromov-Witten side – such as descendent invariants and Virasoro constraints – whose analogs on the Donaldson-Thomas side are still poorly understood or unproven. Third, it is expected that, at least for Calabi-Yau threefolds, there should be a higher-rank version of the correspondence. However, there are nontrivial philosophical obstacles to how to phrase such a conjecture precisely. Finally, and most obviously, there is the question of proving the conjecture for all threefolds and giving a satisfying, geometric explanation for its structure. While this last question seems to me quite formidable, the previous questions are useful steps towards this goal.

2. Quantum cohomology of Hilbert schemes of points on a surface

Given an algebraic surface, the Hilbert scheme of points is a smooth variety parametrizing collections of points on the surface. There is a rich representation-theoretic structure to the classical cohomology ring of these varieties developed by Nakajima, Grojnowski, and others in the past decade [2, 14]. The quantum cohomology ring of an algebraic variety is a deformation of the classical cohomology ring whose structure constants are given by counting rational curves through various cycles. A question I am interested in is to what extent can one extend the structure of the classical cohomology to the quantum deformation of this ring. There is one family of examples where we are able to do this. In joint work with Oblomkov [11], we have given a description of this quantum deformation for certain noncompact surfaces known as A_n-resolutions; in this case, we observe a relation between the quantum deformation, affine Lie algebras, and the quantum Calogero-Sutherland integrable system. It is still unclear from our work whether any of this structure arises for a general surface.

A related question in this theory is its connection to the GW/DT correspondence described in the last section. In a sequence of papers [6, 10, 11], we prove the following result.

Theorem 2.1. *For an A_n-resolution X, the quantum cohomology of the Hilbert scheme of X is equivalent to both the Gromov-Witten theory and Donaldson-Thomas theory of the threefold $X \times \mathbf{P}^1$.*



This triangle of equivalent theories plays an essential role in proving the GW/DT correspondence for toric varieties. Unfortunately, it is easy to see that this triangle does not persist for a general surface. Is there a way of correcting the statement so that it is valid for all algebraic surfaces? In certain cases, I have proven a precise correction that I conjecture is valid for all Fano surfaces, but the general case is again unclear.

3. Exact calculations

If one is interested in Gromov-Witten theory with domain curves of genus $g \geq 1$, there are very few results that apply for general varieties. In joint work with Pandharipande [12], we give a series of techniques for attacking this question modelled on cut-and-paste results from classical topology. Our main result involves *relative* Gromov-Witten theory, which enumerates curves mapping to X with prescribed ramification over a smooth divisor $D \subset X$.

Theorem 3.1. *The relative Gromov-Witten theory of the pair (X, D) can be effectively and uniquely reconstructed from the Gromov-Witten theories of X and D and the map*

$$H^*(X, \mathbb{Q}) \to H^*(D, \mathbb{Q}).$$

In a sense, this theorem shows that all geometric information contained in these relative invariants can be recovered from usual Gromov-Witten theory. As a corollary, we have the following result which we view as analogous to the Mayer-Vietoris theorem in classical topology. Suppose we have a degeneration of a smooth variety V to a transverse union $V_1 \cup_W V_2$ of two varieties V_i along a smooth divisor W so that the total space of the degeneration is smooth.

Corollary 3.2. *The Gromov-Witten theory of the nonvanishing cohomology of V can be uniquely and effectively reconstructed from the Gromov-Witten theory of V_1, V_2, and W and the maps*

$$H^*(V_1, \mathbb{Q}) \to H^*(W, \mathbb{Q}) \text{ and } H^*(V_2, \mathbb{Q}) \to H^*(W, \mathbb{Q}).$$

Using these results, we construct a genus-independent algorithm for studying complete intersections and cyclic covers. As a special consequence, this gives the first known mathematical determination of the Gromov-Witten theory of the Calabi-Yau quintic hypersurface in $\mathbf{P}^4$ in all genera. In other cases, we are able to execute this algorithm to sufficient depth to make contact with string-theoretic predictions [13]. However, the algorithm is still too involved to implement with any frequency, so I would like to investigate whether it can be simplified at all. A specific goal of interest in mathematical physics is the extension of our Calabi-Yau calculations to still-higher genera and, if possible, the determination of closed formulas in all genera.

A second question that this work raises is whether we can use these degeneration techniques to attack universal conjectures in Gromov-Witten theory, i.e. statements that are expected to hold for all algebraic varieties. Starting from toric varieties, where these statements are often already proven, it is in principle possible to access any variety via degeneration techniques. It may therefore be possible to prove these conjectures by starting from the toric situation and showing they are compatible with our cut-paste techniques.

4. Surfaces with $p_g > 0$

While the taxonomy of algebraic surfaces is rather wild, their Gromov-Witten theory is very well-behaved. In recent work of Lee and Parker [5], it is proven that there exists a universal theory for Kähler surfaces $p_g > 0$ that is insensitive to all but the coarsest invariants of the surface. This is in some sense a partial generalization of work of Taubes relating certain Gromov-Witten invariants to Seiberg-Witten invariants of symplectic 4-manifolds. What is unresolved in this work is any description of what this universal theory looks like. One reason for this is that, even with techniques mentioned above, computations are still quite difficult so there is a shortage of reliable information. Based on examples described in the last section, Pandharipande and I have conjectured a precise form of this universal theory in low degree [13]. I would like to investigate the higher degree situation. A good answer to this question should eventually lead to a complete solution to the Gromov-Witten theory of all algebraic surfaces.

References

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