A brief description of my research

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In dynamical systems, which is the field I have done most of my work in, we study actions of groups on spaces in various categories (e.g. for measure spaces this is the topic of ergodic theory, for spaces with a smooth structure (manifolds) smooth dynamics, for a general topological space topological dynamics, and there is even a rich and interesting theory of Borel actions on Borel spaces). The main feature which differentiates between dynamical systems and other disciplines is that in dynamical systems one is interested in actions which have very complicated orbits (so, in particular, actions of compact groups which typically have nice orbits are not interesting from this viewpoint).

Of particular interest to me are the connections between dynamical systems and other disciplines, notably number theory, autmorphic forms, and quantum chaos. Many of these applications involve classifying invariant probability measures or closed invariant sets for a specific concrete action on a specific concrete space. This classification is interesting in the cases where these invariant measures and sets are scarce and well understood. A very notable example is G.A. Margulis proof of the long-standing Oppenheim conjecture in 1986 which involved precisely such a classification.

A bit of background

In order to prove in Oppenheim conjecture, Margulis studied the action of the group $\mathrm{SO}(2,1)$ of matrices preserving an indefinite quadratic form in three variables on the space of unimodal lattices in $\mathbb{R}^3$ (which can also be identified as the locally homogeneous space $\mathrm{SL}(3,\mathbb{Z})\backslash\mathrm{SL}(3,\mathbb{R})$). An important feature of this group is that it is generated by unipotent elements. Following several contributions, culminating in Ratner's proof of Raghunathan's conjectures, we now have a very good understanding of the action of groups generated by unipotents on locally homogeneous spaces.

While actions of large groups (such as $\mathrm{SO}(2,1)$) are covered by this theory, the core of Ratner's theorems and the other results on groups generated by unipotents is the study of actions of a one parameter unipotent group, which already exhibits scarcity of invariant sets and measures.

In 1967 (many years before Margulis's proof of the Oppenheim conjecture, and Ratner's theorems) Furstenberg discovered a surprising phenomenon: while there are plenty of closed sets and probability measures on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ invariant under the map $\times_m : x \mapsto 3x \bmod 1$ (for any integer m), there are very few closed sets invariant under both $\times_m$ and $\times_n$ for n, m which are not powers of the same integer (i.e. $\log n / \log m \notin \mathbb{Q}$; this condition is called multiplicative independence): indeed the only infinite such set is $\mathbb{T}$ itself. In this case, this scarcity of closed invariant sets is only manifests for the full action and does **not** come from individual elements of the action. The measure theoretic analog of this statement, namely that the only non-atomic $\times_m, \times_n$-invariant measure on $\mathbb{T}$ is Lebesgue measure, is a well-known conjecture of Furstenberg.

Furstenberg's theorem and conjecture serve as a paradigm for more complicated $\mathbb{Z}^d$ or $\mathbb{R}^d$ for $d > 1$ where each one-dimensional subgroup has a big zoo of closed invariant sets and invariant probability measures, but at least conjecturally the invariant sets/measures for the full action are severely restricted: either they are nice (algebraic) as in Ratner's theorem, or they arise from some pathologic situations where the action degenerates into a one-dimensional one. The following is a simple (to state) conjecture due to Margulis: in the space of unimodal lattices in $\mathbb{R}^n$ (which can be identified with $\mathrm{SL}(n,\mathbb{Z})\backslash \mathrm{SL}(n,\mathbb{R})$) any bounded orbit of the full diagonal subgroup A of $\mathrm{SL}(n,\mathbb{R})$ is automatically closed. A solution to this conjecture will settle a long-standing and well-known conjecture of Littlewood regarding simultaneous Diophantine approximations, namely that for every $\alpha, \beta \in \mathbb{R}$ it holds that $\underline{\lim}\, n \left\|n\alpha\right\| \left\|n\beta\right\| = 0$.

The best result to date towards Furstenberg's conjecture is due to Johnson and Rudolph. It states that if μ is $\times_m, \times_n$-ergodic for m, n multiplicatively independent then either the entropy of μ with respect to any single element of the action is zero or μ is Lebesgue measure.

Katok and Spatzier discovered that Rudolph's proof can be modified to work also in a much more general setting. But the result one gets in most cases is substantially weaker: in addition to the entropy assumption, in most cases one needs to assume in addition an assumption regarding mixing properties of some one-dimensional sub actions. Unfortunately, and unlike the entropy assumption, for most applications I know of the mixing assumption is impossible to verify.

Very recently, Einsiedler and Katok managed to remove this ergodicity assumption in some cases, in particular for the A action on $\mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})$. In order to do this, they need a stronger assumption about the entropy: that **every** one parameter group acts with positive entropy.

Quantum Unique Ergodicity

Much on my work has been motivated by the quantum unique ergodicity conjecture of Rudnick and Sarnak. This conjecture states that if M is a compact manifold of negative sectional curvature and ϕ_i an orthonormal sequence of eigenfunction of the Laplacian then $|\phi_i|^2\, dvol_M$ converges in the weak star topology to the normalized Riemannian volume on M. Schnirelman, Colin de Verdiere and Zelditch showed that if ϕ_i is a complete orthonormal sequence sorted by eigenvalue then this convergence holds on average.

A special case which attracted considerable attention has been the case where M is an arithmetic quotient of the hyperbolic plane (and in this context, it is natural to consider general finite area quotients which may have cusps and so fail to be compact). These surfaces have additional symmetries, the Hecke operators, which commute with the Laplacian. Joint eigenfunction of the Laplacian and the Hecke operators are called Hecke Maas forms, and are of great interest to number theorists. Numerical evidence suggests that eigenfunction of Laplacian are also automatically Hecke eigenfunctions, and it is easy to see that the Hecke Maas forms span the discrete spectrum of the Laplacian. The arithmetical case of the quantum unique ergodicity question is whether $|\phi_i|^2\, dvol_M$ converges in the weak star topology to the normalized Riemannian volume when ϕ_i are Hecke Maas forms.

Assuming GRH (a generalization of the Riemann Hypothesis), Watson has been able to establish arithmetic quantum unique ergodicity.

By the work of Schnirelman, Colin de Verdiere and Zelditch the quantum unique ergodicity question becomes a question about measures invariant under the geodesic flow on $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$. However, the fact that ϕ_i are also Hecke eigenfunctions give additional information. Improving on a technique by a Rudnick and Sarnak, jointly with Bourgain we have shown that the measures which arise in this way cannot be too degenerate, and in particular have positive entropy (for the experts I should add that this holds for all ergodic components).

Motivated by work of Host, I have found a condition which I call Hecke recurrence which guarantees that a measures invariant under the geodesic flow and with positive entropy is Lebesgue measure. The proof uses in an essential way ideas of Ratner: indeed, it can be loosely described as an attempt to prove Ratner's theorem in a case where the measure in question is certainly not invariant under the relevant flow. In order to carry out the proof some new techniques regarding non-measure preserving dynamics, indeed dynamics where even measure class is not preserved, are developed, including a new type of ergodic theorem which is joint with Rudolph.

The invariant measures on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ that arise in the arithmetic context from weak star limits as above are easily seen to satisfy this condition of Hecke recurrence. Thus one gets an unconditional solution to the arithmetic quantum unique ergodicity conjecture in the compact case. In the general finite volume case this situation also isn't too bad: one gets that any arithmetic quantum limit as above is a scalar multiple of the volume (though hypothetically this scalar may be zero).

Other related results of mine

The techniques developed for studying quantum unique ergodicity gives the following general theorem: let μ be a measure on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ invariant and ergodic under the full diagonal group. Then if there is **at least one** one parameter subgroup of the full diagonal group with positive entropy then μ is Haar-Lebesgue measure. It should be noted that the techniques of Einsiedler and Katok are unapplicable in this case.

The scope of applicability of the new techniques is not limited to such products (which is not surprising, since it is based by the techniques of Ratner which are very general). I am now writing with Einsiedler and Katok a proof of the following substantial strengthening of their theorem which I quoted above, namely that if μ is invariant and ergodic under the action of the full diagonal group on $\mathrm{SL}(n,\mathbb{Z})\backslash \mathrm{SL}(n,\mathbb{R})$ then if there is **at least one** one parameter subgroup of the full diagonal group with positive entropy then μ is algebraic (if n is prime, the only possibility is Haar-Lebesgue measure). This will imply that the set of exceptions to Littlewood's conjecture has Hausdorff dimension zero. This is also related to previous work I have carried out with Barak Weiss.

I have also worked on studying measures invariant under groups of automorphism is of the torus or more generally a compact connected abelian group. With Klaus Schmidt we have shown that techniques of measure rigidity give some unexpected (at least to me) results where they should not be relevant, namely the study of measures invariant under one irreducible automorphism of the torus **if** this automorphism happens to be non-hyperbolic. With Einsiedler, I have also been able to extend Rudolph's theorem to general totally irreducible $\mathbb{Z}^d$ actions. The restriction to totally irreducible actions, i.e. actions for which no finite union of infinite closed subgroups of the torus is invariant, is only to make the analogy to Rudolph's theorem accurate; our techniques deal with general $\mathbb{Z}^d$ actions by automorphisms of compact connected abelian groups.

What I plan to work on next

The great challenge in this field is to progress beyond the positive entropy case. As the reader no doubt noticed, we have already manage to work with "less" entropy then was previously required, but the general case seems very hard (or perhaps I should say very interesting and challenging).

There are lots of things to do even in the positive entropy case. In particular, there are many fascinating questions about analogous questions for actions of automorphisms of totally disconnected abelian groups such as the famous example due to Ledrappier.

The many deep results on unipotent flows, and the new if partial results on Cartan-type flows have found several applications in diverse areas. It is my belief that the potential of these techniques is still much larger, and I intend to continue trying to apply these dynamical techniques wherever I can.