Stable birational invariants and the Lüroth problem

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Definition: (*Stable birational equivalence*)

- Two varieties X, Y (say, over $\mathbb{C}$) are stably birational if $X \times \mathbb{P}^r$ is birational to $Y \times \mathbb{P}^s$ for some r, s.
- X is stably rational if it is stably birational to a point.

- **Stable birational invariants**: Equivalently, should be invariant under $X \rightsquigarrow X \times \mathbb{P}^r$ and under birational equivalence.

Example

- X smooth projective, $H^0(X, \Omega_X^{\otimes k})$, $k \geq 0$, is a stable birational invariant of X.
- X smooth projective over $\mathbb{C}$; then $\pi_1(X_{an})$ is a stable birational invariant.

These invariants are trivial for rationally connected (RC) varieties over $\mathbb{C}$:

Theorem

If X is smooth proj. RC over $\mathbb{C}$, then
(i) $H^0(X, \Omega_X^{\otimes k}) = 0$, $k > 0$.
(ii) (Serre) X_{an} is simply connected.

Property (i) conjecturally characterizes RC varieties over $\mathbb{C}$ (Mumford's conjecture). True in dimension ≤ 2:

Theorem (Castelnuovo)

(Over $\mathbb{C}$) If $\dim X = 1$ or $\dim X = 2$ and $H^0(X, \Omega_X^{\otimes k}) = 0$ for $k > 0$, then X is rational (in particular X is stably rational and RC).

Definition: (*Unirational varieties*)

X is unirational if there exists a dominant rational map $\phi : \mathbb{P}^N \dashrightarrow X$.

• **Obvious implications:**

Rational $\Rightarrow$ Stably rational $\Rightarrow$ Unirational $\Rightarrow$ Rationally connected.

• The general question is how strict are these implications. The most important one (and completely open) is:

Conjecture

There exist rationally connected varieties which are not unirational.

The other implications are known to be strict starting from dimension 3:

• **Unirational non-rational varieties**:
(1) **Iskovskikh-Manin**: Consider the group $\mathrm{Bir}(X)$ of birational self-maps $X \dashrightarrow X$. If X is rational, this group is enormous.

Theorem (Iskovskikh-Manin 1971)

Certain smooth quartic hypersurfaces $X \subset \mathbb{P}^4_{\mathbb{C}}$ are unirational. Any such X has $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ (a finite group), hence is not rational.

(2) **Clemens-Griffiths**.

• *The intermediate Jacobian*. $\dim X = 3$, X RC. $J^3(X) =$ complex torus $H^{1,2}(X)/H^3(X,\mathbb{Z})$. This is a ppav (use $<,>_X$ for the polarization).

Theorem (Clemens-Griffiths 1970)

(i) If X is rational then $J^3(X)$ is isomorphic as a ppav to a product of Jacobians of curves.
(ii) A smooth cubic hypersurface in $\mathbb{P}^4$ is not rational because it does not satisfy criterion (i).

• (i) is used by Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer to prove: *There exist stably rational non-rational varieties*.

Artin-Mumford double solid. Let X be a desingularization of $Y=$ double cover of $\mathbb{P}^3$ with equation $u^2 = f(x_0, \ldots, x_3)$, where $\deg f = 4$ and the quartic $f = 0$ has 10 nodes in special position: projecting from a node, the $K3$ is a ramified double cover of the plan. One wants the ramification curve to be the union of two cubics.

- X is unirational.

Theorem (Artin-Mumford 1972)

X as above has nonzero 2-torsion in $H^3(X, \mathbb{Z})$.

Theorem (Artin-Mumford 1972)

The group $\mathrm{Tors}(H^3(X,\mathbb{Z}))$ *is a stable birational invariant.*

• Thus a stably rational smooth projective variety has no torsion in $H^3(X,\mathbb{Z})$ and the Artin-Mumford double solid above is not stably rational.

Proof of Thm: Use the fact that $H^1(B,\mathbb{Z})$ is always torsion free. So $\mathrm{Tors}(H^3(X,\mathbb{Z}))$ is invariant under $X \rightsquigarrow X \times \mathbb{P}^r$.
For birational invariance, if $Y \cong_{birat} X$, there is a degree 1 morphism $Y' \to X$, where Y' is a blow-up of Y, so $H^3(X,\mathbb{Z}) \hookrightarrow H^3(Y',\mathbb{Z})$ and $\mathrm{Tors}(H^3(Y',\mathbb{Z})) = \mathrm{Tors}(H^3(Y,\mathbb{Z}))$.

• X is algebraic over $\mathbb{C}$ so X (or rather its set of points) has two topologies: "us" and "Zar". Continuous map: $\pi : X_{us} \to X_{Zar}$.

• Let A be an abelian group. The Leray spectral sequence associated to π and A with term $E_2^{p,q} = H^p(X_{Zar}, R^q\pi_* A)$ is the Bloch-Ogus spectral sequence.

Definition: (*Unramified cohomology*, Colliot-Thélène and Ojanguren 1988)

• $\mathcal{H}^i(A) := R^i\pi_* A$.(This is a sheaf on X_{Zar}).

• $H^i_{nr}(X, A) := H^0(X_{Zar}, \mathcal{H}^i(A))$.

- The groups $H^i_{nr}(X, A)$ are stable birational invariants.
- $H^i_{nr}(X, A)$ vanishes for $i > dim\, X$.
- $H^i_{nr}(X, \mathbb{Z})$ vanishes for $i > 0$, X RC. (Easy with $\mathbb{Q}$-coeffts, hard with $\mathbb{Z}$-coeffts (Bloch-Kato conjecture)).

Example

- If X is RC, $H^2_{nr}(X, \mathbb{Q}/\mathbb{Z}) = \mathrm{Tors}(H^3(X, \mathbb{Z}))$ =Brauer group of X.
- (C-T-V '12) If X is RC, $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hdg}^4(X, \mathbb{Z})/H^4(X, \mathbb{Z})_{alg}$.

Theorem (Colliot-Thélène-Ojanguren 1988)

There exist 6-dimensional unirational varieties with $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \neq 0$. Any such X is not stably rational.

• Clemens-Griffiths works only in dimension 3. Iskovskikh-Manin does not work on some interesting examples (like quartic double solids). They do not address the stable rationality problem.

• The Artin-Mumford invariant detects stable irrationality, but it is deformation invariant. For unirational threefolds, this is the only possibly nonzero unramified coh. group.

Theorem (Voisin 2006)

Let X be a uniruled threefold. Then any integral Hodge class on X is algebraic. Thus by CT-V, $H^i_{nr}(X, \mathbb{Q}/\mathbb{Z}) = 0$ for $i \geq 3$, X a RC 3-fold.

On the other hand:

Conjecture

(stable) rationality is not a deformation invariant property.

Example

Some cubic fourfolds are rational. It is generally believed that the very general cubic fourfold is not rational.

• X over K. $\mathrm{CH}_0(X)$ is the group $\mathcal{Z}_0(X)/\mathcal{Z}_0(X)_{rat}$ of 0-cycles on X modulo rational equivalence. $\mathcal{Z}_0(X)_{rat}$ is generated by $\operatorname{div}\phi$, $\phi =$ rational function on $C \subset X$, everything defined over K.

• If X is RC over $\mathbb{C}$, $\mathrm{CH}_0(X) = \mathbb{Z}$. Indeed, for any $x, y \in X$ there is a $\mathbb{P}^1$ in X passing through x and y: then $x - y = \operatorname{div}\phi$ on this curve.

• For $X = \mathbb{P}^n$, much more is true, because even over a non-alg. closed field L, $\mathrm{CH}_0(\mathbb{P}^n_L) = \mathbb{Z}$.

• So if X is rational, or stably rational, for any field L containing $\mathbb{C}$, $\mathrm{CH}_0(X_L) = \mathbb{Z}$ (one then says that "$\mathrm{CH}_0(X)$ is *universally trivial*"). (this notion is due to Merkureev, Auel-CT-Parimala).

• Let $L := \mathbb{C}(X)$. The diagonal of $X \times X$, seen over the generic point, provides a L-point $\delta_L \in X_L(L)$. If X has universally trivial CH_0, this point is rationally equivalent over L to the constant point $x_L := Spec(L) \times x \in X_L(L)$.

Lemma (Auel-Colliot-Thélène-Parimala 2013)

$\mathrm{CH}_0(X)$ *is universally trivial iff* $\delta_L - x_L = 0$ *in* $\mathrm{CH}_0(X_L)$.

• $Spec(L) \times X \subset X \times X$ is the limit of $U \times X$, $U \subset X$ Zariski open dense subset.

• $x_L \in \mathrm{CH}_0(X_L) = \mathrm{CH}^n(X_L)$ is the restriction to $Spec(L) \times X$ of $X \times x$, and $\delta_L \in \mathrm{CH}^n(X_L)$ is the restriction of Δ_X. Hence $\delta_L - x_L = 0$ in $\mathrm{CH}_0(X_L)$ iff for some dense open $U \subset X$, $\Delta_X - X \times x$ vanishes in $\mathrm{CH}^n(U \times X)$ and thus by localization exact sequence

$$(*) \qquad \Delta_X = X \times x + Z \text{ in } \mathrm{CH}^n(X \times X)$$

with Z supported on $D \times X$, where $D = X \setminus U$.

• (*) is called a **Chow-theoretic decomposition of diagonal**. This notion is due to Bloch-Srinivas (Voisin with $\mathbb{Z}$-coeffts).

• A weaker version is the corresponding **cohomological decomposition** of the diagonal:

$$[\Delta_X] = [X \times x] + [Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}).$$

with Z supported on $D \times X$.

• The existence of such decompositions is a necessary criterion for stable rationality.

Lemma

If X admits a cohomological (or Chow) decomposition of the diagonal, $\mathrm{Tors}(H^3(X,\mathbb{Z}))=0$.

Sketch of proof. Indeed, let $j:\widetilde{D}\to X$ be a desingularization of D, with a lift $\widetilde{Z}\in \mathrm{CH}^n(\widetilde{D}\times X)$. Then

$$(j,Id_X)_*([\widetilde{Z}])=[\Delta_X]-[X\times x] \text{ in } H^{2n}(X\times X,\mathbb{Z}).$$

Hence, for any $\alpha\in H^3(X,\mathbb{Z})$,

$$\alpha=([\Delta_X]-[X\times x])^*\alpha=j_*([\widetilde{Z}]^*\alpha) \text{ in } H^3(X,\mathbb{Z}),$$

with $[\widetilde{Z}]^*\alpha\in H^1(\widetilde{D},\mathbb{Z})$.
But $H^1(\widetilde{D},\mathbb{Z})$ has no torsion, so $\alpha=0$ if $\alpha\in\mathrm{Tors}(H^3(X,\mathbb{Z}))$.

Theorem (Voisin 2013)

Let $\mathcal{X} \to B$ be a flat morphism of relative dim ≥ 2 onto a smooth curve B over $\mathbb{C}$; let $0 \in B$. Assume the general fiber $\mathcal{X}_t$ is smooth and $\mathcal{X}_0$ has at worst ordinary quadratic singularities.
(i) If $\mathcal{X}_t$ has a Chow-theoretic decomposition of diagonal for $t \neq 0$, then so does the desingularization $\widetilde{\mathcal{X}_0}$.
(ii) If $\mathcal{X}_t$ has a cohomological decomposition of diagonal for $t \neq 0$, then so does $\widetilde{\mathcal{X}_0}$, assuming the even degree integral cohomology of $\widetilde{\mathcal{X}_0}$ is algebraic.

Sketch of proof of (i) For general $t \in B$, there exist $D_t \subset \mathcal{X}_t$, $Z_t \subset D_t \times \mathcal{X}_t$, and $x_t \in \mathcal{X}_t$ such that $\Delta_{\mathcal{X}_t} = \mathcal{X}_t \times x_t + Z_t$ in $\mathrm{CH}(\mathcal{X}_t \times \mathcal{X}_t)$. We pass to a finite cover B' of B so as to have the data above in family and specialize to $\mathcal{X}_0 \times \mathcal{X}_0$ and this gives a decomp of the diagonal for $\mathcal{X}_0$. Taking proper transforms in $\widetilde{\mathcal{X}_0} \times \widetilde{\mathcal{X}_0}$, one gets:
$\Delta_{\widetilde{\mathcal{X}_0}} = \widetilde{\mathcal{X}_0} \times x_0 + Z + Z'$ in $\mathrm{CH}(\widetilde{\mathcal{X}_0} \times \widetilde{\mathcal{X}_0})$, with Z supported on $D_0 \times \widetilde{\mathcal{X}_0}$ and Z' supported on $E \times \widetilde{\mathcal{X}_0} \cup \widetilde{\mathcal{X}_0} \times E$, $E=$ except. divisor of resolution. Then use that E is a union of quadrics to decompose Z'.

Remark

In dim 3, de Fernex-Fusi have a similar degeneration result for rationality. A crucial point is to have restrictions on the singularities.

Application of main Thm

Theorem (Voisin 2013)

The very general k-nodal double solid with $k \leq 7$ is not stably rational. (On the other hand, it has no torsion in H^3 by Endrass.)

Sketch of proof

• The Artin-Mumford threefold does not admit a Chow-theoretic decomposition of the diagonal because $\mathrm{Tors}(H^3(X,\mathbb{Z})) \neq 0$ (use the lemma).

• On the other hand, it is the desingularization of a 10-nodal quartic double solid.

• So by the degeneration result, it suffices to show that the very general $k \leq 7$-nodal quartic double solid can be specialized to the Artin-Mumford double solid.

• Colliot-Thélène and Pirutka: Prove the degeneration results under much less restrictive assumptions on singularities. Application: Very general quartic hypersurfaces are not stably rational.

• Using this method, Beauville proves stable irrationality of very general sextic double solids, quartic double fourfolds or fivefolds.

• Totaro uses the above improvements and Kollár's specialization to char. $\neq 0$ to prove stable irrationality of very general hypersurfaces in Kollár's range.

• Hasset-Tschinkel provide a geometric criterion for stable irrationality of very general conic bundles over surfaces.

- The degeneration argument also applies to prove that the very general $k \leq 7$-nodal double solid X does not admit a **cohomological** decomposition of the diagonal. We now want to make this non-existence result more explicit.

- X of dim n with $H^{3,0}(X) = 0$. Then abelian variety (intermediate Jacobian) $J^3(X) = H^3(X, \mathbb{C})/(H^{2,1}(X) \oplus H^3(X, \mathbb{Z})) = H^{n-1,n-2}(X)^*/H_{2n-3}(X, \mathbb{Z})$ with Abel-Jacobi map

$$\Phi_X : \mathrm{CH}^2(X)_{hom} \to J^3(X), \quad z \mapsto \int_\gamma \in H^{n-1,n-2}(X)^*, \, \partial\gamma = z.$$

Theorem (Bloch-Srinivas)

$\Phi_X : \mathrm{CH}^2(X)_{hom} \to J^3(X)$ *is a group isomorphism for any rationally connected variety* X*.*

• The left hand side is not an algebraic variety (it is a limit of quotients of algebraic varieties by equivalence relation), while the right hand side is an algebraic variety. Φ_X is algebraic in a certain functorial sense.

Question. *Does there exist a universal codimension* 2 *cycle* $Z \in \mathrm{CH}^2(J^3(X) \times X)$ *i.e:*

$$\Phi_Z : J^3(X) \to J^3(X), \ \ t \mapsto \Phi_X(Z_t),$$

is the identity?

Theorem (Voisin 2012 and 2014)

Let X be a rationally connected threefold. Then X admits a cohomological decomposition of the diagonal iff:
(i) $\mathrm{Tors}(H^3(X,\mathbb{Z})) = 0$.
(ii) There exists a universal codimension 2 cycle on $J^3(X) \times X$.
(iii) There exists a 1-cycle $z \in J^3(X)$ of the minimal class, ie $[z] = \theta^{g-1}/(g-1)!$, $g = dim\, J^3(X)$.

Hence (i), (ii), (iii) are necessary criteria for stable rationality. (i) is Artin-Mumford's criterion. (iii) generalizes Clemens-Griffiths' criterion.

Corollary

The very general 7-nodal double solid X does not have a universal codimension 2 cycle on $J^3(X) \times X$.

Indeed, $\mathrm{Tors}(H^3(X, \mathbb{Z})) = 0$ by Endrass, and $dim\, J^3(X) = 3$ so $J^3(X)$ is a Jacobian, hence (iii) is satisfied. But X does not admit a cohomological decomposition of the diagonal.

Theorem (Voisin 2014)

Let $X \subset \mathbb{P}^n$ be a smooth cubic hypersurface of odd dimension or a cubic fourfold. Then X admits a cohomological decomposition of the diagonal iff X admits a Chow-theoretic decomposition of the diagonal.

The proof uses the observation that $X^{[2]}$ is birational to the projective bundle over X with fiber over x the lines in $\mathbb{P}^n$ passing through x, by the map

$(x, y) \mapsto (l_{x,y}, z),\ l_{x,y} =< x, y >, z + x + y = l_{x,y} \cap X.$

- Recall $\mathrm{CH}_0(X)$ being universally trivial is equivalent to X admitting a Chow-theoretic decomposition of the diagonal.

Theorem (Voisin 2014)

Let X be a smooth cubic threefold. Then $\mathrm{CH}_0(X)$ is universally trivial iff there exists a 1-cycle on $J^3(X)$ in the class $\theta^4/4!$.

- Whether this is satisfied or not is a very classical open problem.
- There is a countable union of subvarieties of codimension ≤ 3 in the moduli space of X where this is satisfied.

Theorem (Voisin 2014)

Let X be a cubic fourfold. Assume X is special in the sense of Hassett, with discriminant not divisible by 4. Then $\mathrm{CH}_0(X)$ is universally trivial.

Both results use the previous theorem. One shows that under the assumptions made, X admits a cohomological decomposition of the diagonal, hence a Chow-theoretic one.