RESEARCH STATEMENT

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1. Thesis research

My field of thesis research was Floer homotopy, which lies at the intersection of gauge theory and algebraic topology.

Floer homotopy is an idea introduced by Cohen, Jones, and Segal. In geometry, there are various examples of functionals on an infinite-dimensional configuration space $\mathcal{B}$, which give rise to a sequence of abelian groups called Floer homology. These groups are obtained by a Morse-theoretic construction of counting critical points and gradient flow lines between them. Floer homology groups have been intensely studied and have had an impact on numerous areas of mathematics. They appear in symplectic geometry, as well as in gauge theory (using either the Yang-Mills or the Seiberg-Witten equations). Cohen, Jones, and Segal asked the following question:

Problem 1. *In each of these settings, is there a natural space (or a more general algebraic-topologic object) whose homology is the Floer homology?*

One cannot expect such an object to be a space in general, or even a spectrum, but rather an algebraic-topologic gadget called a pro-spectrum. Furthermore, there are at least two difficulties involved in its construction: one is a purely algebraic-topologic obstruction which lives in $KO^1(\mathcal{B})$, usually in the image $K^1(\mathcal{B}) \to KO^1(\mathcal{B})$, and the other has to do with anaytical problems related to noncompactness and "bubbling".

In the case of the Seiberg-Witten equations on 3-manifolds, the obstruction is zero in many cases, and no bubbling appears. However, there are many other analytical difficulties related to the construction of the Floer homology itself. The Seiberg-Witten equations have had a tremenduous impact on our understanding of 4-dimensional manifolds. In dimension 3, one expects a TQFT (topological quantum field theory). A TQFT is basically a functor from the category of 3-dimensional closed manifolds (possibly with some extra structures) and cobordisms to the category of vector spaces (Floer homology groups) and linear maps. A fundamental problem is the following:

Problem 2. *Construct a 3+1 dimensional TQFT starting from the Seiberg-Witten equations.*

The main difficulty is to find an appropriate set of perturbations for the Seiberg-Witten equations which would guarantee a strong genericity condition. This problem is the object of recent work of Kronheimer and Mrowka. An alternate construction, in the form of a version of symplectic Floer homology, was introduced by Ozsváth and Szabó in a series of papers. Their theory is expected to give rise to the same invariants as Seiberg-Witten gauge theory.

In my thesis, I have offered a new approach to Problem 2 above. At the same time, this provides an answer to Problem 1 in the case of Seiberg-Witten theory on 3-manifolds with $b_1 = 0$. The result is a "spectrum-valued TQFT":

Theorem 1. *(i) To every closed, oriented 3-manifold Y with $b_1(Y) = 0$ and spin^c structure $\mathfrak{c}$ on Y, one can associate an S^1-equivariant suspension spectrum $SWF(Y, \mathfrak{c})$, well-defined up to canonical equivalence. Its homology is Seiberg-Witten Floer homology.*

(ii) To every spin^c cobordism $(X, \hat{\mathfrak{c}})$ between 3-manifolds Y_1 and Y_2 there is an associated morphism

$$\mathcal{D}_{X,\hat{\mathfrak{c}}} : SWF(Y_1, \mathfrak{c}_1) \to \Sigma^* SWF(Y_2, \mathfrak{c}_2).$$

Here Σ^* denotes suspension by a certain representation of S^1, depending on X.

(iii) (Gluing theorem) Given spin^c cobordisms $(X_1, \hat{\mathfrak{c}}_1)$ between Y_1 and Y_2, and $(X_2, \hat{\mathfrak{c}}_2)$ between Y_2 and Y_3, the morphism for the composite cobordism $(X, \hat{\mathfrak{c}}) = (X_1 \cup X_2, \hat{\mathfrak{c}}_1 \cup \hat{\mathfrak{c}}_2)$ is:

$$\mathcal{D}_{X,\hat{\mathfrak{c}}} = \mathcal{D}_{X_2,\hat{\mathfrak{c}}_2} \circ \Sigma^* \big(\mathcal{D}_{X_1,\hat{\mathfrak{c}}_1}\big).$$

The main idea used in the proof is the technique of finite dimensional approximation. This was introduced by Furuta to prove his 10/8-Theorem about the homotopy type of spin 4-manifolds. Later on, Bauer and Furuta have constructed a refinement of the Seiberg-Witten invariant for closed 4-manifolds in the form of an element in the equivariant cohomotopy groups of spheres. $\mathcal{D}_{X,\hat{\mathfrak{c}}}$ can be thought of as a generalization of the Bauer-Furuta invariant to the case of 4-manifolds with boundary.

2. FUTURE PLANS

There is still much to be understood about Floer homotopy in other settings. The first case to be considered is that of the Seiberg-Witten equations on 3-manifolds with $b_1 > 0$. As suggested by Furuta, the relevant concept there seems to be that of a pro-spectrum with parametrized universe, a generalization of the notion of spectrum. It would be worth seeing whether the same idea can be used to construct homotopy versions of the symplectic and instanton Floer homologies.

Of particular importance is Ozsváth-Szabó Floer homology, which has numerous applications to low-dimensional topology and is easier to compute than its Seiberg-Witten analogue. Creating a homotopy version of Ozsváth-Szabó theory is related to understanding its A_∞ structure, i.e. to counting pseudoholomorphic polygons in the relevant symplectic manifold. In the case of double branched covers of S^3 over a link, Ozsväth and Szabó have found a relation between the A_∞ structure and an interesting combinatorial link invariant, Khovanov homology. I plan to explore this circle of ideas and study the connections between the various Floer homologies that one can associate to a link in S^3.