Summary of Research Results

The Strong Perfect Graph Theorem

with N.Robertson, P.Seymour, R.Thomas, *Manuscript, May 2002*

The *chromatic number* of a graph G is the minimum number of colors needed to color the vertices of G in such a way that no two adjacent vertices receive the same color. Clearly it is at least the size of the largest complete subgraph in G.

A graph G is *perfect* if for every induced subgraph H, the chromatic number of H equals the size of the largest complete subgraph of H, and G is *Berge* if no induced subgraph of G is an odd cycle of length at least 5 or the complement of one. The "strong perfect graph conjecture" (introduced by Berge in 1961 [2]) asserts that a graph is perfect if and only if it is Berge.

A stronger conjecture was made recently by Conforti, Cornuéjols and Vušković - that every Berge graph either falls into one of a few basic classes, or it has a kind of separation that cannot occur in a minimal imperfect graph.

We prove both these conjectures.

Recognition Algorithm for Berge Graphs

with Paul Seymour *submitted*

One important aspect of perfect graphs is that the problems of finding the chromatic number and the size of the maximum clique, which are NP-complete in general, can be solved in polynomial time for perfect graphs (a result due to Grötschel, Lovász and Schrijver [5]).

In a joint paper with Paul Seymour we give an algorithm to test if a graph G is Berge, with running time $O(|V(G)|^9)$. This is independent of the recent proof of the strong perfect graph conjecture. The algorithm uses the following result:

Cleaning for Bergeness

with Gérard Cornuéjols, Xinming Liu, Paul Seymour, Kristina Vušković

submitted

A vertex v is *major* for C if its neighbors in C do not lie in a 2-edge path of C. In this paper we give a polynomial time algorithm, which with input a graph G, either decides that G is not Berge, or outputs polynomially many subsets of $V(G)$, with the following property: that if C is a shortest odd hole in G, and every major vertex has at least four neighbors in C, then one of the subsets is disjoint from C and contains every vertex that is major for C.

Berge Trigraphs

in preparation

A *trigraph* T is a complete graph with the edge set partitioned into three sets: *strong edges, strong non-edges and switchable edges.* A trigraph is *Berge* if for every subset S of switchable edges the graph G whose edge set is the union of S with the set of all strong edges of T is Berge.

I have a proof of the following result for Berge trigraphs:

The decomposition result we have for Berge graphs extends (with slight modifications) to trigraphs i.e. either T belongs to one of a few basic classes or T has a decomposition, where all the "important" edges and non-edges in the decomposition are strong edges and strong non-edges of T respectively.

In the proof of the Strong Perfect Graph Theorem we used 3 kinds of decompositions: skew-partitions, 2-joins and M-joins. The result about Berge trigraphs implies that the M-join decomposition is in fact unnecessary.

Another consequence of the result about Berge trigraphs is the following: if G is a Berge graph then either it belongs to one of a few basic classes, or it admits a skew partition or an M-join, or it admits a 2-join so that neither half of it is just an induced path, or the complement of G admits a 2-join (possibly with one half of it just an induced path).

Triangulated Spheres and Colored Cliques

with R. Aharoni, A. Kotlov,

Discrete and Computational Geometry, 28 (2002) 223-229

Aharoni and Haxell ([1]) proved a generalization of Hall's theorem to families of hypergraphs. The generalization provides a sufficient condition for the existence of systems of disjoint representatives (SDR's) in families of hypergraphs. The proof used Sperner's lemma and the existence of particular triangulations of the n-dimensional simplex, satisfying certain special conditions.

Later, Meshulam [6] proved an extension of this result, an interpolation between it and another known sufficient condition for the existence of SDR's (implicitly proved in [4]). His proof used homology and the nerve theorem.

In this paper we show how the triangulations method can be used to derive Meshulam's results. The proof is based on results on extensions of triangulations from the sphere to the full ball. A typical result of this type is that any triangulation of the $n-1$-dimensional sphere S^{n-1} can be extended to a triangulation of the ball B^n, by adding one point at a time, having degree at most $2n$ to its predecessors.

Future Research

In my future research I plan to continue working on problems in structural graph theory and graph coloring.

The theorem in [3] states that every graph with no odd hole or antihole either belongs to one of a few basic classes or has a certain kind of decomposition. This result is strong enough to prove the Strong Perfect Graph Conjecture, but does not give an actual way to build all perfect graph starting from basic pieces. One of the directions that I am planning to work in is the problem of strengthening the decomposition theorem in [3] to obtain a complete structural description of perfect graphs.

References

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[2] C. Berge, Les Problèmes de Coloration en Théorie des Graphes, *Publ. Inst. Stat. Univ. Paris*, 9:123–160, 1960.

[3] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas The Strong Perfect Graph theorem, *Manuscript, June2002*

[4] P.E. Haxell, A condition for matchability in hypergraphs, *Graphs Combin.* 11 (1995), no. 3, 245–248.

[5] M. Grötschel, L. Lovász, and A. Schrijver, Polynomial algorithms for perfect graphs, In C. Berge and V. Chvátal, editors, *Topics on Perfect Graphs (Ann. Discrete Math. 21)*, pages 325–356. North-Holland, Amsterdam, 1984.

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