Tim Austin: Research Statement

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As a graduate student I have largely worked in two areas: ergodic theory and its relations to additive combinatorics; and metric geometry and geometric group theory. Here I will describe some of my results in these areas and the future projects to which they may lead; and will then describe in more general terms some other related subjects in which I hope to develop a greater interest in the future.

Ergodic theory and additive combinatorics

Much the largest part of my energy as a graduate student has been spent on the ergodic theory of multiple ergodic averages and their relations to additive combinatorics.

Interest in these stems from the following remarkable result of Szemerédi ([58]), which confirmed an old conjecture of Erdős and Turán.

Theorem 1 (Szemerédi's Theorem) *For any* $\delta > 0$ *and* $k \geq 1$ *there is some* $N_0 \geq 1$ *such that if* $N \geq N_0$ *then any* $A \subseteq \{1, 2, 3, \ldots, N\}$ *with* $|A| \geq \delta N$ *includes a nontrivial* k*-term arithmetic progression:* $A \supseteq \{a, a+r, \ldots, a+(k-1)r\}$ *for some* $a \in \{1, 2, \ldots, N\}$ *and* $r \geq 1$.

Shortly after the appearance of Szemerédi's ingenious combinatorial proof, in 1977 Furstenberg gave a new proof of the above theorem ([31]) using a superficially quite different approach, relying on a conversion to a problem about probability-preserving dynamical systems.

Such a system consists of a probability space (X, Σ, μ) together with an invertible, measurable, μ-preserving transformation $T : X \to X$. Furstenberg proved that all such systems enjoy a property of 'multiple recurrence':

Theorem 2 (Multiple Recurrence Theorem) *Whenever* (X, Σ, μ) *and* T *are as above,* $A \in \Sigma$ *has* $\mu(A) > 0$ *and* $k \geq 1$*, then*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} \mu(A \cap T^r(A) \cap \cdots \cap T^{(k-1)r}(A)) > 0.$$

In particular, there is some $r \geq 1$ such that

$$\mu(A \cap T^r(A) \cap \cdots \cap T^{(k-1)r}(A)) > 0.$$

Shortly after this, Furstenberg and Katznelson realized that the same basic method could be modified to apply to collections of commuting measure-preserving transformations.

Theorem 3 (Multidimensional Multiple Recurrence Theorem) *If (X, Σ, μ) is a probability space, T_1, T_2, ..., T_k are commuting measurable invertible μ-preserving self-maps of X and $A \in \Sigma$ has $\mu(A) > 0$, then*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} \mu(A \cap T_1^r(A) \cap \cdots \cap T_k^r(A)) > 0.$$

They were also able to convert this back into a multidimensional combinatorial result generalizing Szemerédi's Theorem, asserting the presence of a nondegenerate right-angled simplex of the form $\{\mathbf{a}, \mathbf{a} + (r, 0, \ldots, 0), \ldots, \mathbf{a} + (0, 0, \ldots, 0, r)\}$ inside any dense subset of a sufficiently large discrete box $\{1, 2, \ldots, N\}^d$.

This ergodic theoretic approach to results in additive combinatorics has since developed into a whole subdiscipline, sometimes termed 'ergodic Ramsey Theory'; see, for instance, Bergelson's survey [20]. One further result due to Furstenberg and Katznelson is also worth mentioning explicitly: in [32] they prove the following density version of the classical Hales-Jewett Theorem [38]:

Theorem 4 (Density version of the Hales-Jewett Theorem) *For any $\delta > 0$ and $k \geq 1$ there is some $N_0 \geq 1$ such that if $N \geq N_0$ then any $A \subseteq [k]^N$ with $|A| \geq \delta k^N$ includes a **combinatorial line**: a subset $L \subseteq [k]^N$ of the form*

$$L = \{w \in [k]^N : \ w|_{[N]\setminus J} = w_0, \ \& \ w_j \text{ is the same letter for all } j \in J\},$$

for some fixed nonempty $J \subseteq [N]$ and $w_0 \in [k]^{[N]\setminus J}$.

In fact, this result implies most of the other main results in density Ramsey Theory, including Szemerédi's Theorem and its multidimensional generalization; see, for instance, Graham, Rothschild and Spencer [36].

In addition to achieving some striking new combinatorial results, ergodic Ramsey Theory has also motivated new ergodic theoretic questions, and has witnessed an ongoing interplay between insights into these two aspects of the subject.

One basic question that was resolved only recently is whether the 'multiple ergodic averages' studied in Theorems 2 and 3 above actually converge (that is, whether '$\liminf$' can be replaced with '$\lim$'). In the case of Theorem 2, this was finally shown to be so by Host and Kra in [43], following the establishment of several special cases and related results over two decades in [25, 26, 27, 63, 33, 41] (see also Ziegler's paper [64] for another proof). The more general setting of Theorem 3 was then settled by Tao in [60]. However, while those earlier works together develop a large body of ergodic theoretic machinery for the analysis of these averages, Tao departs quite markedly from those approaches and effectively converts the problem of convergence into a finitary assertion concerning averages of $[-1, 1]$-valued functions on large finite grids $\{1, 2, \ldots, N\}^d$.

In [10] I gave a new proof of Tao's Theorem using classical ergodic theoretic machinery. It turns out that this convergence can be proved relatively quickly using a version of the older approaches, with the one new twist that starting from a system of commuting transformations of interest $T_1, T_2, \ldots, T_d \curvearrowright (X, \Sigma, \mu)$ one must first pass to a carefully-chosen *extended* system $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_d \curvearrowright (\tilde{X}, \tilde{\Sigma}, \tilde{\mu})$ (that is, a new system for which the original one is isomorphic to the action of the $\tilde{T}_i$'s on some globally invariant σ-subalgebra of $\tilde{\Sigma}$: in ergodic theoretic terms, the original system is a 'factor' of the new one). If the extension is constructed correctly then the asymptotic behaviour of the multiple ergodic averages associated to it admits a simplification allowing them to be compared with a similar system of averages involving only $k-1$ transformations; from this point convergence in L^2 follows quickly by induction on k. Interestingly, the need for this extension also offers some explanation for the fact that Tao gains an advantage in his approach to Theorem 5 by converting to the finitary combinatorial world: during the course of his proof he constructs new functions from the initial data of the problem in ways that cannot be used to construct *measurable* functions in the ergodic theoretic setting, but suitable measurable functions are available using the larger σ-algebra of the extended system.

In fact, the proof of [10] gives a slight strengthening of Tao's Theorem, in that the convergence is uniform in the location of the interval of averaging:

Theorem 5 (Norm convergence of nonconventional averages: [10]) *For any commuting tuple of invertible measurable μ-preserving transformations $T_1, T_2, \ldots, T_d \curvearrowright (X, \Sigma, \mu)$, any sequence of finite discrete intervals $I_N \subset \mathbb{N}$ with $|I_N| \to \infty$ and any functions $f_1, f_2, \ldots, f_d \in L^\infty(\mu)$, the multiple ergodic averages*

$$\frac{1}{|I_N|} \sum_{n \in I_N} \prod_{i=1}^{d} f_i \circ T_i^n$$

converge in $L^2(\mu)$ as $N \to \infty$.

This recovers the convergence of the scalar averages appearing in Theorem 3 because

$$\frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T_1^n(A) \cap \cdots \cap T_d^n(A)) = \int_X 1_A \cdot \left(\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{d} (f_i \circ T_i^n) \right) \mathrm{d}\mu$$

when $f_1 = f_2 = \ldots = f_d = 1_A$.

Having discovered this new ingredient required to prove Theorem 5, I also realized that the same extended system admits a somewhat simpler description of the limiting value of these scalar averages. These can always be expressed in terms of a certain $(d+1)$-fold self-joining of the system $(\tilde{X}, \tilde{\Sigma}, \tilde{\mu}, \tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_d)$ (which appears already in the works of Furstenberg and Katznelson), and one finds that for the 'improved' extended system this self-joining takes a special form, to which Tao's infinitary analog of the hypergraph removal lemma from [59] can be applied to give a new proof of Theorem 3. I worked out the details of this alternative proof in [8]. Later still I was able to combine an important insight due to the Density Hales-Jewett Polymath project [57] with a related construction (now for a family of stochastic processes obeying a kind of 'self-similarly' rather than a measure-preserving action of a group) to give a new proof of Theorem 4 (although unlike the new proof due to the Polymath project, mine does not give an effective dependence of N_0 on (k, δ)); this is contained in [7].

A different, more ergodic theoretic direction for research extending Theorem 5 concerns the convergence of 'polynomial' versions of these averages. Given a system of commuting transformations $(X, \mu, T_1, T_2, \ldots, T_d)$ as previously, these are averages of the form

$$\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{d} f_i \circ (T_1^{p_{i,1}(n)} \circ T_2^{p_{i,2}(n)} \circ \cdots \circ T_d^{p_{i,d}(n)}),$$

where now the shifts of the functions f_i are considered at times given by the values of polynomials $p_{i,j} : \mathbb{Z} \to \mathbb{Z}$. A far-reaching conjecture by Bergelson and Leibman in [23] asserts that these should also always converge in $L^2(\mu)$ as $N \to \infty$, but most cases of this conjecture remain open. Once again, the positivity of some related averages of measures of intersections corresponds to a combinatorial result about the existence of 'polynomial patterns' in dense subsets of $\{1, 2, \ldots, N\}^d$; see, for instance, Bergelson and Leibman [22].

These averages are much better understood in the special case $T_1 = T_2 = \ldots = T_d$. For these, one particular instance was settled in an influential paper of Furstenberg and Weiss [33], and since then all cases subject to this restriction have been successfully treated in work of Host and Kra [42] and of Leibman [46], building on the identification of characteristic factors as pronilsystems (that is, inverse limits of systems given as rotations on compact homogeneous spaces of nilpotent Lie groups) developed by Host and Kra for their proof in [43] of the corresponding special case of Theorem 5. By enhancing the methods developed for my proof of Theorem 5 for constructing extended systems in which the averages in question behave more simply, I have now been able to prove one new special case of this theorem, concerning the averages

$$\frac{1}{N}\sum_{n=1}^{N}(f_1 \circ T_1^{n^2})(f_2 \circ T_1^{n^2}T_2^n)$$

for commuting T_1 and T_2. That these always converge in $L^2(\mu)$ was proved as the culmination of the sequence of papers [13, 14, 15], which also develop much of the relevant machinery (unfortunately, not all of it) in enough generality to be brought to bear on the general case.

The structural results of those papers allow one to reduce the problem to the analysis of the corresponding averages for some simpler, more restricted class of systems (the 'characteristic factors of the extension'). However, the final systems obtained in the above sequence of three papers still involve a new class of ergodic $\mathbb{Z}^2$-systems, referred to as 'directional CL-systems', which are concrete enough to enable a 'bare-hands' proof of the above convergence, but which otherwise remain largely mysterious. Examples include joinings of certain two-step Abelian isometric systems for which some one-dimensional subaction is trivial, and also nilsystem actions of $\mathbb{Z}^2$, but I do not know whether all examples can be expressed as factors of such joinings. If this is so then the proof of the above convergence could be greatly simplified. It can be shown that all examples can be classified modulo these simpler examples by a certain invariant of the system residing in the third measurable cohomology group of the compact Abelian group of the Kronecker factor of the system, following Moore's definition of measurable cohomology from [50, 51, 52]. However, Moore's cohomology groups are still so poorly understood that it is not clear whether these invariants are ever nontrivial. A proof that they are always trivial would constitute an improved understanding of that cohomology theory. If they are not, then this may point to genuinely new phenomena in the study of higher-dimensional nonconventional averages, which one would expect to have more complicated counterparts for the other cases of the Bergelson-Leibman conjecture that will also need to be understood. Moreover, if such an exotic system could be found, then sampling along its trajectories may also give rise to interesting examples of obstructions to higher-dimensional Gowers uniformity that will shed new light on the effort to improve the known bounds for the multidimensional Szemerédi Theorem (see, for instance, Gowers [35] and Chapters 10 and 11 of Tao and Vu [61]). A further investigation of these issues is one of the most active foci of my current research.

However, it seems likely that many of the above difficulties presented by discrete-time polynomial nonconventional averages simplify for the analogous continuous-time averages

$$\frac{1}{T}\int_0^T \prod_{i=1}^{k}(f_i \circ T^{\mathbf{p}_i(t)})\,\mathrm{d}t$$

corresponding to a jointly measurable action $T : \mathbb{R}^d \curvearrowright (X, \Sigma, \mu)$ and a tuple of polynomials $\mathbf{p}_i : \mathbb{R} \to \mathbb{R}^d$. I suspect that a suitable adaptation of the methods from [10] can be used to prove the norm convergence of these in general, and I am currently working with Vitaly Bergelson to try to prove this, and also generalize this result further to handle (i) probability-preserving actions of a connected nilpotent Lie group G and (ii) maps $p_i : \mathbb{R} \to G$ that are not necessarily polynomial, but only obey certain softer growth conditions on their derivatives (as introduced, for instance, by Bergelson and Knutson in [21]).

In the future, I also plan to pursue some of the related questions of pointwise convergence, which generally seem to be much harder. Aside from the result of Bourgain [24] that the particular averages $\frac{1}{N}\sum_{n=1}^{N}(f_1 \circ T^{an})(f_2 \circ T^{bn})$ converge pointwise almost surely for any $a \neq b$, little is known, and in the long run I plan to spend more time and effort on attempting generalizations of this result. This will require more than just the new techniques indicated above, including more harmonic analytic ideas such as those recently developed by Demeter, Lacey, Tao and Thiele in [29, 28].

Some of the above results on nonconventional averages required a generalization of the classical machinery of Mackey (see [48] and also Section 3.5 of Glasner [34]), Furstenberg ([31]) and Zimmer ([66, 65]) on the analysis of isometric extensions of probability-preserving systems, enabling one to use their results without assuming ergodicity of the overall system. I developed non-ergodic versions of these works and some of their corollaries (such as Mentzen's Theorem [49] classifying automorphisms of such extensions) in [6]. That work has led me to a broader interest in ergodic theory, which I have continued in collaboration with Mariusz Lemańczyk in the paper [16], which is concerned with improving some known methods of analysis for non-isometric extensions of probability-preserving transformations that can be expressed by combining a fixed action of a locally compact Abelian group on the fibre space with a cocycle taking values in that group. Another of my current projects is to use similar machinery to analyze when, given a pair of locally compact second countable groups $H \leq G$ and a probability-preserving jointly measurable action of H, an extension of this H-system can be found for which the action of H can then be embedded into a jointly measurable action of the larger group G. It turns out that if H is cocountable in G then this is always possible, but it seems that in other situations a range of different obstructions can arise.

Over the past three years I have established a number of collaborations within the ergodic theory community, and in addition to the projects described above I plan to continue working actively in this area in the future.

Metric geometry and geometric group theory

During recent years I have also developed an interest in metric geometry, largely in collaboration with Assaf Naor.

Our concrete results to date have focused on finding methods for estimating how well an invariant metric on a discrete group can be embedded into some 'nice' target space, usually a Hilbert space or another well-understood Banach space. On the one hand, embeddability results for groups have in some cases been shown to have far-reaching consequences for their algebraic properties (perhaps most strikingly through Yu's proof that groups admitting coarse embeddings into Hilbert spaces satisfy the Novikov conjecture [62]). On the other hand, the highly symmetric setting of invariant metrics on groups provides a valuable testing ground for the development of more general techniques for establishing bounds on different quantitative measures of the embeddability of a metric space.

Starting in the setting of finite groups, in the joint work [18] with Naor and Alain Valette we gave an explicit construction of a fairly low-distortion embedding of the lamplighter group $(\mathbb{Z}/2\mathbb{Z})\wr(\mathbb{Z}/n\mathbb{Z})$ into Hilbert space, matching a lower bound on the distortion that had previously been obtained by Lee, Naor and Peres [45]. We then switched to the world of infinite groups, where a popular quantitative measure of embeddability for a group Γ with a left-invariant metric ρ into a Banach space $(X, \|\cdot\|)$ is its 'compression exponent', introduced by Guentner and Kaminker in [37]: this is the supremum of the set of $\alpha \in [0, 1)$ for which there are some Lipschitz map $f : \Gamma \to X$ and constant $C > 0$ satisfying $\|f(g) - f(h)\| \geq C\rho(g, h)^\alpha$ for all $g, h \in \Gamma$. In the paper [17] with Naor and Yuval Peres we determined that the infinite lamplighter group $\mathbb{Z} \wr \mathbb{Z}$ has Euclidean compression exponent $\frac{2}{3}$, using a new adaptation of Ball's notion of Markov type to the setting of equivariant group embeddings to provide the upper bounds. Naor and Peres have since developed this technique considerably further to give an analysis of many more wreath products in [53, 54].

An outstanding question in this area due to Arzhantseva, Guba and Sapir [4] was whether every finitely generated amenable group admits a coarse Hilbert space embedding with compression exponent at least $\frac{1}{2}$. If true, this would have amounted to a quantitative strengthening of the classical result that any amenable group admits a coarse embedding into Hilbert space. However, I was recently able to settle this question negatively in [12], in fact constructing a finitely generated amenable group Γ whose left-invariant word metric ρ (for any choice of finite generating set) is such that any Lipschitz embedding $f : \Gamma \to L_p$ into any Lebesgue space with $p < \infty$ must give $\|f(g_n) - f(h_n)\| = \mathrm{O}(\log \rho(g_n, h_n))$ for some sequence of pairs g_n, h_n in Γ with $\rho(g_n, h_n) \to \infty$.

This Γ was constructed as a quotient of a doubly-iterated wreath product over some amenable base group of exponential growth. By choosing the quotient carefully, one can create a left-invariant word metric exhibiting a certain 'collapsing' phenomenon along a sequence of rapidly-increasing distance scales, which gives rise to an embedded sequence of expander graphs whose edge-length in Γ grows only very slowly relative to their numbers of vertices, so that this sequence serves as an obstruction to positive-compression embeddings.

However, having found this example I realized that the same basic method of construction (that is, as a 'slightly collapsed' wreath product) could also be used to address another, rather older conjecture in geometric group theory: the Atiyah Conjecture on the rationality of L^2-Betti numbers (see Atiyah [5] and Chapter 10 of Lück [47]). After a standard conversion into a purely group-theoretic question, this asserted that for any rational group ring element $A \in \mathbb{Q}\Gamma$ for a discrete group Γ, interpreted as an operator on $\ell^2(\Gamma)$ via the regular representation, the kernel $\ker A$ should have von Neumann dimension $\dim_{L\Gamma} \ker A$ relative to the group von Neumann algebra $L\Gamma$ that is a rational number.

I give a family of counterexamples to this conjecture in [11]. One of the chief difficulties in addressing this conjecture is that these von Neumann dimensions are generally very hard to compute, and so it is difficult to find one that is simple enough to evaluate explicitly but nevertheless gives an irrational number. A central innovation of [11] is that the examination of quotients of a wreath product circumvents this problem, by instead giving rise to a whole uncountable collection of examples (in fact, they come naturally parameterized by the power set $\mathcal{P}(\mathbb{N})$), such that for the correct choice of the parameters describing these subgroups it can be shown that the resulting von Neumann dimensions are all distinct (in particular, they define a strictly increasing function $\mathcal{P}(\mathbb{N}) \to \mathbb{R}$ for the lexicographic ordering of $\mathcal{P}(\mathbb{N})$). It follows that many of these values are irrational — even transcendental — without the need to evaluate any of them explicitly.

However, in this case the construction cannot yet give examples among amenable groups, since this time the wreath products are taken over the free group $\mathbf{F}_2$. Two immediate questions that I hope to examine in the near future are whether a more delicate implementation of this method can be made to work using an amenable group as base; and just what range of values of von Neumann dimensions can be produced in this way (the present method gives some Cantor set of values lying in $[0, 1]$). A more ambitious question asks how much can be recovered allowing only finitely *presented* groups. Of course, there are only countably many of these, so most of the examples from [11] must fall outside this class, and the restriction of Atiyah's conjecture to this class remains open. Thomas Schick has recently suggested to me that one may nevertheless be able to produce a large range of computable numbers using an adaptation of my construction, and I hope to work on this question more in the future.

More generally I should like to know how this construction can be combined with other known ideas of geometric group theory to study this and related conjectures (such as the Baum-Connes conjecture) for new classes of groups. This line of research has stimulated my interest in geometric group theory, and alongside my ongoing collaboration with Assaf Naor on metric geometry (we currently have another project underway to examine the embeddability properties of Wasserstein metrics on spaces of probability measures) I hope to move further into this area in the future.

Other plans for the future

One of the first projects I undertook as a graduate student was to try to understand how the theory of exchangeable arrays of random variables could be related to extremal hypergraph theory in much the same way as the ergodic theory of $\mathbb{Z}$-actions is related to additive combinatorics via the Furstenberg correspondence principle. This led to the survey paper [9] and the application of these ideas to prove a general property-testing result for directed coloured multi-hypergraphs in the joint work [19] with Tao, before I spent more time working with the ergodic theory of $\mathbb{Z}^d$-actions.

However, more recently I have returned to exchangeability theory with a rather different motivation: that of understanding exchangeable random *measures*. Given a (say, finite) alphabet A and a countably infinite index set S, these are probability distributions $\mathbb{P}$ on the standard Borel space $\mathrm{Pr}\ A^S$ of Borel probability measures on A^S (that is, random measures on A^S) that are invariant for the action of the permutation group $\mathrm{Sym}(S)$ on $\mathrm{Pr}\ A^S$ obtained by composition with the coordinate-permuting action on A^S. I believe that a structure theorem for these similar to the original results of Aldous [2, 3], Hoover [39, 40] and Kallenberg [44] on exchangeable arrays (and also related to the Dovbysh-Sudakov representation for exchangeable random infinite nonnegative-definite matrices [30, 55]) should be available. In addition to the intrinsic interest of such a result, I am interested in exchangeable random measures because they seem to offer a natural class of limit objects for certain diluted mean-field spin glass models, and so could play a similar rôle to the Random Overlap Structures of Aizenman, Sims and Starr [1] in the case of the Sherrington-Kirkpatrick model.

This idea has been motivated in part by several stimulating discussions with Dmitry Panchenko and Michel Talagrand, and by their joint paper [56] in which the beginnings of a Parisi-like ansatz for diluted spin-glass models are established (including an upper bound on the free energy that they conjecture to be tight), but with a rather complicated description of the relevant limit objects in terms of towers of random probability measures.

More generally, I have recently begun to learn more about the recent explosion in activity around spin glasses (largely due to Talagrand), with the ambition that together with my background in dynamical systems this will enable me to contribute to this theory in the future.

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