FUTURE PLANS

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The main result of my thesis is the following.

Theorem 1. *For every positive integer n or for $n = \infty$, and for any $k \leq n$, the map*

$$\pi : G(k,n) \to \|\operatorname{MacP}(k,n)\|$$

is a homotopy equivalence.

In order to explain further topics of related research, I should first put the above result in context. In 1991, MacPherson introduced combinatorial differential (CD) manifolds, which are purely combinatorial objects which are meant to model smooth manifolds. An n-dimensional CD manifold is a pseudomanifold of dimension n (that is, a simplicial complex each of whose maximal simplices is n-dimensional and each of whose $(n-1)$-simplices is contained in exactly two maximal simplices) furnished with some combinatorial data which is supposed to model the tangent bundle. Here, the combinatorial data comes in the form of a family of "oriented matroids." An oriented matroid is a combinatorial model for an arrangement of finitely many vectors in a real vector space; in this setting, one should think of them as sections of the tangent bundle of the manifold. MacPherson also demonstrated that a smoothly triangulated manifold gives rise to a CD structure on the underlying simplicial complex, leading to the hope that one could obtain a functor from the category of smoothly triangulated manifolds to CD manifolds.

He further outlined a purely combinatorial notion of bundles, which he called matroid bundles. It was subsequently shown by Anderson and Davis that the set of isomorphism classes of rank k matroid bundles over a complex B is classified by homotopy classes of maps from B to the space $\|\operatorname{MacP}(k,\infty)\|$, which is the geometric realization of a partially ordered set of oriented matroids. Thus, Theorem 1 tells us that the contravariant functors

$$B \longmapsto \left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of rank } k \\ \text{matroid bundles over } B \end{array} \right\}$$

and

$$B \longmapsto \left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of rank } k \\ \text{vector bundles over } B \end{array} \right\}$$

are naturally isomorphic.

Hence, we have a geometric notion, the CD manifold, which carries with it a bundle theory that is isomorphic to the theory of real vector bundles. It is therefore natural to try to determine what other properties of smooth manifolds are captured in the CD world. The first step in this program is to find an appropriate definition of maps of CD manifolds. We begin this by constructing the groupoid of CD manifolds and their isomorphisms; while this is perhaps only a part of the goal, the definition of isomorphisms of CD manifolds is an important step. Indeed, it allows us to begin to compare diffeomorphism classes of smooth manifolds to isomorphism classes of CD manifolds, which will hopefully provide combinatorial tools and insights into the classification of manifolds.

Secondly, one of the most interesting circles of questions in differential topology concerns the structure of diffeomorphism groups of manifolds, and we now have a combinatorial object, the isomorphism group of a CD manifold, which might give a new approach to this question; this seems like perhaps the most promising line of potential research. More precisely, let M be a smooth manifold with a fixed smooth triangulation, that is, a fixed homeomorphism $\eta : X \to M$ where X is a simplicial complex and η is smooth on closed simplices. Then, as stated above, η gives rise to a CD structure on M. We now have the following diagram of "forgetful" inclusions.

$$(1) \qquad \begin{array}{ccc} CD(M) & \xhookrightarrow{\iota_{CD}} & PL(M) \\ & & \Big\downarrow \iota_{PL} \\ Diff(M) & \xhookrightarrow{\iota_{Diff}} & PDiff(M) \end{array}$$

Here, for any category CAT, the symbol $CAT(M)$ denotes the automorphism group of M in CAT. The symbols CD, PL, and $Diff$, denote the categories of CD, PL, and smooth manifolds, and the symbol $PDiff$ denotes the category of piecewise-differentiable manifolds; in other words, $PDiff(M)$ consists of all homeomorphisms $M \to M$ which are smooth on the simplices of some triangulation of M subordinate to η.

Now, it follows from the work of Lashof and Rothenberg that the right-hand map ι_{PL} of the square (1) is a homotopy equivalence and thus admits a homotopy inverse $p_{PL} : PDiff(M) \to PL(M)$. Our propsed method of studying $Diff(M)$ goes via the program outlined in the following conjecture.

Conjecture 2. The following relationships between the CD, PL, and smooth categories are satisfied.

(1) There is a map $\kappa : CD(M) \to Diff(M)$ making the square (1) homotopy commutative.
(2) The map p_{PL} can be chosen so that the image of $p_{PL} \circ \iota_{Diff}$ lies in the image of ι_{CD}.
(3) The composition

$$CD(M) \xrightarrow{\kappa} Diff(M) \xrightarrow{p_{PL} \circ \iota_{Diff}} CD(M)$$

is homotopic to the identity.

Thus, $CD(M)$ is a (homotopy) retract of $Diff(M)$.

If we could show that Conjecture 2 holds, then an analysis of the homotopy type of $CD(M)$ might have a great deal of impact on the study of $Diff(M)$. Furthermore, it does not seem entirely far-fetched to hope that the diagram (1) is actually homotopy Cartesian; this would of course imply that the the map $\kappa : CD(M) \to Diff(M)$ is a weak equivalence, which would be extremely interesting.

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