p-adic Hodge theory

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Algebraic Geometry
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If $C = \widehat{\overline{\mathbb{Q}_p}}$, then $C^\flat = \widehat{\overline{\mathbb{F}_p((t))}}$, where t corresponds to $(p, p^{1/p}, \ldots)$.

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The map $C^\flat \to C : x \mapsto x^\sharp$ is analytic, highly non-algebraic.

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As $x \mapsto x^\sharp$ is non-algebraic, need to formalize this in an analytic world.

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Note: If $L = C^\flat$ is of characteristic p, last condition is equivalent to requiring R perfect.

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Theorem (S., 2011)

The functor $R \mapsto R^\flat$ is an equivalence between the category of perfectoid C-algebras and the category of perfectoid $C^\flat$-algebras.

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Define general perfectoid spaces by gluing affinoid perfectoid spaces.

Corollary

The categories of perfectoid spaces over C *and* $C^\flat$ *are equivalent.*

Example

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The inverse limit

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char p geometry as infinite covering of char 0 geometry.

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This is closely related to Faltings's celebrated "almost purity theorem".

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$$H^0(X_{\text{proét}}, \hat{\mathcal{O}}_X^+) = R \ ,$$
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Remains to compute

$$H^i_{\mathrm{cont}}(\mathbb{Z}_p, \mathcal{O}_C\langle T^{\pm 1/p^\infty}\rangle) = \widehat{\bigoplus}_{j \in \mathbb{Z}[1/p]} H^i_{\mathrm{cont}}(\mathbb{Z}_p, \mathcal{O}_C \cdot T^j) \ .$$

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Fix a compatible system of p-power roots of unity $\zeta_p, \zeta_{p^2}, \ldots \in \mathcal{O}_C$. Then the generator $\gamma = 1 \in \mathbb{Z}_p$ acts via

$$\gamma \cdot T^j = \zeta_{p^m}^n T^j \ , \ j = n/p^m \ , \ n, m \in \mathbb{Z} \ .$$

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Then $H^i_{\mathrm{cont}}(\mathbb{Z}_p, \mathcal{O}_C \cdot T^j)$ computed by the complex

$$(\mathcal{O}_C \cdot T^j \xrightarrow{\gamma - 1} \mathcal{O}_C \cdot T^j) \cong (\mathcal{O}_C \xrightarrow{\zeta_{p^m}^n - 1} \mathcal{O}_C)\ .$$

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End result.

$$H^0(X_{\text{proét}}, \hat{\mathcal{O}}_X^+) = \widehat{\bigoplus}_{j \in \mathbb{Z}} \mathcal{O}_C T^j = R \ ,$$

$$\begin{aligned} H^1(X_{\text{proét}}, \hat{\mathcal{O}}_X^+) &= \widehat{\bigoplus}_{j \in \mathbb{Z}} \mathcal{O}_C T^j \oplus \bigoplus_{j = n/p^m \in \mathbb{Z}[1/p] \setminus \mathbb{Z}} (\mathcal{O}_C / (\zeta_{p^m}^n - 1)) T^j \\ &= \Omega^1_{R/\mathcal{O}_C} \oplus (p^{1/(p-1)}\text{-torsion}) \ . \end{aligned}$$

Almost finite generation: Local case

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For the proof, redo the computation in any dimension, or use the Künneth formula.

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The key point is that the transition maps

$$H^j(V_{i,\mathrm{\acute{e}t}}, \mathcal{O}_X^+/p) \to H^j(U_{i,\mathrm{\acute{e}t}}, \mathcal{O}_X^+/p)$$

have almost finitely generated image (over $\mathcal{O}_C$).

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Enough to prove similar result for

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There, use Artin–Schreier sequence

$$0 \to \mathbb{F}_p \to \mathcal{O}_X^+/p \to \mathcal{O}_X^+/p \to 0 \ .$$

The Hodge–Tate decomposition

Recall that we want to prove the Hodge–Tate decomposition, for a proper smooth rigid-analytic variety X_0 over K with $X = X_0 \otimes_K C$,

$$H^i(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \cong \bigoplus_{j=0}^{i} H^{i-j}(X, \Omega_X^j)(-j) \ .$$

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At this point, we have an isomorphism

$$H^i(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \cong H^i(X_{\text{proét}}, \hat{\mathcal{O}}_X) \ ,$$

where $\hat{\mathcal{O}}_X = \hat{\mathcal{O}}_X^+[1/p]$.

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The local computation of the cohomology of $\hat{\mathcal{O}}_X$ in terms of differentials gives the *Hodge–Tate spectral sequence*

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Remark. One can show that the Hodge–Tate spectral sequence degenerates always, for a proper smooth rigid-analytic variety X over C. However, it does not canonically split.