

The representation theory of big groups has been the main domain of my interest over the last five years. The term "big group" means that the elementary (irreducible) representations of the group depend on infinitely many continuous parameters. The group $S(\infty)$ consisting of finite permutations of the natural numbers is the simplest example of a big group. It is known that the elementary representations for $S(\infty)$ are parametrized by the points of an infinite-dimensional simplex.

The development of any type of representation theory is usually split into two stages. The first stage is a description of all "elementary" representations. The second stage consists of constructing certain "natural" representations of the group and decomposing them into irreducible ones. The second stage is usually referred to as noncommutative harmonic analysis.

For big groups the first result for the first stage belongs to E. Thoma who in 1964 classified the elementary representations for $S(\infty)$. When I started my work the first stage was essentially complete for most interesting big groups due to efforts of many people (S. Kerov, A. Kirillov, G. Olshanski, A. Vershik, D. Voiculescu and others). However, concerning the second stage, little was known.

One reason for that is the fact that the regular representation of $S(\infty)$, which is the most obvious "natural" one, turns out to be irreducible. Thus, the situation is very different from the classical one — the decomposition of the regular representation of finite and compact groups into irreducibles plays a central role in the representation theory of such groups.

In 1993, Kerov, Olshanski, and Vershik constructed a certain 2-parameter deformation of the regular representation which had all the properties one would want from a "natural" representation. The decompositions of these generalized regular representations into irreducibles are governed by certain measures (called spectral measures) which are supported on an infinite-dimensional space. A large part of my initial research work was devoted to the study of these measures [5, 7, 8, 10, 14].

The main result (which is the subject of my thesis) is a complete description of these spectral measures, see [5, 14]. The measures are interpreted as stochastic point processes on the punctured real line and all correlation functions of these processes are explicitly computed.

Unexpectedly, the correlation functions turn out to have a determinantal form, exactly the same as the point processes that arise in Random Matrix Theory describing the spectra of random matrices. In another direction, this connection led me to the explicit computation of local correlation functions in a certain random matrix model, see [4].

The above computation of spectral measures was the first result which connected the representation theory of the infinite symmetric group with determinantal point processes. Later G. Olshanski and I found another approach to this result, see [8]. The key claim is that a certain mixture of restrictions of the generalized regular representations to all finite symmetric groups generates a probability distribution on all partitions which in turn can be viewed as a determinantal point process on a one-dimensional lattice.

Subsequently, A. Okounkov, G. Olshanski, and I applied the same approach to the Plancherel measure on symmetric groups [9]. This, in particular, gave a proof of the Baik–

Deift–Johansson conjecture, which states that the asymptotic behavior of the properly scaled lengths of the first rows of Young diagrams of order n , distributed according to the Plancherel measure, coincides with that of the first properly scaled eigenvalues of $n \times n$ Hermitian matrices from the Gaussian Unitary Ensemble as $n \rightarrow \infty$. (For the individual distributions of the first and the second rows, the conjecture was proved earlier by J. Baik, P. Deift, and K. Johansson.) Different independent proofs of this conjecture have also been given by A. Okounkov and K. Johansson.

Further developments have led to interesting connections with the theory of Toeplitz determinants [11] and integrable systems (integrable operators and Riemann–Hilbert problem) [12].

One direction of my current research is the so-called discrete Riemann–Hilbert problem [12]. It arose naturally in our work on the infinite symmetric group. As was observed later, the problem of constructing soliton solutions in the classical integrable systems has a very similar setting. This connection looks very promising to me. It should allow to apply directly the methods of soliton theory, especially the methods of asymptotic analysis, to the representation theoretic and combinatorial problems.

Currently we are also involved in extending the calculation for $S(\infty)$ to the infinite unitary group $U(\infty)$. The family of generalized regular representations for $U(\infty)$ turns out to be richer than that for $S(\infty)$. Moreover, in a certain limit the spectral measures for $U(\infty)$ degenerate to the $S(\infty)$ spectral measures. But at the same time the situation becomes more complicated and requires new techniques like the theory of discrete orthogonal polynomials of hypergeometric type and certain elements of ergodic theory.

The most promising part of this new direction of research is a wider connection with other domains of mathematics, especially with integrable systems and combinatorics. The new point processes arising from the representation theory seem to provide new insights into questions like “Why do Painleve equations appear in Random Matrix Theory?” or “What is behind the asymptotic equivalence of combinatorial and random matrix models?”

I strongly believe that further development of the representation theory of big groups will not just provide new interesting problems within representation theory itself but will also have an impact on other domains of mathematics.