Faltings heights and Zariski density of CM abelian varieties

Shou-Wu Zhang
Algebraic Geometry, Salt Lake City
July 31, 2015

References

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$$\mathbb{S}_1 = \mathbb{G}_m \backslash \mathcal{W} \overset{j}{\simeq} \mathbb{A}^1, \qquad \mathbb{S}_1(\mathbb{C}) = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H} \overset{j}{\simeq} \mathbb{C}$$

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$$j(E_{a,b}) = 1728 \frac{4a^3}{4a^3 + 27b^2}, \quad j(E_\tau) = e^{-2\pi i \tau} + 744 + 196884 e^{2\pi i \tau} + \cdots$$

Elliptic curves with complex multiplications.

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In this description, $\mathrm{End}(E) \subset \mathcal{O}_K$ is an order, and

$$CM(R) := \{[E] : \mathrm{End}(E) = R\}$$

is a PHS of $\mathrm{Pic}(R)$ under the action $I * \mathbb{C}/\Lambda = \mathbb{C}/(I \cdot \Lambda)$.

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Theorem (Weber–Fueter)

1. *All $j(E), E \in CM(R)$ are algebraic integers and conjugates to each other;*
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For example, if $R = \mathbb{Z}[1 + \sqrt{-163})/2]$, then $h(R) = 1$, and

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This gives an approximation:

$$e^{\pi\sqrt{163}} = 640320^3 + 743.99999999999925007..., \qquad (\textit{Ramanujan})$$

Degree and distribution of CM points

The schemes $CM(R)$ has dimension 0. As $\mathrm{disc} R \longrightarrow \infty$, the degree $h(R)$ grows as follows:

$$h(R) = |\mathrm{disc} R|^{1/2+o(1)}, \quad \text{(Brauer–Siegel)}.$$

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The subschemes $CM(R)$ in $\mathbb{S}_1$ are equidistubted with respect to the Poincaré measure:

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Question

For a prime p, are $CM(R)$'s equidistributed on the p-adic analytic space $\mathbb{S}_1^{\mathrm{an}}$ with respect to some measure?

Siegel moduli spaces

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Theorem (Shimura–Taniyama)

The $[A] \in \mathbb{S}_g(\mathbb{C})$ is defined over a number field, more precisely,

$$[A] \in \mathbb{S}_g(K(\Phi)^{\mathrm{ab}}), \qquad K(\Phi) := \mathbb{Q}(\mathrm{tr}\Phi(x), x \in K).$$

Special subvarieties

The $\mathcal{H}_g$ can be identified with the $\mathrm{GSp}(2g,\mathbb{R})_+$- conjugacy class of

$$h_g : \mathbb{C}^\times \longrightarrow \mathrm{GSp}(2g,\mathbb{R})_+ \qquad x+yi \mapsto \begin{pmatrix} x1_g & y1_g \\ -y1_g & x1_g \end{pmatrix}^{-1}.$$

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CM points are exactly the zero dimensional special subvarieties of $\mathbb{S}_g$.

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Conjecture (Edixhoven)

There are positive c_g, δ_g such that for any CM abelian variety A of dim g, with the center Z of $\mathrm{End}(A)$,

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Equidistribution Problem

The Galois orbits for a generic sequence of CM points on a special variety X/L are equidistruted with respect to some v-adic measure for each place v of L.

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Theorem

The AO conjecture holds for mixed Shimura varieties of abelian type.

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$$\textit{Faltings height of } A = h(A) := \frac{1}{[K:\mathbb{Q}]} \deg \overline{\omega}(\mathcal{A}).$$

Assume $\mathcal{A}$ is semiabelian, then height is invariant under base change.

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A: a CM abelian variety of type (O_E, Φ).

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Question

Is there a arithmetic/geometric formula for every derivative $L^{(n)}(0,\eta)$?

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Combine to obtain $\deg[A] >> |\mathrm{disc} E|^{c_g/4+o_g(1)}$. □

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When A has CM, $\mathrm{disc}(A) = 1$; for the second term, apply either Kronecker–Limit or Chowla–Selberg formula.

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This series is proportional to the Eisenstein series of weight 2. □

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Proposition

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