Past and Current Research: An outline of my papers.

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1 Introduction

The greatest part of my energy during the past years has been directed towards proving a conjecture of Deser and Schwimmer, originally formulated in high energy physics [9]. I was able to prove this conjecture in a special setting in my thesis [1], and recently have completed a 582-page manuscript where I prove it in full generality.

In section 2, I formulate the Deser-Schwimmer conjecture [9], and then give a very brief outline of the two published papers [3], [4] and of the preprint [5]. Some of the motivation behind this conjecture can be found in the introduction of [3]. In section 3, I briefly discuss some work on conformally invariant differential operators [6] and then my recent preprint [7] with Rafe Mazzeo on complete minimal surfaces in hyperbolic manifolds.

2 The Deser-Schwimmer Conjecture.

2.1 Formulation and Background.

The Deser-Schwimmer conjecture has a simple formulation, for which only a little background is required.

Background: Given any Riemannian manifold (M^n, g), a *local Riemannian invariant* is a scalar quantity $L(g)$ (which is assumed to depend polynomially on the components of the curvature tensor R_{ijkl} and its covariant derivatives $\nabla^{(m)}_{r_1 \dots r_m} R_{ijkl}$) whose value should be invariant under changes of coordinates. It is then known classically that $L(g)$ can be expressed as a linear combination of complete contractions in the tensor $\nabla^{(m)} R$ (i.e. covariant derivatives of the curvature tensor):

$$L(g) = \sum_{l \in L} a_l contr^l (\nabla^{(m_1)} R \otimes \dots \otimes \nabla^{(m_a)} R) \tag{1}$$

We restrict attention to Riemannian invariants of weight $-n$ (where n is the dimension of M^n–this means that $L(t^2 g) = t^{-n} L(g)$, for any constant $t > 0$).

Now, the Deser-Schwimmer conjecture concerns "global conformal invariants", which are defined to be local Riemannian invariants $L(g)$ whose *integral* over compact manifolds (M^n, g) is conformally invariant. In other words:

Definition 1 *A Riemannian invariant $P(g)$ of weight $-n$ (n even) will be called a* **global conformal invariant** *if for every compact orientable Riemannian n-manifold (M^n, g), the value of the integral $\int_{M^n} P(g) dV_g$ remains invariant under conformal re-scalings of the metric g.*

We also recall that a *local conformal invariant* of weight $-n$ is a Riemannian invariant $W(g)$ for which $W(e^{2\phi}g) = e^{-n\phi}W(g)$ for every Riemannian metric g and every $\phi \in C^\infty(M^n)$. Furthermore, recall the Pfaffian of the curvature tensor $\mathrm{Pfaff}(R_{ijkl})$, whose integral on a compact Riemannian manifold gives the Euler characteristic: $\int_{M^n} \mathrm{Pfaff}(R_{ijkl}) dV_{g^n} = \frac{2^n \pi^{\frac{n}{2}} (\frac{n}{2}-1)!}{2(n-1)!} \chi(M^n)$.

The Deser-Schwimmer conjecture:

Conjecture 1 *Let $P(g)$ be a global conformal invariant. Then there exists a local conformal invariant $W(g)$, a Riemannian vector field $T^i(g)$ and a constant $(Const)$ so that we can write:*

$$P(g) = W(g) + div_i T^i(g) + C \cdot \mathrm{Pfaff}(R_{ijkl}). \tag{2}$$

The main aim of the papers [3], [4], [5] was to prove the following theorem:

Theorem 1 *Conjecture 1 is true.*

Remark: It is interestng to note that a question which is strongly analogous to the Deser-Schwimmer conjecture has appeared in CR-geometry, in the context of the diagonal asymptotics of the Szegö kernel of strictly pseudo-convex CR-manifolds, see [10] for more details.

2.2 Synopsis of my papers on the Deser-Schwimmer conjecture:

On the decomposition of Global Conformal Invariants I, (65 pages) to appear in Annals of Mathematics, math.DG/0509571:

This paper introduces the first (and maybe most fundamental) tool in the proof of the Deser-Schwimmer conjecture: Consider any global conformal invariant $P(g)$ and define the differential operator $I_g(\phi)$ via the formula $I_g(\phi) = e^{n\phi}P(e^{2\phi}g) - P(g)$. The global invariance of $P(g)$ implies that $\int_{M^n} I_g(\phi) dV_g = 0$ for every compact (M^n, g) and every $\phi \in C^\infty(M^n)$. The main theorem in [3] proves a simple, usable formula which explicitly expresses $I_g(\phi)$ as a divergence: $I_g(\phi) = div_i T^i_g(\phi)$, in a highly non-trivial way. This formula is called the "super divergence formula" and is quite universal: Given any local invariant $L_g(\phi)$

whose integral is zero on any compact manifold, the method developed in [3] can be applied to explicitly express $L_g(\phi)$ as a divergence.

The first step in the derivation of the super divergence formula goes as follows: Starting from the integral equation $\int_{M^n} I_g(\phi)dV_g = 0$, we derive a sequence of new integral equations $\int_{M^N} I_g(\phi)dV_g = 0$ for any higher dimension $N \geq n$. Using a conformal transformation and successive integrations by parts, we then derive a sequence of *local equations* $I_g(\phi) = div_i(T_N)^i_g(\phi)$, where the coefficients of the vector fields $(T_N)^i_g(\phi)$ depend on the dimension N. Passing to a limit $N \to \infty$ we derive the "simple divergence formula"; after quite some work, we then refine this "simple divergence formula" to derive the "super divergence formula".

On the decomposition of Global Conformal Invariants II, Adv. in Math. **206** (2006), 466-502, math.DG/0509572:

This work builds on the super divergence formula introduced in [3] to prove the Deser-Schwimmer conjecture in a special case, where the global conformal invariant $P(g)$ locally depends only on the curvature tensor (without derivatives). In that case, I show that $P(g)$ can be written as $P(g) = W(g) + (Const)\cdot$ Pfaff(R_{ijkl}), where $W(g)$ is a local conformal invariant which locally depends only on the Weyl curvature (without derivatives).

The proof of this decomposition is an inductive one: We assume that we know the coefficient of one particular complete contraction in $P(g)$. We then employ the super divergence formula (applied to $I_g(\phi)$), and we iteratively determine the coefficients of *all other* complete contractions in $P(g)$, *up to a local conformal invariant* $W(g)$. The proof then concludes by proving that *up to a local conformal invariant*, $P(g)$ must necessarily be equal to a scalar multiple of the Pfaffian.

The Decomposition of Global Conformal Invariants: On a conjecture of Deser and Schwimmer. arXiv:0711.1685, 582 pages.

An outline of the argument: The Deser-Schwimmer conjecture is proven by a multiple induction. At the roughest level, the induction works as follows: We consider a global conformal invariant $P(g)$, which we write out as a linear combination of complete contractions, $P(g) = \sum_{l\in L} a_l \cdot contr^l(\nabla^{(m_1)}R \otimes \dots \otimes \nabla^{(m_a)}R)$. (We will denote the complete contractions by $C^l(g)$ for short).

The different complete contractions $C^l(g)$ appearing above can be grouped up into "categories" according to certain algebraic features of the tensors involved. Accordingly, we write: $P(g) = \sum_{t=1}^T \sum_{l\in L^t} a_l C^l(g)$, where the terms indexed in the same index set L^t belong to the same category (and vice versa), and $\bigcup_{t=1}^T L^t = L$. Now, the categories of complete contractions are naturally graded: A given category of complete contractions will be "better" or "worse" than any other given category. The "best" category is the one where no curvature factor in the complete contractions $C^l(g)$ is differentiated. The main step of our proof then works as follows:

We assume that in $P(g)$ (expressed as in the previous paragraph), for each pair $1 \leq \alpha < \beta \leq T$ the category of terms indexed in L^β is "worse" than the category of terms indexed in L^α. (Therefore, in particular the "worst" category of complete contractions in $P(g)$ is the category $\sum_{l\in L^T} a_l C^l(g)$).

The main step of our induction is to prove that *unless the complete contractions* $C^l(g), l \in L^T$ *have only undifferentiated curvature factors*, there exists a local conformal invariant $W(g)$ and a divergence of a vector field $div_i T^i(g)$ so that:

$$\sum_{l\in L^T} a_l C^l(g) - W(g) - div_i T^i(g) = \sum_{l\in L^{new}} a_l C^l(g), \tag{3}$$

where the complete contractions in the RHS of the above belong to categories that are all "better" than the category $\sum_{l\in L^T} a_l C^l(g)$.

Iterative repetition of this "main step" shows that there exists a local conformal invariant $W(g)$ and a divergence $div_i T^i(g)$ so that:

$$P(g) - W(g) - div_i T^i(g) = \tilde{P}(g), \tag{4}$$

where $\tilde{P}(g)$ is a linear combination of terms with only (undifferentiated) curvature factors. Furthermore, $\tilde{P}(g)$ *is also a global conformal invariant.* Therefore, invoking the main theorem of [4], we derive that $\tilde{P}(g)$ can be written in the form:

$$\tilde{P}(g) = W'(g) + (Const) \cdot \text{Pfaff}(R_{ijkl}), \tag{5}$$

where $W'(g)$ is a local conformal invariant[1]. Therefore, combining (4) and (5) we derive the Deser-Schwimmer conjecture.

The main difficulty: The inductive argument presented above is hard to implement, for one main reason: Whereas the super divergence formula is a powerful tool which expresses the operator $I_g(\phi) = e^{n\phi}P(e^{2\phi}g) - P(g)$[2] as a divergence, if one attempts to apply this formula straightforwardly to $P(g)$, one finds that one can subtract a divergence from $P(g)$ and cancel out the "worst piece" in $P(g)$, *but one will also introduce correction terms which will not necessarily be "better" than the ones we have cancelled out.* Therefore, in order to make the argument close up, one needs to make a more subtle use of the super divergence formula for $I_g(\phi)$. This is why the paper is 582 pages long.

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[1] $\text{Pfaff}(R_{ijkl})$ is the Pfaffian of the curvature tensor.

[2] Notice that $I_g(\phi)$ "measures" the non-conformally invariant piece of $P(g)$.

2.3 Work on minimal surfaces in hyperbolic space and on conformally invariant differential operators.

On conformally invariant differential operators in odd dimensions Proc. Natl. Acad. Sci. USA **100** (2003), no. 8, 4409–4410 and **On conformally invariant differential operators**, submitted, arxiv math.DG/0608771.

The above two papers consider the problem of constructing conformally invariant differential operators (acting on scalar functions) and then proving that all operators with this invariance arise via this construction. This question fits into the broader problem of understanding the local invariants of conformal structures. In the first paper above, I present a construction of invariant operators and explain that the methods of [8] can be adapted to prove that all such operators arise via this construction (for odd dimensions, apart from certain exceptional cases). In the second paper, I introduce a simple and direct method which circumvents the representation-theoretic methods developed in [8] for such problems, and prove that all invariant operators arise via this construction, when the weight of the operators is bounded by the dimension–this new approach captures certain cases that were left open in [8].

Renormalized area and complete minimal surfaces in hyperbolic manifolds, preprint (joint with R. Mazzeo).

This paper considers complete minimal surfaces in hyperbolic manifolds (primarily $\mathbb{H}^3$) with a prescribed asymptotic boundary at the conformal infinity $\mathbb{S}^2$. The main object of our study is the *renormalized area* functional[3] on the moduli space of such minimal surfaces.

We derive a formula which expresses the renormalized area via a *convergent* integral: For any minimal surface Y with an asymptotic boundary γ, its renormalized area is essentially equivalent to its total extrinsic curvature squared: $Ren.Area[Y] = -2\pi\chi(Y) - \frac{1}{2}\int_Y |II_Y|^2 dV_Y$. (This formula could lead to a confirmation of a certain conjecture of A.M. Polyakov, [11]).

Another result we derived concerns minimal surfaces which are *critical points* for the renormalized area functional[4]. We showed that the only minimal surfaces which are critical w.r.t. the renormalized area functional are the totally geodesic hyperplanes (i.e., in the upper half-space model for $\mathbb{H}^3$, the half-spheres which bound a round circle). The proof of this result relied on an "asymptotic maximum principle" for minimal surfaces with asymptotic boundaries which we proved.

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[3] Roughly speaking, the renormalization of area is performed by considering the area $Area\{Y \bigcap B(p,M)\}$ ($B(p,M)$ stands for the ball in $\mathbb{H}^3$ with center p and radius M), taking the expansion of this area as $M \to \infty$ and then picking out the *finite term* in this expansion.

[4] In other words, minimal surfaces Y for which the first variation of renormalized area will be zero under any first-order perturbation of Y.

References

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